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THE EQUIVARIANT COHOMOLOGY OF WEIGHTED PROJECTIVE SPACES

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ABSTRACT. We describe the integral equivariant cohomology of a weighted projective space in terms of piecewise polynomials as well as by generators and relations. Unlike the ordinary integral cohomology, this ring distinguishes among weighted projective spaces. We also prove a Chern class formula for weighted projective bundles.

1. INTRODUCTION

Let $\chi = (\chi_0, \dots, \chi_n)$ be a vector of positive natural numbers. The associated weighted (or "twisted") projective space is the quotient

$$\mathbb{P}(\chi) = S^{2n+1}/S^1\langle \chi_0, \dots, \chi_n \rangle, \tag{1.1}$$

where the numbers in angles indicate the weights with which S^1 acts on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$,

$$g \cdot (x_0, \dots, x_n) = (g^{\chi_0} x_0, \dots, g^{\chi_n} x_n). \tag{1.2}$$

Note that $\mathbb{P}(\chi)$ is equipped with an action of the n-dimensional torus

$$T = (S^1)^{n+1}/S^1\langle \chi_0, \dots, \chi_n \rangle, \tag{1.3}$$

where the quotient is defined as before. Different weights vectors may give the same weighted projective space; we will come back to this in Section 5

Kawasaki [Kw] has computed the ordinary cohomology ring of $\mathbb{P}(\chi)$ with integer coefficients. Additively, the cohomology is isomorphic to that of ordinary complex projective space, but the multiplication is different. More precisely, if c_1 is a generator of the group $H^2(\mathbb{P}(\chi))$, then $H^*(\mathbb{P}(\chi))$ is generated as a ring by the elements

$$c_m = \frac{\operatorname{lcm}\{\, \prod_{i \in I} \chi_i : |I| = m \,\}}{\operatorname{lcm}(\chi_0, \dots, \chi_n)^m}\, c_1^m \in H^{2m}(\mathbb{P}(\chi)) \tag{1.4}$$

with $1 \le m \le n$, with the obvious multiplication.

In this note we study $H_T^*(\mathbb{P}(\chi))$, the T-equivariant cohomology of $\mathbb{P}(\chi)$ with integer coefficients. Our main result, Theorem 3.7, describes $H_T^*(\mathbb{P}(\chi))$ in terms of generators and relations.

Two applications are given: Firstly, we show how to essentially recover the weight vector χ from $H_T^*(\mathbb{P}(\chi))$. This implies that different weighted projective spaces have different integral equivariant cohomologies rings. Secondly, we consider weighted projective bundles. The Chern classes of a complex vector bundle $E \to X$ of rank $n+1$ can be defined by the formula

$$\xi^{n+1} + c_1(E)\, \xi^n + \cdots + c_n(E)\, \xi + c_{n+1}(E) = 0, \tag{1.5}$$

where $-\xi$ is the first Chern class of the canonical line bundle over the associated projective bundle $\mathbb{P}(E)$. Al Amrani [A] has stated a generalisation of this formula to

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weighted projective bundles and proved it in a special case. Theorem 6.2 establishes such a formula in general.

We consider our computation as lying in the realm of toric topology; we shall elaborate on this theme in a subsequent document, where we discuss relationships with weighted lens spaces, homotopy colimits, the Bousfield–Kan spectral sequence, and weighted face rings.

2. From equivariant cohomology to piecewise polynomials

By a *ring* we always mean a graded commutative ring with unit element. Almost all rings we consider happen to be concentrated in even degrees, so that they are commutative in the ordinary sense.

Remark 2.1. Before starting in earnest, we would like to point out that there are several ways to describe the divisibility of the powers c_1^m in $H^*(\mathbb{P}(\chi))$. Kawasaki looks at the p-contents of the weights for each prime p separately. If $q_0(p), \ldots, q_n(p)$ are the p-contents, in increasing order, of the weights, then

$$c_m = \prod_{p \text{ prime}} \frac{q_n(p)\cdots q_{n-m+1}(p)}{q_n(p)^m}\, c_1^m. \tag{2.1}$$

Al Amrani [A, Sec. I.5] considers sets I of size $m+1$ instead of m and writes

$$c_m = \frac{\operatorname{lcm}\{\, \gcd\{\chi_i : i \in I\}^{-1} \prod_{i\in I} \chi_i : |I| = m+1 \,\}}{\operatorname{lcm}(\chi_0, \ldots, \chi_n)^m}\, c_1^m. \tag{2.2}$$

Taking $m+1$ elements and dividing by their greatest common divisor removes the smallest p-content for each prime p. Taking the least common multiple over all I then gives the product of the m largest p-contents of all weights, as in the denominators of formulas (1.4) and (2.1).

Now let $\iota\colon \mathbb{P}(\chi) \to \mathbb{P}(\chi)_T$ be the inclusion of a fibre into the Borel construction.

Lemma 2.2. *$H_T^*(\mathbb{P}(\chi))$ is a free $H^*(BT)$-module of rank $n+1$. Moreover, as a ring it is generated by the image of $H^2(BT)$ in $H_T^2(\mathbb{P}(\chi))$ together with any subgroup $A \subset H_T^*(\mathbb{P}(\chi))$ which surjects onto $H^{>0}(\mathbb{P}(\chi))$ under ι^*.*

Proof. According to Kawasaki, $H^*(\mathbb{P}(\chi))$ is free over $\mathbb{Z}$ and concentrated in even degrees. Hence, the Serre spectral sequence of the fibration $\mathbb{P}(\chi) \hookrightarrow \mathbb{P}(\chi)_T \to BT$ degenerates at the E_2 level, and $H_T^*(\mathbb{P}(\chi)) \cong H^*(\mathbb{P}(\chi)) \otimes H^*(BT)$ as $H^*(BT)$-modules by the Leray–Hirsch theorem. This isomorphism is induced by any (additive) section to ι^*. Since we can assume this section to take values in A, our claim is proven. □

The equivariant cohomology of ordinary complex projective space, which corresponds to the case of all weights being equal to 1, is well-known. A convenient description of it comes from the theory of toric varieties.

To wit, the space $\mathbb{P}(\chi)$ is an n-dimensional projective toric variety. It is defined by any complete simplicial fan Σ spanned by vectors $v_0, \ldots, v_n \in N = \mathbb{Z}^n$ with the following properties, cf. [Fu, Sec. 2.2] or the nice overview in [Ks, Sec. 4.1]:

(1) The vectors $v_0, \ldots, v_n$ span N.
(2) They satisfy the relation

$$\chi_0 v_0 + \cdots + \chi_n v_n = 0. \tag{2.3}$$

The equivariant cohomology of ordinary projective n-space can be described as the integral Stanley–Reisner algebra of the fan Σ,

$$\mathbb{Z}[\Sigma] = \mathbb{Z}[a_0, \ldots, a_n]/(a_0 \cdots a_n), \tag{2.4}$$

where each generator a_i has cohomological degree 2. In other words, the only relation between the generators a_i is

$$\prod_{i=0}^{n} a_i = 0. \tag{2.5}$$

For the general case, we will give a similar description of $H_T^*(\mathbb{P}(\chi))$ in terms of generators and relations. Our main tool will be piecewise polynomials, to which we turn now.

A function $f\colon N \to \mathbb{Z}$ is called *piecewise polynomial* if on (the lattice points in) each cone $\sigma \in \Sigma$ it coincides with some (globally defined) polynomial $g \in \mathbb{Z}[N]$.

Proposition 2.3. *$H_T^*(\mathbb{P}(\chi))$ is isomorphic as $H^*(BT)$-algebra to the algebra of piecewise polynomials on Σ.*

Under this isomorphism, the cup product corresponds to the usual pointwise multiplication of functions, and the canonical map $H^*(BT) \to H_T^*(\mathbb{P}(\chi))$ to the inclusion of (globally defined) polynomials.

Proof. Set $X = \mathbb{P}(\chi)$. Since $H_T^*(X)$ is free over $H^*(BT)$ and moreover all isotropy groups of X are connected (as for any toric variety), the so-called Chang–Skjelbred sequence

$$0 \longrightarrow H_T^*(X) \xrightarrow{j^*} H_T^*(X^T) \xrightarrow{\delta} H_T^{*+1}(X_1, X^T) \tag{2.6}$$

is exact (Franz–Puppe [FP]). Here X^T denotes the T-fixed points, X_1 the union of X^T and all 1-dimensional orbits, j the inclusion $X^T \to X$ and δ the differential of the long exact cohomology sequence for the pair (X_1, X^T).

The piecewise polynomials are a way to represent the kernel of the map δ. (The first to observe this relation between the Chang–Skjelbred sequence and piecewise polynomials was probably Brion.) It goes as follows: Write $\mathcal{O}_\sigma$ for the orbit under the complexification $T_\mathbb{C}$ of T corresponding to $\sigma \in \Sigma$, and $\mathbb{Z}[\sigma]$ for the polynomials with integer coefficients on the linear hull of σ. Note that a polynomial on the linear hull of σ is uniquely defined by its restriction to σ.

For full-dimensional $\sigma \in \Sigma^n$ we have

$$H_T^*(\mathcal{O}_\sigma) = H^*(BT) = \mathbb{Z}[\sigma]. \tag{2.7}$$

In one dimension lower for $\tau \in \Sigma^{n-1}$ denote the isotropy group of $\mathcal{O}_\tau$ by T_τ. Note that the action of the circle T/T_τ on $\bar{\mathcal{O}}_\tau$ is isomorphic to the standard action of S^1 on $\mathbb{CP}^1$, whose fixed points we write as 0 and ∞. We get

$$H_T^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) = H^*(BT_\tau) \otimes H_{T/T_\tau}^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \tag{2.8}$$

and

$$H_{T/T_\tau}^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \cong H_{S^1}^*(\mathbb{CP}^1, \{0, \infty\}) \cong \mathbb{Z}[+1], \tag{2.9}$$

hence

$$H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \cong \mathbb{Z}[\tau]. \tag{2.10}$$

Moreover, it turns out that for a facet τ of σ the differential

$$H_T^*(\mathcal{O}_\sigma) \to H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \tag{2.11}$$

is the canonical restriction $\mathbb{Z}[\sigma] \to \mathbb{Z}[\tau]$, multiplied by ± 1 depending on the orientation of the interval $\mathcal{O}_\tau/T \approx (0, \infty)$ implicitly chosen above.

As result we find that the differential δ from (2.6) is a signed sum of restrictions of polynomials,

$$\delta\colon \bigoplus_{\sigma \in \Sigma^n} \mathbb{Z}[\sigma] \to \bigoplus_{\tau \in \Sigma^{n-1}} \mathbb{Z}[\tau], \tag{2.12}$$

where the component in $\mathbb{Z}[\tau]$ is the difference of the restrictions of the polynomials on the two full-dimensional cones having τ as their common facet. Hence, the kernel consists of those collections of polynomials on the full-dimensional cones which glue along their common facets. But this is the same as requiring that the polynomials collect along *any* intersection $\tau = \sigma \cap \sigma'$ of two cones σ, $\sigma' \in \Sigma$. The reason is that σ and σ are connected by a sequence of cones, each containing τ. (In other words, Σ is a hereditary fan, cf. [BR].) We get therefore that the kernel of δ is the set of the piecewise polynomial functions on Σ, i.e., the functions which are polynomial on each $\sigma \in \Sigma$. □

Remark 2.4. The integral equivariant cohomology of any smooth, not necessarily compact toric variety X_Σ is given by the integral Stanley–Reisner algebra of Σ or, equivalently, by the piecewise polynomials on Σ [BDP], [DJ], [Br]. Similarly, for a simplicial fan Σ the rational equivariant cohomology $H_T^*(X_\Sigma;\mathbb{Q})$ is given by the rational Stanley–Reisner algebra $\mathbb{Q}[\Sigma]$, which is again isomorphic to the piecewise polynomials on Σ with rational coefficients, cf. [Fu, p. 107]. Note that the latter result applies in particular to weighted projective spaces.

A canonical isomorphism between the Stanley–Reisner algebra of Σ and the algebra of piecewise polynomials on Σ can be defined by assigning the Courant function a_ρ associated with the ray ρ to the Stanley–Reisner generator corresponding to ρ. This function a_ρ is the piecewise linear function on Σ that assumes the value 1 on the generator of ρ and 0 on all other rays. It is well-defined because the smoothness of X_Σ implies that the rays of any cone in Σ can be completed to a basis of the lattice N.

We also remark that Payne [P] has shown that for *any* fan Σ the ring of integral piecewise polynomials is isomorphic to the equivariant Chow ring of X_Σ (see also [KP]).

3. Generators of the ring of piecewise polynomials

For $i = 0, \dots, n$ we will write $\sigma_i \in \Sigma$ for the full-dimensional cone spanned by all fan generators except v_i. Moreover, given a piecewise polynomial f, we will denote the unique polynomial which coincides with f on σ_i by $f^{(i)}$. We call a piecewise polynomial *reduced* if it is not divisible, in the ring of (integral) piecewise polynomials, by any rational prime.

Let b_{ij}, $i \neq j$, be the reduced linear function that assumes a positive value on v_i and vanishes on all v_k, $i \neq k \neq j$.

Lemma 3.1. *We have*

$$b_{ij}(v_i) = \frac{\chi_j}{\gcd(\chi_i,\chi_j)} = \frac{\operatorname{lcm}(\chi_i,\chi_j)}{\chi_i} \tag{3.1}$$

for $i \neq j$.

Proof. Applying b_{ij} to the relation (2.3) yields

$$\chi_i\, b_{ij}(v_i) = -\chi_j\, b_{ij}(v_j). \tag{3.2}$$

Since b_{ij} is reduced and v_i and v_j span $N/\ker b_{ij} \cong \mathbb{Z}$, $b_{ij}(v_i)$ and $b_{ij}(v_j)$ must be coprime. This implies the claimed formula. □

Proposition 3.2. *The functions b_{ij}, $i \neq j$, generate the linear functions.*

Proof. Because multiplying all weights by a constant factor changes neither the fan Σ nor the functions b_{ij}, we may assume that the greatest common divisor of the weights equals 1.

For given j, let N_j be the span of the linear independent set $V_j = \{v_i : i \neq j\}$ and $N_j^\vee$ its dual. By Lemma 3.1, the restriction of each b_{ij}, $i \neq j$, to N_j is divisible by a divisor of χ_j, and this quotient is an element of the basis dual to V_j.

Denote by M_j the sublattice generated by the b_{ij}, $i \neq j$, inside the dual $N^\vee$ of N, and by M the one generated by all M_j. Our goal is to show that $N^\vee = M$.

We have

$$N_j^\vee / N^\vee = (N_j^\vee / M_j) \,/\, (N^\vee / M_j). \tag{3.3}$$

Therefore, the order of $N^\vee/M_j$ divides that of $N_j^\vee/M_j$, which itself divides χ_j^n by what we have said so far. Hence, the order of $N^\vee/M_j$ divides χ_j^n as well, and the same applies to $N^\vee/M$ since

$$N^\vee / M = (N^\vee / M_j) \,/\, (M / M_j). \tag{3.4}$$

This implies that the order of $N^\vee/M$ divides the greatest common divisor of all χ_j^n, which we assumed to be 1. □

Denote by a_i the *(integral) Courant function* corresponding to v_i. By this we mean the reduced piecewise linear function that assumes a positive value on v_i and vanishes on all v_j for $j \neq i$.

Lemma 3.3. *Together with the linear functions, each a_i generates the piecewise linear functions.*

Proof. Let f be piecewise linear. Then $f - f^{(i)}$ vanishes on σ_i, hence is a multiple of a_i. □

Lemma 3.4. *We have*

$$a_i(v_i) = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i} \qquad \text{and} \qquad a_i^{(j)} = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}(\chi_i, \chi_j)}\, b_{ij} \tag{3.5}$$

for $i \neq j$.

Proof. By Lemma 3.1 we get a well-defined piecewise linear function f by setting $f^{(i)} = 0$ and $f^{(j)}$ as given above for $j \neq i$. This function assumes a positive value on v_i. Moreover, it is reduced: if, for a prime p, the maximal p-content occurs in χ_i, then no $f^{(j)}$, $j \neq i$, is divisible by p, and if it occurs in χ_j, $j \neq i$, then this $f^{(j)}$ is not divisible by p. Therefore, f is equal to a_i. □

Lemma 3.5. *We have*

$$b_{ij} = \frac{\operatorname{lcm}(\chi_i, \chi_j)}{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}\, (a_i - a_j) \tag{3.6}$$

for $i \neq j$.

Proof. We have

$$a_i^{(j)} = -a_j^{(i)} = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}(\chi_i, \chi_j)}\, b_{ij} \tag{3.7}$$

by Lemma 3.4, and $a_i^{(i)} = -a_j^{(j)} = 0$, hence, $(a_i - a_j)^{(i)} = (a_i - a_j)^{(j)}$. By Lemma 3.3, each piecewise linear function is the sum of a linear function and a multiple of a Courant function. For a Courant function, the restrictions to any two maximal cones are distinct linear functions. Hence $a_i - a_j$ is in fact linear and divisible as claimed. □

We now consider higher-degree analogues of the Courant functions a_i.

Lemma 3.6. *For a subset $I \subset \{0, \dots, n\}$ of size $m > 0$, the function $\prod_{i \in I} a_i$ is divisible by*

$$\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}}. \tag{3.8}$$

Proof. We look at each prime p separately. If the maximal p-content occurs in some χ_j with $j \notin I$, then there is nothing to prove because the p-content of the expression above is 1. We can therefore assume that it occurs in some χ_k, $k \in I$.

Choose an $i \in I$ and denote the p-contents of χ_k and χ_i by q_k and q_i, respectively. Then all $a_i^{(j)}$, $j \notin I$, are divisible by q_k/q_i, which is the p-content of

$$\frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}}. \tag{3.9}$$

Taking the product over all $i \in I$ finishes the proof. □

Hence, for $I \subset \{0, \dots, n\}$ with $|I| = m > 0$ we may define the piecewise polynomial

$$a_I = \Big(\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \Big)^{-1} \prod_{i \in I} a_i \tag{3.10}$$

of polynomial degree m (and cohomological degree $2m$).

Theorem 3.7. *The functions a_I, $\emptyset \neq I \subsetneq \{0, \dots, n\}$, and b_{ij}, $i \neq j$, generate $H_T^*(\mathbb{P}(\chi))$ as a ring. The only relations are* (2.5), (3.6) *and* (3.10).

Proof. Since there are no more relations between the a_I and the b_{ij} in $H_T^*(X_\Sigma; \mathbb{Q})$, the same is true in $H_T^*(X_\Sigma; \mathbb{Z})$, which injects into $H_T^*(X_\Sigma; \mathbb{Q})$ because it is free over $\mathbb{Z}$. It remains to show that these elements are indeed ring generators.

By Proposition 3.2, the b_{ij} generate the linear functions, which are the image of $H^2(BT)$ in $H_T^2(\mathbb{P}(\chi))$. Hence, by Lemma 2.2, it suffices to show that the subgroup generated by the a_I surjects onto $H^*(\mathbb{P}(\chi))$. In other words, we have to show that c_m lies in the span of $\{\iota^*(a_I) : |I| = m\}$ for each $m \geq 1$.

For $m = 1$, this is true by Lemma 3.3 because we know ι^* itself to be surjective. Moreover, Lemma 3.5 implies that all elements a_i are mapped to the same element of $H^2(\mathbb{P}(\chi))$. This must necessarily be a generator, which we can assume to be c_1 (instead of $-c_1$).

For $1 < m \leq n$, we get that

$$\iota^*(a_I) = \Big(\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \Big)^{-1} \prod_{i \in I} \iota^*(a_i) \tag{3.11a}$$

$$= \Big(\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \Big)^{-1} c_1^m \tag{3.11b}$$

$$= \frac{\prod_{i \in I} \operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}}{\operatorname{lcm}\{\prod_{j \in J} \chi_j : |J| = m\}} c_m. \tag{3.11c}$$

We have to show that these multiples of c_m generate $H^{2m}(\mathbb{P}(\chi))$. We look again at each prime p separately and take as I the set of indices corresponding to m weights with greatest p-content. (This set need not be unique.) Since this is also the set J which maximises the p-content of the denominator of the last formula above, we conclude that for each p there appears a multiple of c_m whose p-content is 1. In other words, the greatest common divisor of all multiples is 1, which was to be shown. □

4. An example

We illustrate the results of the preceding section by an example. This will also show that the elements b_{ij} cannot be omitted in general.

We take $\chi = (1, 2, 3, 4)$ and choose $v_0 = (-2, -3, -4)$, $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$. Writing a piecewise polynomial as $f = (f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)})$ and the canonical basis of $N^\vee$ as (x, y, z), we get

$$a_0 = (0, -6x, -4y, -3z), \tag{4.1a}$$

$$a_1 = (6x, 0, 6x - 4y, 6x - 3z), \tag{4.1b}$$

$$a_2 = (4y, -6x + 4y, 0, 4y - 3z), \tag{4.1c}$$

$$a_3 = (3z, -6x + 3z, -4y + 3z, 0). \tag{4.1d}$$

While a single component $a_i^{(j)}$, $i \neq j$, may not be reduced, all components $a_i^{(j)}$ for different j together have no non-trivial divisor. This changes if we consider a product $\prod_{i \in I} a_i$ because then the components $a_i^{(j)}$ with $j \in I$ become irrelevant. This is the essence of Lemma 3.6. In this example we get

$$a_{01} = a_0 a_1, \quad a_{02} = a_0 a_2/3, \quad a_{03} = a_0 a_3/2, \tag{4.2a}$$

$$a_{12} = a_1 a_2/3, \quad a_{13} = a_1 a_3/4, \quad a_{23} = a_2 a_3/6, \tag{4.2b}$$

$$a_{012} = a_0 a_1 a_2/9, \quad a_{013} = a_0 a_1 a_3/8, \tag{4.2c}$$

$$a_{023} = a_0 a_2 a_3/36, \quad a_{123} = a_1 a_2 a_3/72. \tag{4.2d}$$

Any globally linear function obtained from the a_i's is a linear combination of

$$a_1 - a_0 = 6x, \quad a_2 - a_0 = 4y, \quad a_3 - a_0 = 3z, \tag{4.3a}$$

$$a_2 - a_1 = 4y - 6x, \quad a_3 - a_1 = 3z - 6x, \quad a_3 - a_2 = 3z - 4y. \tag{4.3b}$$

Note that for example the lower row is redundant, as are most of the a_I's. But there is no canonical choice of a minimal set of generators.

5. Recovering the weights

It is clear from the definition (1.1) that $\mathbb{P}(\chi)$ does not change if all weights are multiplied by the same factor. In particular, one may always divide the weights by their greatest common divisor, as we have done in the proof of Proposition 3.2. Moreover, if all weights except one are divisible by some prime p, one may divide these weights by p as well. Using the toric description of $\mathbb{P}(\chi)$, this is easy to see: If all weights except χ_i are divisible by p, then, by equation (2.3), v_i must be divisible by p as well. The fan Σ and, consequently, the toric variety $X_\Sigma = \mathbb{P}(\chi)$ do not change if v_i is replaced by v_i/p and χ by $(\chi_0/p, \ldots, \chi_i, \ldots, \chi_n/p)$.

By repeating these simplifications, one can always achieve that for each prime p there are at least two weights not divisible by p. This new weight vector is uniquely defined by χ (up to order); the corresponding weights are called *normalised*.

Theorem 5.1. *The graded ring $H_T^*(\mathbb{P}(\chi))$ determines the normalised weights up to order.*

Note that this is false for ordinary cohomology because the divisibility rule (1.4) does not take into account how the distributions of the weights' p-contents are related for different p. For example, the graded rings $H^*(\mathbb{P}(1, 2, 3))$ and $H^*(\mathbb{P}(1, 1, 6))$ are isomorphic, but $\mathbb{P}(1, 2, 3)$ and $\mathbb{P}(1, 1, 6)$ are not homeomorphic (for instance because $\mathbb{P}(1, 2, 3)$ has two singular points while $\mathbb{P}(1, 1, 6)$ only has one).

Proof. We will assume that χ is normalised.

We start by observing that the relation (2.5) characterises the generators a_i in the following sense: If $f_0, \ldots, f_n \in H_T^2(\mathbb{P}(\chi))$ are reduced (in particular, non-zero) and satisfy the relation $f_0 \cdots f_n = 0$, then, up to order and sign, they are equal to the a_i's.

To see this, we look at the f_i's as piecewise polynomials. For each cone σ_j, there must be an f_i vanishing on σ_j. If an f_i vanished on two cones, then, by a reasoning analogous to that done in the proof of Lemma 3.5, this f_i would be identically zero. Hence, each f_i vanishes on exactly one σ_j, and being reduced, it must be equal to $\pm a_j$.

Knowing the a_i's, we also know, for each prime p, the greatest power q_{ij} of p dividing $a_i - a_j$. By Lemma 3.5, this is the p-content of

$$\frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}(\chi_i, \chi_j)}. \tag{5.1}$$

Because we assume the weights to be normalised, a greatest such number, say q_{jk}, corresponds to two weights χ_j and χ_k not divisible by p. The p-content of χ_i, $j \neq i \neq k$, is finally equal to q_{jk}/q_{ik}. This determines the weights completely. □

6. Weighted projective bundles

Let $L_i \to X$, $i = 0, \ldots, n$, be a complex line bundle over some base X, and let $E = L_0 \oplus \cdots \oplus L_n \to X$ be their direct sum. The torus $\tilde{T} = (S^1)^{n+1}$ acts on E and on the corresponding sphere bundle $S(E)$ in a canonical fashion. The associated *weighted projective bundle* with weights χ is the quotient

$$\mathbb{P}(E, \chi) = S(E)/S^1\langle \chi_0, \ldots, \chi_n \rangle. \tag{6.1}$$

For the universal bundle $E = (E\tilde{T} \times \mathbb{C}^{n+1})/\tilde{T} \to X = B\tilde{T}$ the same reasoning as in the proof of Lemma 2.2 gives that $\mathbb{P}(E, \chi)$ is a free $H^*(X)$-module of rank $n+1$. Since any direct sum of line bundles is induced from the universal bundle, the same holds in general by the naturality of the Serre spectral sequence.

If all weights are equal to 1, then $\mathbb{P}(E, \chi)$ is an ordinary projective bundle, and its cohomology is generated by an element $\xi \in H^2(\mathbb{P}(E, \chi))$ and its powers, subject only to relation (1.5). Note that (1.5) can equivalently be written in the form

$$\prod_{i=0}^{n} \bigl(\xi + c_1(L_i)\bigr) = 0. \tag{6.2}$$

In [A, Ch. III] Al Amrani stated the following generalisation of (6.2) to weighted projective bundles and proved it under the assumption that the weights form a divisor chain: One has

$$\prod_{i=0}^{n} \Bigl(\xi + \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i}\, c_1(L_i)\Bigr) = 0 \tag{6.3}$$

for a certain $\xi \in H^2(\mathbb{P}(E, \chi))$ which restricts to $c_1 \in H^2(\mathbb{P}(\chi))$ under the inclusion of a fibre $\mathbb{P}(\chi) \hookrightarrow \mathbb{P}(E, \chi)$.

Again by naturality, it suffices to verify (6.3) for $X = B\tilde{T}$, that is, inside the equivariant cohomology $H^*_{\tilde{T}}(\mathbb{P}(\chi))$, where the $\tilde{T}$-action on $\mathbb{P}(\chi)$ is induced by the projection

$$\tilde{T} \to T = \tilde{T}/S^1\langle \chi_0, \ldots, \chi_n \rangle. \tag{6.4}$$

Not surprisingly, there is a description of $H^*_{\tilde{T}}(\mathbb{P}(\chi))$ in terms of piecewise polynomials.

Let $\pi \colon \tilde{N} = \mathbb{Z}^{n+1} \to N$ be the map which sends the canonical basis vector e_i to v_i, $i = 0, \ldots, n$. This is the quotient of $\tilde{N}$ by the line through $u = \chi_0 e_0 + \cdots + \chi_n e_n$. We write the basis dual to (e_i) as (x_i). Let $\tilde{\Sigma} = \pi^{-1}(\Sigma) = \{\pi^{-1}(\sigma) : \sigma \in \Sigma\}$ be the pull-back of Σ. Note that the cones in the fan $\tilde{\Sigma}$ are not pointed (they all contain the line through u), but they still enjoy the property that their intersections are again in $\tilde{\Sigma}$. We denote by π^* the pull-back of piecewise polynomials from Σ to $\tilde{\Sigma}$.

Lemma 6.1. *$H^*_{\tilde{T}}(\mathbb{P}(\chi))$ is isomorphic as $H^*(B\tilde{T})$-algebra to the algebra of piecewise polynomials on $\tilde{\Sigma}$. Moreover, the map $H^*_T(\mathbb{P}(\chi)) \to H^*_{\tilde{T}}(\mathbb{P}(\chi))$ induced by* (6.4) *corresponds to π^*.*

Proof. We have

$$H^*_T(\mathbb{P}(\chi)) \to H^*_{\tilde{T}}(\mathbb{P}(\chi)) = H^*_T(\mathbb{P}(\chi)) \otimes H^*(BS^1) \tag{6.5}$$

because $\tilde{T} \cong T \times S^1$ and this circle acts trivially on $\mathbb{P}(\chi)$. The freeness of $H^*_T(\mathbb{P}(\chi))$ over $H^*(BT)$ implies that $H^*_{\tilde{T}}(\mathbb{P}(\chi))$ is free over $H^*(B\tilde{T})$.

We can therefore imitate the reasoning of Proposition 2.3. We have for $\tilde{\sigma} \in \tilde{\Sigma}$ with $k = \operatorname{codim} \tilde{\sigma} \le 1$ (in fact, for all $\tilde{\sigma}$) that

$$H^{*+k}_{\tilde{T}}(\bar{\mathcal{O}}_{\tilde{\sigma}}, \partial\mathcal{O}_{\tilde{\sigma}}) \cong \mathbb{Z}[\tilde{\sigma}], \tag{6.6}$$

and the boundary map in cohomology corresponds up to sign to the restriction of polynomials. Hence, the integral Chang–Skjelbred sequence is exact, which in turn permits us to identify $H^*_T(\mathbb{P}(\chi))$ with the piecewise polynomials.

Since for all $\sigma \in \Sigma$ and $\tilde{\sigma} = \pi^{-1}(\sigma)$ the map

$$H^*_T(\bar{\mathcal{O}}_\sigma, \partial\mathcal{O}_\sigma) \to H^*_{\tilde{T}}(\bar{\mathcal{O}}_{\tilde{\sigma}}, \partial\mathcal{O}_{\tilde{\sigma}}) \tag{6.7}$$

corresponds to the pull-back of functions by π, the same applies to the restriction of T-equivariant cohomology to $\tilde{T}$. □

Let ξ be the piecewise linear function on $\tilde{N}$ defined by $\xi(u) = -\operatorname{lcm}(\chi_0, \dots, \chi_n)$ and $\xi(e_i) = 0$ for all i. (This is not a contradiction because no cone contains all e_i's.) Equivalently,

$$\xi^{(i)} = -\frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\chi_i}\, x_i \tag{6.8}$$

for all i.

Theorem 6.2. *This ξ restricts to $c_1 \in H^*(\mathbb{P}(\chi))$ and satisfies equation* (6.3).

Proof. We have

$$\pi^*(a_i)^{(j)} = \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\chi_i}\, x_i - \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\chi_j}\, x_j \tag{6.9a}$$

$$= \frac{\operatorname{lcm}(\chi_0, \dots, \chi_n)}{\chi_i}\, x_i + \xi^{(j)} \tag{6.9b}$$

because the right-hand side vanishes on u and assumes the same value on e_k as a_i does on v_k. Since ξ differs from $\pi^*(a_i)$ by a linear function, it restricts to the same element $c_1 \in H^*(\mathbb{P}(\chi))$ as $\pi^*(a_i)$ and a_i. Using $c_1(L_i) = x_i$, we conclude finally that equation (6.3) is nothing but the pull-back of relation (2.5). □

References

[A] A. Al Amrani, Cohomological study of weighted projective spaces, pp. 1–52 in: S. Sertöz (ed.), Algebraic geometry, Proc. Bilkent Summer School (Ankara 1995), *Lect. Notes Pure Appl. Math.* **193**, Dekker, New York 1997

[BDP] E. Bifet, C. De Concini, C. Procesi, Cohomology of regular embeddings, *Adv. Math.* **82** (1990), 1–34

[BR] L. J. Billera, L. L. Rose, Modules of piecewise polynomials and their freeness, *Math. Z.* **209** (1992), 485–497

[Br] M. Brion. Piecewise polynomial functions, convex polytopes and enumerative geometry, pp. 25–44 in: P. Pragacz (ed.), *Parameter spaces*, Banach Cent. Publ. **36**, Warszawa 1996

[DJ] M. W. Davis, T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.* **62** (1991), 417–451

[FP] M. Franz, V. Puppe, Exact cohomology sequences with integral coefficients for torus actions, *Transformation Groups* **12** (2007), 65–76

[Fu] W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, Princeton 1993.

[Ks] A. M. Kasprzyk, Toric Fano varieties and convex polytopes, PhD thesis, University of Bath 2006, available at `http://www.math.unb.ca/~kasprzyk/research/pdf/Thesis.pdf`

[KP] E. Katz, S. Payne, Piecewise polynomials, Minkowski weights, and localization on toric varieties, preprint arXiv:math/0703672

[Kw] T. Kawasaki, Cohomology of twisted projective spaces and lens complexes, *Math. Ann.* **206** (1973), 243–248

[P] S. Payne, Equivariant Chow cohomology of toric varieties, *Math. Res. Lett.* **13** (2006), 29–41

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