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Second order structures for sprays and connections on Fréchet manifolds

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Abstract

Ambrose, Palais and Singer [6] introduced the concept of second order structures on finite dimensional manifolds. Kumar and Viswanath [23] extended these results to the category of Banach manifolds. In the present paper all of these results are generalized to a large class of Fréchet manifolds. It is proved that the existence of Christoffel and Hessian structures, connections, sprays and dissections are equivalent on those Fréchet manifolds which can be considered as projective limits of Banach manifolds. These concepts provide also an alternative way for the study of ordinary differential equations on non-Banach infinite dimensional manifolds. Concrete examples of the structures are provided using direct and flat connections.

Keywords: Banach manifold, Fréchet manifold, Hessian structure, Christoffel structure, connection, spray, dissection, geodesic, ordinary differential equations.

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1 introduction

The study of infinite dimensional manifolds has received much interest due to its interaction with bundle structures, fibrations and foliations, jet fields, connections, sprays, Lagrangians and Finsler structures (11, 14, 7, 8, 10, 18 and 21). In particular, non-Banach locally convex modelled manifolds have been studied from different points of view (see for example 2, 4, 11, 12, 19 and 27). Fréchet spaces of sections arise naturally as configurations of a physical field and the moduli space of inequivalent configurations of a physical field is the quotient of the infinite-dimensional configuration space \( \mathcal{X} \) by the appropriate symmetry gauge group. Typically, \( \mathcal{X} \) is modelled on a Fréchet space.
of smooth sections of a vector bundle over a closed manifold. For example, see Omori [25, 26].

The second order structures introduced by Ambrose et al. [6] for finite dimensional manifolds were extended by Kumar and Viswanath [23] for Banach modelled manifolds. They proved that Hessian structures, sprays, dissections and (linear) connections are in a one-to-one correspondence. However, there these concepts have to be supported by a Christoffel bundle and vector fields. In this paper, following the lines of [23], we first construct the concepts of Christoffel bundle and fields for a class of projective limit Fréchet manifolds. Then, we identify it with the other structures, i.e. connections, Hessian structures and sprays.

One of the main problems in the study of non-Banach modelled manifolds $M$ is the pathological structure of the general linear group $GL(F)$ of a non-Banach space $F$. $GL(F)$ serves as the structure group of the tangent bundle $TM$, similar to finite dimensional and Banach cases, but it is not even a reasonable topological group structure within the Fréchet framework (see [16, 18]). Moreover, for a Fréchet space $F$, $L(F)$, the space of linear maps on $F$, is not in general a Fréchet space. The same problem holds for the space of bilinear maps $L^2(F,F) = \{ B; B : F \times F \rightarrow F, B \text{ is linear} \}$.

If one follows the classical procedure to define the notion of Christoffel bundle or Hessian structures, then $L^2(F,F)$ will appear as the corresponding fibre type. As stated in Section 2, these problems are overcome by replacing $L^2(F,F)$ with an appropriate Fréchet space. Another serious drawback in the study of Fréchet manifolds and bundles is the fact that there is no general solvability theory for differential equations ([27]). This problem also can be overcome if we restrict ourselves to the category of those Fréchet manifolds which can be considered as projective limits of Banach corresponding factors. To eliminate these difficulties, we endow $TM$ with a generalized vector bundle structure. (Note that Galanis in [16] proved a similar result but with a different definition for tangent bundle). In the sequel we construct the Christoffel bundles, connections, Hessian structures, sprays and dissections. It is shown in this way that all the results stated in [6] and [23] hold in the category of projective limit manifolds.

Our approach here gives the opportunity to study the problems related to ordinary differential equations that arise via geometric objects on manifolds. For example, geodesics with respect to connections and sprays, and parallel transport are discussed. Finally, the associated structures for flat and direct connections are introduced.

## 2 Christoffel bundle

Most of our calculus is based on [5] and [24]. Let $E$ be a real Banach space, $M$ a Hausdorff paracompact smooth manifold and $m$ a point of $M$. The tangent bundle of $M$ is defined as follows: $TM = \bigcup_{m \in M} T_m M$, where $T_m M$ is considered as the set of equivalence classes of all triples $(U, \varphi, e)$, where $(U, \varphi)$ is a chart of $M$ around $m$ and $e$ is an element of the model space $E$ in which $\varphi U$ lies. $TM$ is a vector bundle on $M$ with structure group $GL(E)$ ([24]).

We summarise our basic notations about a certain rather wide class of Fréchet manifolds, namely those which can be considered as projective limits of Banach manifolds. Let $\{(M_i, \varphi_{ij})\}_{i,j \in \mathbb{N}}$ be a projective system of Banach
manifolds with $M = \lim_{\leftarrow} M_i$ such that for every $i \in \mathbb{N}$, $M_i$ is modelled on the Banach space $E_i$ and $\{(E_i, \rho_{ji})\}_{i \in \mathbb{N}}$ forms a projective system of Banach spaces. Furthermore suppose that for each $m = (m_i)_{i \in \mathbb{N}} \in M$ there exists a projective system of local charts $\{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$ such that $m_i \in U_i$ and $U = \lim_{\leftarrow} U_i$ is open in $M$ (see [4]).

It is known that for a Fréchet space $F$, the general linear group $GL(F)$ cannot be endowed with a smooth Lie group structure. It does not even admit a reasonable topological group structure. The problems concerning the structure group of $TM$ can be overcome by the replacement of $GL(F)$ with the following topological group (and in a generalized sense it is also a smooth Lie group):

$$H_0(F) = \{((f_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} GL(E_i) : \lim_{\leftarrow} f_i \text{ exists}\}.$$

More precisely $H_0(F)$ is isomorphic to the projective limit of the Banach Lie groups

$$H_0^i(F) = \{(f_1, f_2, ..., f_i) \in \prod_{k=1}^{i} GL(E_k) : \rho_{jk} \circ f_j = f_k \circ \rho_{jk}, (k \leq j \leq i)\}.$$

Under these notations the following basic theorems hold (compare with [16]).

**Theorem 2.1.** If $\{M_i\}_{i \in \mathbb{N}}$ is a projective system of manifolds then $\{TM_i\}_{i \in \mathbb{N}}$ is also a projective system with limit (set-theoretically) isomorphic to $TM = \lim_{\leftarrow} TM_i$.

**Theorem 2.2.** $TM = \lim_{\leftarrow} TM_i$ has a Fréchet vector bundle structure on $M = \lim_{\leftarrow} M_i$ with structure group $H_0(F)$.

Let $L(E, E)$ be the space of continuous linear maps from a Banach space $E$ to $E$ and let $L^2(E, E)$ be the space of all continuous bilinear maps from $E \times E$ to $E$. For $m \in M$ and every chart $(U, \varphi)$ at $m$, consider the triples of the form $(U, \varphi, B)$ where $B \in L^2(E, E)$.

**Definition 2.3.** Two triples $(U, \varphi, B_1)$ and $(V, \psi, B_2)$ are called equivalent at $m$ if

$$B_2(DF(u).e_1, DF(u).e_2) = DF(u).B_1(e_1, e_2) + D^2F(u)(e_1, e_2),$$

where $u = \varphi m$, $F = \psi \circ \varphi^{-1}$ and $e_1, e_2 \in E$.

It can be checked that this is an equivalence relation. Each equivalence class is called a Christoffel element at $m$ and a typical element is denoted by $\gamma$. Let $(U, \varphi)$ be a fixed chart at $m$. Define the mapping

$$C_\varphi : C_m \longrightarrow L^2(E, E)$$

$$\gamma \longmapsto (\varphi m, B)$$

where $C_m$ is the set of all Christoffel elements at $m$ and $(U, \varphi, B) \in \gamma$. Then $C_\varphi$ is a bijection, which endows $CM = \bigsqcup_{m \in M} C_m$ with a $C^\infty$-atlas. (For more details see [23]).

From [23] we have the result:

...
Theorem 2.4. The family \{\{CU, C\varphi\}: (U, \varphi) is a chart on M\} is a $C^\infty$-atlas for CM.

We emphasise again at this point that for a Fréchet space $F$, $L^2(F, F)$ does not need to be a Fréchet space in general. Hence, the classical procedure for $CM$ for a non-Banach Fréchet manifold $M$, does not yield a Fréchet manifold (nor bundle) structure. To overcome this obstacle we use the Fréchet space:

$$\mathcal{H}^2(F, F) := \{(B_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} L^2(\mathbb{E}_i, E_i) : \lim_{i \to \infty} B_i \text{ exists}\}.$$

$\mathcal{H}^2(F, F)$ is isomorphic to the projective limit of Banach spaces

$$\mathcal{H}^2(F, F) := \{(B_1, ..., B_i) \in \prod_{k=1}^i L^2(\mathbb{E}_k, E_k) : B_k \circ (\rho_{jk} \times \rho_{jk}) = \rho_{jk} \circ B_j, (k \leq j \leq i)\}.$$

Let \{\{M_i\}_{i \in \mathbb{N}}\} be a projective system of Banach manifolds as introduced earlier, $B, \bar{B} \in \mathcal{H}^2(F, F)$ and $(U = \lim_{i \to \infty} U_i, \varphi = \lim_{i \to \infty} \varphi_i), (V = \lim_{i \to \infty} V_i, \psi = \lim_{i \to \infty} \psi_i)$ two corresponding charts.

**Definition 2.5.** Two triples $[U, \varphi, B]$ and $[V, \psi, \bar{B}]$ are equivalent if, for every $i \in \mathbb{N}$, $[U_i, \varphi_i, B_i]$ and $[V_i, \psi_i, \bar{B}_i]$ are equivalent.

By these means one can show that $CM$ is endowed with a Fréchet manifold structure modelled on $F \times \mathcal{H}^2(F, F)$.

**Proposition 2.6.** If $\{\{M_i\}_{i \in \mathbb{N}}\}$ is a projective system of manifolds and $\lim_{i \to \infty} CM_i$ exists then $\lim_{i \to \infty} CM_i = CM(\lim_{i \to \infty} M_i)$ (set-theoretically).

**Proof.** If we consider

$$Q : C(\lim_{i \to \infty} M_i) \rightarrow \lim_{i \to \infty} (CM_i)$$

$$[U, \varphi, B] \mapsto ([U_i, \varphi_i, B_i])_{i \in \mathbb{N}}$$

then $Q$ is well defined. $Q$ is one to one since $Q([U, \varphi, B]) = Q([\bar{U}, \bar{\varphi}, \bar{B}])$ yields;

$$[U_i, \varphi_i, B_i] = [\bar{U}_i, \bar{\varphi}_i, \bar{B}_i], \quad i \in \mathbb{N}.$$

Consequently $[U, \varphi, B] = [\lim_{i \to \infty} U_i, \lim_{i \to \infty} \varphi_i, \lim_{i \to \infty} B_i] = [\lim_{i \to \infty} [U_i, \varphi_i, B_i]] = [\lim_{i \to \infty} [\bar{U}_i, \bar{\varphi}_i, \bar{B}_i]] = [\bar{U}, \bar{\varphi}, \bar{B}]$. Then $Q$ is also surjective since for every $([U_i, \varphi_i, B_i])_{i \in \mathbb{N}}$ in $\lim_{i \to \infty} (CM_i)$, $Q(a) = ([U_i, \varphi_i, B_i])_{i \in \mathbb{N}}$ where $a = [\lim_{i \to \infty} U_i, \lim_{i \to \infty} \varphi_i, \lim_{i \to \infty} B_i]$.

Therefore, $Q$ is a bijection between $CM$ and $\lim_{i \to \infty} (CM_i)$.

The functions

$$\xi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times L^2(\mathbb{E}, \mathbb{E})$$

$$\gamma \mapsto (m, B), \quad \alpha \in I$$

with $\gamma \in C_m$, $(U_\alpha, \varphi_\alpha, B) \in \gamma$, define a family of trivializations under which $(CM, M, \pi)$ becomes a fibre bundle ($\pi$ is the natural projection).

In the next theorem the concept of $(CM, M, \pi)$ is generalized to a Fréchet manifold $M = \lim_{i \to \infty} M_i$. 
Theorem 2.7. If $CM = \lim M_i$ exists, then it admits a Fréchet fibre bundle structure on $M = \lim M_i$ with fibre type $\mathcal{H}^2(F,F)$.

Proof. Let $\mathcal{A} = \{(U_\alpha = \lim U_{\alpha}^i, \varphi_\alpha = \lim \varphi_\alpha^i)\}$ be an atlas for $M = \lim M_i$. Then, for every $i \in \mathbb{N}$, $(CM_i, M_i, \pi_i)$ is a fibre bundle with fibres of type $L^2(E_i, E_i)$ and trivializations:

$$\xi_\alpha^i : \pi_\alpha^{-1}(U_\alpha^i) \rightarrow U_\alpha^i \times L^2(E_i, E_i)$$

$$\gamma_i : (\xi_i) \rightarrow (m_i, B_i)$$

Suppose that $\{c_{ij}\}_{i,j \in \mathbb{N}}$, $\{\varphi_{ji}\}_{i,j \in \mathbb{N}}$ and $\{\rho_{ji}\}_{i,j \in \mathbb{N}}$ are the connecting morphisms of the projective systems $CM = \lim CM_i$, $M = \lim M_i$ and $F = \lim E_i$ respectively. Since $\varphi_{ji} \pi_j = \pi_i c_{ij}$, $\{\pi_i\}_{i \in \mathbb{N}}$ is a projective system of maps. For every $\alpha \in I$, $\{\xi_\alpha^i\}_{i \in \mathbb{N}}$ is a projective system and $\pi = \lim \pi_i : CM \rightarrow M$ serves as the projection map. On the other hand, $\xi := \lim \xi_\alpha^i : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathcal{H}^2(F,F)$ is a diffeomorphism since it is a projective limit of diffeomorphisms. $\square$

For an open subset $U$ in $E$, define a Christoffel map $\Gamma$ on $U$ to be a smooth mapping $\Gamma : U \rightarrow L^2(E, E)$ and for every chart $(U, \varphi)$ of $M$ a Christoffel map is locally a smooth mapping $\Gamma_\varphi : \varphi U \rightarrow L^2(E, E)$.

Definition 2.8. $M$ is endowed with a Christoffel structure $\{\Gamma_\varphi\}$ if for every choice of charts $(U, \varphi)$ and $(V, \psi)$ with $U \cap V \neq \emptyset$, the following relation holds true:

$$\Gamma_\varphi(v)(DF(u).e_1, DF(u).e_2) = DF(u)\Gamma_\psi(u)(e_1, e_2) + D^2F(u).(e_1, e_2)$$

where $e_1, e_2 \in E$, $\varphi m = u, \psi m = v$ and $F = \psi \circ \varphi^{-1}$.

For $v, w \in TM$ we can express this condition as follows:

$$\Gamma_\varphi(v)(v_\varphi, w_\varphi) = DF(u)\Gamma_\psi(u)(v_\psi, w_\psi) + D^2F(u).(v_\psi, w_\psi)$$

where $\varphi m = u, v_\varphi = DF(u).v_\psi, w_\varphi = DF(u).w_\psi, v = [U, \varphi, v_\varphi]$ and $w = [U, \varphi, w_\varphi]$ (see also (23)).

In a similar manner one can define the Christoffel map for the non-Banach case as follows: Let $U = \lim U_i$ be an open subset of $F = \lim E_i$. A Christoffel map on $U = \lim U_i$ is a projective limit smooth mapping $\Gamma = \lim \Gamma_i : U \rightarrow \mathcal{H}^2(F,F)$. Note that for each chart $(U = \lim U_i, \varphi = \lim \varphi_i)$ of $M$, $\lim \Gamma_i := \Gamma_\varphi : \varphi U \rightarrow \mathcal{H}^2(F,F)$ defines a Christoffel map on $U$. Now we can state the following definition for Fréchet manifolds.

Definition 2.9. $M = \lim M_i$ is endowed with a Christoffel structure $\{\Gamma_\varphi = \lim \Gamma_\varphi_i\}$ if for every pair of charts $(U = \lim U_i, \varphi = \lim \varphi_i)$ and $(V = \lim V_i, \psi = \lim \psi_i)$ around $m = (m_i)_{i \in \mathbb{N}}$ the following relation is satisfied:

$$\Gamma_\varphi(v)(DF(u).e_1, DF(u).e_2) = DF(u)\Gamma_\psi(u)(e_1, e_2) + D^2F(u).(e_1, e_2),$$

where $e_1 = (e_1^i)_{i \in \mathbb{N}}, e_2 = (e_2^i)_{i \in \mathbb{N}} \in F$, $\lim \varphi_i m_i = \lim u_i = u$, $\lim \psi_i m_i = \lim v_i = v$ and $F = \lim F_i = \lim \psi_i \circ \varphi_i^{-1}$. For $v, w \in TM = \lim TM_i$ this condition takes the form

$$\Gamma_\varphi(v)(v_\varphi, w_\varphi) = DF(u)\Gamma_\psi(u)(v_\psi, w_\psi) + D^2F(u).(v_\psi, w_\psi)$$

where $\lim \varphi_i m_i = \lim u_i = u, v_\varphi = \lim DF_i(u_i).v_{\varphi_i}, w_\psi = \lim DF_i(u_i).w_{\varphi_i}, v = ([U_i, \varphi_i, v_{\varphi_i}])_{i \in \mathbb{N}}$ and $w = ([U_i, \varphi_i, w_{\varphi_i}])_{i \in \mathbb{N}}$. 

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5
3 Connections and Hessian structures

A connection on $M$ by Koszul’s definition (see [15]) is a smooth mapping

$$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M)$$

$$(X,Y) \rightarrow \nabla_X Y$$

such that on every local chart $(U, \varphi)$ on $M$, there exists a smooth map $\Gamma_\varphi : \varphi U \rightarrow L^2(\mathbb{E}, \mathbb{E})$ with

$$(\nabla_X Y)(\varphi m) = DY_\varphi(\varphi m).X_\varphi(\varphi m) - \Gamma_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m)); \forall m \in U.$$

We prove in the sequel that if $\nabla$ is a connection on $M$, then $\{\Gamma_\varphi\}$ forms a Christoffel structure on $M$. Conversely if $\{\Gamma_\varphi\}$ is a Christoffel structure on $M$ and $X, Y \in \chi(U)$, then a connection $\nabla$ can be defined by

$$(\nabla_X Y)(m) = T\varphi^{-1}[DY_\varphi(\varphi m).X_\varphi(\varphi m) - \Gamma_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m))]$$

(see [23]).

Before proceeding to results, it is necessary to prove the following.

**Theorem 3.1.** The limit $\nabla = \lim_{i \in \mathbb{N}} \nabla_i$ of a projective system of connections $\{\nabla_i\}_{i \in \mathbb{N}}$ is a connection on $M = \varprojlim M_i$.

**Proof.** For $i \leq j$, let $(U_j, \varphi_j)$ be a chart of $M_j$ around $m_j$ and $(U_i, \varphi_i)$ be a chart of $M_i$ at $\varphi_i m_j = m_i$. Moreover for every $i \in \mathbb{N}$, let $X_{\varphi_i} : \varphi_i U_i \rightarrow \mathbb{E}_i$ be the local principal part of $X_i \in \chi(U_i)$. Since $\nabla$ is a smooth mapping as a projective limit of smooth factors, to prove the theorem it suffices to check that

$$\rho_{ji} \circ \nabla_{X_{\varphi_j}} Y_{\varphi_j} = \nabla_{X_{\varphi_i}} Y_{\varphi_i} \circ \rho_{ji}.$$  

The last equality holds since for $m_j \in U_j$;

$$\rho_{ji} \circ \nabla_{X_{\varphi_j}} Y_{\varphi_j}(\varphi_j m_j) = \rho_{ji}D_{\varphi_j}(\varphi_j m_j).X_{\varphi_j}(\varphi_j m_j) - \rho_{ji} \Gamma_{\varphi_j}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))$$

$$= \nabla_{X_{\varphi_i}} Y_{\varphi_i}(\varphi_i m_i) = \nabla_{X_{\varphi_i}} Y_{\varphi_i} \circ \rho_{ji}(\varphi_j m_j)$$

Note that

$$* = \frac{d}{dt}\rho_{ji}Y_{\varphi_j}(\varphi_j m_j + tX_{\varphi_j}(\varphi_j m_j))|_{t=0}$$

$$= \frac{d}{dt}Y_{\varphi_i} \rho_{ji}(\varphi_j m_j + tX_{\varphi_j}(\varphi_j m_j))|_{t=0}$$

$$= \frac{d}{dt}Y_{\varphi_i}(\varphi_i m_i + tX_{\varphi_i}(\varphi_i m_i))|_{t=0}$$

$$= DY_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i)$$

and

$$** = \Gamma_{\varphi_i}(\varphi_i m_i)(\rho_{ji} \times \rho_{ji})(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))$$

$$= \Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i))$$

$\square$
Based on Theorem 3.1, we may now establish several important properties.

**Theorem 3.2.** If $\nabla = \lim \nabla_i$ is a connection on $M = \lim M_i$, then $\{\Gamma_\varphi = \lim \Gamma_{\varphi_i}\}$ forms a Christoffel structure on $M$.

**Proof.** Let $(U=\lim U_i, \varphi=\lim \varphi_i), (V=\lim v_i, \psi=\lim \psi_i)$ be two charts through $m = (m_i)_{i \in \mathbb{N}} \in M$ and $\lim \varphi_i m_i = \lim u_i = u$, $F = \lim F_i = \lim (\psi_i \circ \varphi_i^{-1})$.

Furthermore, suppose that $X_\varphi = \lim X_{\varphi_i}$, then

$$[DY_\psi \cdot X_\varphi](F(u)) = [D \lim Y_\psi \cdot \lim X_{\varphi_i}](\lim F_i(u)) = \lim [[DY_\psi \cdot X_{\varphi_i}](F_i(u_i))]$$

$$= \lim DY_{\psi_i}(F_i(u_i)) \cdot X_{\varphi_i}(F_i(u_i)) = \lim DY_{\psi_i}(F_i(u_i)) \cdot DF_i(u_i).X_{\varphi_i}(u_i)$$

$$= \lim DY_{\psi_i}(F_i(u_i)).X_{\varphi_i}(u_i) = \lim DF_i(Y_{\varphi_i}(u_i)) \cdot X_{\varphi_i}(u_i)$$

$$= \lim [D^2 F_i(u_i)(X_{\varphi_i}, Y_{\varphi_i}) + DF_i(u_i).DY_{\varphi_i}(u_i).X_{\varphi_i}(u_i)].$$

But

$$(\nabla_X Y)_\varphi \circ F = \lim [(\nabla_{X_Y} Y)_\varphi \circ F_i] = \lim [(DY_{\varphi_i} \cdot X_{\varphi_i}) \circ F_i - \Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i})]$$

$$= \lim [D^2 F_i(X_{\varphi_i}, Y_{\varphi_i}) + DF_i(DY_{\varphi_i} \cdot X_{\varphi_i}) - \Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i})],$$

hence

$$\Gamma_{\varphi}(X_{\varphi_i}, Y_{\varphi_i}) = \lim [\Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i})] = \lim [D^2 F_i(X_{\varphi_i}, Y_{\varphi_i}) + DF_i(DY_{\varphi_i} \cdot X_{\varphi_i}) - \Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i})],$$

hence

$$\Gamma_{\varphi}(X_{\varphi_i}, Y_{\varphi_i}) = \lim [\Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i})] = \lim [D^2 F_i(X_{\varphi_i}, Y_{\varphi_i}) + DF_i(DY_{\varphi_i} \cdot X_{\varphi_i}) - \Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i})],$$

i.e. $\{\Gamma_\varphi = \lim \Gamma_{\varphi_i}\}$ forms a Christoffel structure on $M = \lim M_i$. \hfill $\square$

**Remark 3.3.** The converse also of Theorem 3.2 can be obtained by setting

$$(\nabla_X Y)(m) = \lim [T_{\varphi_i^{-1}}[DY_{\varphi_i}(\varphi_i m_i) \cdot X_{\varphi_i}(\varphi_i m_i)$$

$$- \Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i))],$$

where $\{\Gamma_\varphi = \lim \Gamma_{\varphi_i}\}$ is a Christoffel structure on $M$. Moreover for $f = \lim f_i \in C^\infty(M)$ and $X, Y \in \chi(M)$, $\nabla$ satisfies the following conditions:

(i) $\nabla$ is real linear in $X$ and $Y$,

(ii) $\nabla_{fX} Y = f\nabla_X Y$,

(iii) $\nabla_X (fY) = f\nabla_X Y + (Xf)Y$.

In anticipation of the sequel, a Hessian structure on $M$ is a mapping $H : f \mapsto Hf$, which associates to every $f \in C^\infty(M)$ a covariant 2-tensor $Hf$ on $M$ such that on a local chart $(U, \varphi)$ of $M$ and for every $X, Y \in \chi(M)$, there exists a smooth map $\Gamma_\varphi : \varphi U \mapsto \mathbb{L}^2(\mathbb{R}, \mathbb{R})$ with

$$[Hf(X, Y)]_{\varphi}(\varphi m) = D^2 f_{\varphi}(\varphi m)(X_{\varphi}(\varphi m), Y_{\varphi}(\varphi m)) + DF_{\varphi}(\varphi m).\Gamma_{\varphi}(\varphi m)(X_{\varphi}(\varphi m), Y_{\varphi}(\varphi m)).$$

It turns out that $Hf$ is a Hessian structure on $M$ if and only if $M$ admits the Christoffel structure $\{\Gamma_\varphi\}$. Moreover, there is a one-to-one correspondence between Hessian structures and connections given by $Hf(X, Y) = X(Y(f)) - (\nabla_X Y)f$. (For more details see [23]).

Here we study the above results for projective limit manifolds. However, we should consider just the smooth functions and smooth vector fields such that $\mathcal{F}(M) = \{(f_i)_{i \in \mathbb{N}} : f_i : M_i \mapsto \mathbb{R}$ is continuous and $\lim f_i$ exists$\}$ and $\mathcal{G}(M) = \{(X_i)_{i \in \mathbb{N}} : X_i$ is a vector field on $M_i$ and $\lim X_i$ exists$\}$ respectively.
Proposition 3.4. The limit of a projective system of Hessian structures on \( \{M_i\}_{i \in \mathbb{N}} \) is a Hessian structure on \( M = \lim M_i \).

Proof. For every \( i \in \mathbb{N} \), let \( f_i \in C^\infty(M_i) \) and \( X_i, Y_i \in \chi(M_i) \). Consider a chart \((U_i, \varphi_i)\) on \( M_i \). Assume that \( \Gamma_{\varphi_i} : \varphi_i U_i \rightarrow L^2(\mathbb{E}_i, \mathbb{E}_i) \) is a smooth map such that

\[
[H_i f_i(X_i, Y_i)]_\varphi, \varphi_i m_i) = D^2 f_i(x, \varphi_i m_i) (X_{\varphi_i}, Y_{\varphi_i}, (\varphi_i m_i)) + D f_i(x, \varphi_i m_i) \Gamma_{\varphi_i} (\varphi_i m_i), (X_{\varphi_i}, Y_{\varphi_i}, (\varphi_i m_i)).
\]

Hence we must check that for \( j \geq i \), \([H_j f_j(X_j, Y_j)]_\varphi = [H_i f_i(X_i, Y_i)]_\varphi \circ \rho_{ji} \).

For \( m_j \in U_j \):

\[
[H_j f_j(X_j, Y_j)]_\varphi (\varphi_j m_j) = D^2 f_j(\varphi_j m_j)(X_{\varphi_j}, Y_{\varphi_j}, (\varphi_j m_j)) + D f_j(\varphi_j m_j) \Gamma_{\varphi_j} (\varphi_j m_j), (X_{\varphi_j}, Y_{\varphi_j}, (\varphi_j m_j)).
\]

\[
= [H_i f_i(X_i, Y_i)]_\varphi, \varphi_i m_i) = [H_i f_i(X_i, Y_i)]_\varphi \circ \rho_{ji} (\varphi_j m_j).
\]

Note that:

\[
D f_j(\varphi_j m_j)(X_{\varphi_j}, Y_{\varphi_j}, (\varphi_j m_j)) = D(f_i(\varphi_i \circ \rho_{ji})(\varphi_j m_j))(X_{\varphi_j}, Y_{\varphi_j}, (\varphi_j m_j))
\]

\[
= \frac{d}{dt} (f_i(\varphi_i \circ \rho_{ji})(\varphi_j m_j + tX_{\varphi_j}(\varphi_j m_j)))_{t=0}
\]

\[
= \frac{d}{dt} f_i(\varphi_i m_i + tX_{\varphi_i}(\varphi_i m_i))_{t=0} = D f_i(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)),
\]

and consequently

\[
* = D^2 (f_i \circ f_j^{-1})(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))
\]

\[
= D(D(f_i(\varphi_i \circ \rho_{ji})(\varphi_j m_j))(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j)))[Y_{\varphi_j}(\varphi_j m_j)]
\]

\[
= D(D(f_i(\varphi_i m_i))(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)))[Y_{\varphi_i}(\varphi_i m_i)]
\]

\[
= D^2 f_i(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)).
\]

Moreover

\[
** = D(f_i(\varphi_i \circ \rho_{ji})(\varphi_j m_j))(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))
\]

\[
= \frac{d}{dt} (f_i(\varphi_i \circ \rho_{ji})(\varphi_j m_j + tY_{\varphi_j}(\varphi_j m_j))(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j)))_{t=0}
\]

\[
= \frac{d}{dt} f_i(\varphi_i m_i + t\Gamma_{\varphi_i}(\varphi_i m_i)(\rho_{ji} \circ \rho_{ji}(X_{\varphi_i}(\varphi_j m_j), Y_{\varphi_i}(\varphi_j m_j))))_{t=0}
\]

\[
= D f_i(\varphi_i m_i). \Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)).
\]

Hence \( \lim[H_i f_i(X_i, Y_i)]_\varphi = [H f(X, Y)]_\varphi \) where \( f \in \mathcal{F}(M) \), \( \varphi = \lim \varphi_i \) and \( X, Y \in \mathcal{G}(M) \)

Next, Theorem 3.5 proves that there is a one-to-one correspondence between Hessian structures and connections on Fréchet manifolds.

\( \square \)
Theorem 3.5. Let $\nabla = \lim \nabla_i$ be a connection on $M = \lim M_i$, and $Hf(X,Y) := X(Y(f)) - (\nabla XY)f$. Then $H$ is a Hessian structure on $M$. Conversely the connection which obtained as projective limit of connections arises from a Hessian structure.

Proof. Let $v, w \in T_mM = \lim T_{m_i}M_i$ and $(U = \lim U_i, \varphi = \lim \varphi_i)$ be a chart around $m = (m_i)_{i \in \mathbb{N}}$. Consider vector fields $\lim X_i, \lim Y_i \in \mathfrak{X}(\lim U_i)$ with $\lim X_i(m_i) = v$ and $\lim Y_i(m_i) = w$. Suppose $\nabla = \lim \nabla_i$ be a connection on $M = \lim M_i$, then $\{\Gamma_{\varphi} = \lim \Gamma_{\varphi_i}\}$ is a Christoffel structure on $M$. Hence

$$X(Y(f))(m) - (\nabla_X Y).f(m) = X_\varphi(Y_\varphi(f_\varphi))(\varphi m) - (\nabla_X Y)_\varphi(f_\varphi)(\varphi m)$$

$$= \lim X_\varphi(Y_\varphi(f_\varphi))(\varphi m) - (\nabla_X Y)_\varphi(f_\varphi)(\varphi m)$$

$$= \lim[Df_\varphi(Y_\varphi)(\varphi m)](X_\varphi, (\varphi m)),$$ 

$$= \lim[Df_\varphi(Y_\varphi)(\varphi m)](X_\varphi, (\varphi m)),$$ 

$$= \lim[Df_\varphi(Y_\varphi)(\varphi m)](X_\varphi, (\varphi m)),$$ 

$$= \lim[Hf_\varphi(X, Y)](\varphi m) = [Hf(X,Y)](\varphi m).$$

Conversely if $Hf = \lim Hf_i$ is a Hessian structure on $M = \lim M_i$ then $\{\Gamma_{\varphi} = \lim \Gamma_{\varphi_i}\}$ forms a Christoffel structure on $M$. Now we have

4 Sprays

Definition 4.1. A spray $\zeta$ is a second order vector field on $M$ such that on a local chart $(U, \varphi)$ it is determined by a smooth mapping $\Gamma_{\varphi} : \varphi U \rightarrow L^2(\mathfrak{E}, \mathfrak{E})$ in the following way:

$$[\zeta(v)]_{\varphi}(\varphi m, v_\varphi) = (v_\varphi, \Gamma_{\varphi}(\varphi m)(v_\varphi, v_\varphi)); \quad m \in U, \quad v \in T_mM$$

(see [23]). Note that this definition coincides with the one given in [24].

Theorem 4.2. The limit of a projective system of sprays on $M_i$ is a spray on $M = \lim M_i$.

Proof. For every $i \in \mathbb{N}$, let $\zeta_i$ be a second order vector field on $M_i$. Moreover suppose that $(\lim U_i, \lim \varphi_i)$ is a chart of $M = \lim M_i$. Then on the chart $(U_i, \varphi_i)$ on $M_i$, $\zeta_i$ is determined by the map $\Gamma_{\varphi_i} : \varphi_i U_i \rightarrow L^2(\mathfrak{E}_i, \mathfrak{E}_i)$ with the property

$$[\zeta_i(v_i)]_{\varphi_i}(\varphi_i m_i, v_\varphi) = (v_\varphi, \Gamma_{\varphi_i}(\varphi_i m_i)(v_\varphi, v_\varphi)); \quad m_i \in U_i, \quad v_i \in T_{m_i}M_i.$$
To prove the result, it suffices to check that for \( j \geq i, \)
\[
(\rho_j \times \rho_j)[\zeta_j(v_j)]_{\varphi_j} = [\zeta_i(v_i)]_{\varphi_i}(\rho_j \times \rho_j).
\]
Indeed for every \( m_j \in U_j \) and \( v_j = [U_j, \varphi_j, v_{\varphi_j}] \in T_{m_j}M_j \) one obtains:
\[
(\rho_j \times \rho_j)[\zeta_j(v_j)]_{\varphi_j}(\varphi_j m_j, v_{\varphi_j}) = (\rho_j \times \rho_j)(v_{\varphi_j}, \Gamma_{\varphi_j}(\varphi_j m_j)(v_{\varphi_j}) = (v_{\varphi_j}, \Gamma_{\varphi_j}(\varphi_j m_j)(v_{\varphi_j}))
\]
\[
= [\zeta_i(v_i)]_{\varphi_i}(\rho_j \times \rho_j)(\varphi_j m_j, v_{\varphi_j}) = [\zeta_i(v_i)]_{\varphi_i}(\rho_j)(\varphi_j m_j)) = [\zeta_i(v_i)]_{\varphi_i}(\rho_j)(\varphi_j m_j))
\]
As mentioned in [23] if \( \zeta_i \) is a spray on \( M_i \), for every pair of charts \((U_i, \varphi_i)\) and \((V_i, \psi_i)\) of \( M_i \), the transformation formula for \( \Gamma_{\varphi_i} \) is
\[
\Gamma_{\psi_i}(\psi m_i)(v_{\psi_i}, v_{\psi_j}) = D^2 F_i(\varphi_i)(v_{\psi_i}, v_{\psi_j}) + DF_i(\varphi m_i) \Gamma_{\varphi_i}(\varphi m_i)(v_{\psi_i}, v_{\psi_j})
\]
where \( F_i = \psi \circ \varphi_i^{-1} \) and \( v_i = [U_i, \varphi_i, v_{\varphi_i}] \in T_{m_i}M_i \). Suppose that \( \zeta = \lim \zeta_i \) be a spray on \( M = \lim M_i \). Then for charts \((U = \lim U_i, \varphi = \lim \varphi_i)\) and \((V = \lim V_i, \psi = \lim \psi_i)\) at \( m = (m_i)_{i \in \mathbb{N}} \in M \) and \( v = [U, \varphi, v_{\varphi}] \in T_{m_i}M_i \):
\[
\Gamma_{\psi}(\psi m)(v_{\psi}, v_{\psi}) = \lim \Gamma_{\psi_i}(\psi m_i)(v_{\psi_i}, v_{\psi_i})
\]
\[
= \lim [D^2 F_i(\varphi m_i)(v_{\psi_i}, v_{\psi_i}) + DF_i(\varphi m_i) \Gamma_{\varphi_i}(\varphi m_i)(v_{\psi_i}, v_{\psi_i})]
= D^2 F(\varphi m)(v_{\psi}, v_{\psi}) + DF(\varphi m) \Gamma_{\varphi}(\varphi m)(v_{\psi}, v_{\psi})
\]
It means that the spray \( \zeta = \lim \zeta_i \) defines the Christoffel structure \( \{\Gamma_{\varphi} = \lim \Gamma_{\varphi_i}\} \) on \( M = \lim M_i \).
\[
5 \quad \text{Dissections}
\]
The concept of dissection is considered next. Kumar and Viswanath [23] established a one-to-one correspondence between dissections of \( M \) and Christoffel structures on \( M \) for a Banach manifold \( M \). We extend this correspondence to projective limit manifolds.

For \( m \in M \), let \( G_m := \{ f \in C^{\infty}(U_m) : U_m \text{ is a neighbourhood of } m \} \) and \( G^0_m := \{ f \in G_m : f(m) = 0 \} \). Define the space of 1-jets at \( m \), denoted by \( J^1 M \), to be the set of all equivalence classes in \( G^0_m \), where two functions \( f, g \in G^0_m \) are equivalent if on every chart \((U, \varphi)\) of \( M \), the following relation holds true: \( Df(\varphi m) = Dg(\varphi m) \). In a similar way for every chart \((U, \varphi)\) of \( M \), one may define \( J^2 M := \{ [f] \in J^1 M : D^2 f(\varphi m) = D^2 g(\varphi m), \forall g \in [f] \} \). If \( s \in J^2 M \), then the local representation of \( s \) on the chart \((U, \varphi)\) is \( s = \alpha_{\varphi} \oplus B_{\varphi} \in \mathbb{E}^* \oplus L^2_c(\mathbb{E}, \mathbb{R}) \) with transformation rule \( \alpha_{\varphi} = \alpha_{\varphi} \circ \alpha G(v) \) and \( B_{\varphi} = B_{\varphi} \circ (DG(v) \times DG(v)) + \alpha_{\varphi} DG(v) \circ D^2 F(u) \circ (DG(v) \times DG(v)) \), where \( \alpha_{\varphi} \) is the local representation of \( \alpha \in T^*_m M, G = \varphi \circ \psi^{-1}, u = \varphi m \) and \( v = \psi m \) (for more details see [23]).

Definition 5.1. A dissection on \( M \) is a map that to every \( m \in M \) assigns a closed subgroup of \( J^2 M \) say \( D_m \). This is done in such a way that for every chart \((U, \varphi)\) there exists a smooth mapping \( \Gamma_{\varphi} : \varphi U \rightarrow L^2_c(\mathbb{E}, \mathbb{R}) \) such that \( B_{\varphi} = \alpha_{\varphi} \circ \Gamma_{\varphi}(u) \) for \( s \in D_m \) and \( s_{\varphi} = \alpha_{\varphi} \oplus B_{\varphi} \). In other words \([D_m]_{\varphi} = \{ \alpha + \alpha \circ \Gamma_{\varphi}(u) : \alpha \in \mathbb{E}^* \} \) [23].
We extend Kumar and Viswanath’s results to projective limit Fréchet manifolds.

**Proposition 5.2.** If \( \{M_i\}_{i \in \mathbb{N}} \) is a projective system of manifolds and \( \lim_{m} J^2_{m_i} M_i \) exists then \( \lim_{m} J^2_{m_i} M_i = J^2_{\lim m_i} M_i \) (set-theoretically).

**Proof.** Let \( G_m := \{(f_i)_{i \in \mathbb{N}}; f_i : U_{m_i} \to \mathbb{R} \) is continuous and \( \lim f_i \) exists\} and \( G^0_m := \{(f_i)_{i \in \mathbb{N}} \in G_m : f_i(m_i) = 0, \forall i \in \mathbb{N} \}. \) By defining

\[
p : J^2_{m_i} M_i \longrightarrow \lim_{m} J^2_{m_i} M_i \\
[f, m] \longmapsto ([f, m_i])_{i \in \mathbb{N}}
\]

It can be checked that \( p \) is well defined; moreover, \( p \) is one to one since \( p[f, m] = p[g, m] \) yields

\[
[f_i, m_i] = [g_i, m_i], \ i \in \mathbb{N}.
\]

Hence \( [f, m] = [\lim f_i, (m_i)_{i \in \mathbb{N}}] = [\lim f_i, m_i] = [\lim g_i, m_i] = [\lim g_i, (m_i)_{i \in \mathbb{N}}] = [g, m]. \)

Furthermore \( p \) is surjective. In fact if \( ([f_i, m_i])_{i \in \mathbb{N}} \) is an arbitrary element of \( \lim J^2_{m_i} M_i \), we define \( a = [\lim f_i, (m_i)_{i \in \mathbb{N}}] \). Then \( p(a) = ([f_i, m_i])_{i \in \mathbb{N}} \) and therefore \( p \) is an isomorphism between \( J^2_{m_i} M_i \) and \( \lim J^2_{m_i} M_i \). \( \square \)

**Theorem 5.3.** The limit of a projective system of dissections of \( \{M_i\}_{i \in \mathbb{N}} \) is a dissection of \( \lim M_i = M_i \).

**Proof.** For every \( i \in \mathbb{N} \), suppose \( D_{m_i} \) is the closed subgroup of \( J^2_{m_i} M_i \) with the above mentioned properties. Moreover for \( j \geq i \),

\[
B_{\phi_j} \circ \alpha_{\phi_j} \circ (\Gamma_{\phi_j} \circ u_j) = (\alpha_{\phi_i} \circ \rho_{ji}) \circ (\Gamma_{\phi_i} \circ u_i) = (\alpha_{\phi_i} \circ \rho_{ij} \times \rho_{ji})
\]

Therefore \( \lim D_{m_i} \) exists and it is a dissection on \( M = \lim M_i \). \( \square \)

If \( \lim D_{m_i} \) is a dissection of \( \lim M_i = M \) and \( U = \lim U_i, \varphi = \lim \varphi_i \), \( V = \lim V_t, \psi = \lim \psi_t \) are two charts at \( m = (m_i)_{i \in \mathbb{N}} \in M \), then

\[
\Gamma_\psi(v) = \lim_\varphi \Gamma_{\psi_i}(v_i) = \lim_\varphi [D^2 F_i(u_i) \circ (DG_i(v_i) \times DG_i(v_i)) + DF_i(u_i) \circ \Gamma_{\phi_i}(v_i) \circ (DG_i(v_i) \times DG_i(v_i))]
\]

which precisely coincides with the Christoffel structures \( \{\Gamma_{\phi_i} = \lim_{\psi_i} \Gamma_{\phi_i}\} \). (For more details see [23].) Hence we get the following result.

**Corollary 5.4.** There is one-to-one correspondence between dissections and Christoffel structures on \( M = \lim M_i \).
6 Examples

Example 6.1. The direct connection
Let \( G \) be a Banach Lie group with the model space \( E \). Consider the mapping \( \mu : G \times \partial \rightarrow TG \) given by \( \mu(m,v) = T_\mu\lambda_m(v) \), where \( \lambda_m \) is the left translation on \( G \) and \( \partial \) is the Lie algebra of \( G \). According to Vassiliou \[31\], there exists a unique connection \( \nabla^G \) on \( G \) which is \((\mu, id_G)\)-related to the canonical flat connection on the trivial bundle \( L = (G \times \partial, pr_1, G) \). Locally the Christoffel symbols \( \Gamma^G \) of \( \nabla^G \) are given by

\[
\Gamma^G_\varphi(x)(a,b) = Df_\varphi(x)(a, f_\varphi^{-1}(m)(b)); \ x \in \varphi U, \ a,b \in E
\]

where \( f_\varphi \) is the local expression of the isomorphism \( T_0\lambda_x : T_xG \rightarrow T_xG \) and \((U, \varphi)\) chart of \( G \). If \( G = \lim G_i \) is obtained as a projective limit of Banach Lie groups and \( \nabla^{G_i} \) is the direct connection on \( G_i \), then \( \nabla^G = \lim \nabla^{G_i} \) is exactly the direct connection on \( L = (\lim G_i \times \lim \partial, pr_1, \lim G_i) \) \[21\].

Also, \( \nabla^G \) determines a unique spray on \( G = \lim G_i \) locally given by

\[
[k^G_\varphi(m, v_\varphi)] = (v_\varphi, \Gamma^G_\varphi(m)(v_\varphi, v_\varphi)); \ m \in U, \ v \in T_m G.
\]

Moreover, using \( \nabla^G \) the Christoffel structure \( \{\Gamma^G_\varphi\} \) and Hessian structure \( H^G \) are obtained where \( H^G \) is locally given by

\[
[H^G f(X,Y)]_\varphi(m) = D^2 f_\varphi(m)(X_\varphi(m), Y_\varphi(m)) + Df_\varphi(m) \Gamma^G_\varphi(m)(X_\varphi(m), Y_\varphi(m)).
\]

Example 6.2. The flat connection
Let \( M = E \) with the global chart \( (E, id_E) \). The canonical flat connection \( \nabla^C \) on the trivial bundle \( (M \times E, pr_1, M) \) is locally given by the Christoffel structure \( \{\Gamma^C\} \), where \( \Gamma^C(x)(u) = 0 \), for every \( (x,u) \in E \times E \). Let \( M = F = \lim E_i \) and consider it with the global chart \( (F, id_F) = \lim (E_i, id_{E_i}) \). For the canonical flat connection \( \Gamma^C = \lim \Gamma^F_i \) on \( (M \times F, pr_1, M) \), the spray \( \zeta^C \) and the Hessian structure \( H^C \) are given by

\[
[\zeta^C_\varphi(m, v_\varphi)] = (v_\varphi); \ m \in U, \ v \in T_m M
\]

and

\[
[H^C f(X,Y)]_\varphi(m) = D^2 f_\varphi(m)(X_\varphi(m), Y_\varphi(m)).
\]

7 Ordinary differential equations

A curve \( \gamma : (-\varepsilon, \varepsilon) \rightarrow M \) is called autoparallel or a geodesic with respect to the connection \( \nabla \) if \( \nabla_{T\gamma}T\gamma = 0 \) \[32\]. Let \((U, \varphi)\) be a local chart on \( M \) and set \( \gamma_\varphi := \varphi \circ \gamma : (-\varepsilon, \varepsilon) \rightarrow E, \ \gamma'_\varphi(t) := T\gamma_\varphi : (-\varepsilon, \varepsilon) \rightarrow TE \).

In this case the local expression of \( \nabla_{T\gamma}T\gamma = 0 \) takes the form:

\[
\nabla_{T\gamma_\varphi}T\gamma_\varphi(t) = D\gamma_\varphi(t).\gamma'_\varphi(t) - \Gamma_\varphi(\gamma_\varphi(t))[\gamma'_\varphi(t), \gamma'_\varphi(t)] = 0.
\]

Every spray is a second order vector field, hence every integral curve of \( \zeta \) is the canonical lifting of \( \pi \circ \beta \), so \( T(\pi \circ \beta) = \beta \). The curve \( \gamma : (-\varepsilon, \varepsilon) \rightarrow M \)
is a geodesic spray with respect to ζ if $T_γ$ is an integral curve for ζ, namely,
$T_{T_γ(v_i)}T_γ(v_i) = ζ(T_γ(v_i))$, where $v_i \in T_xR$ with $pr_2(v_i) = 1$. In local charts we have:

$$ \left(ζ(T_γ(v_i))\right)_φ(γ_φ(t), D_tγ_φ(v_i)) = \left(γ_φ(t), Γ_φ(γ_φ(t))[D_tγ_φ(v_i), D_tγ_φ(v_i)]\right). $$

and

$$ (T_{T_γ(v_i)}T_γ(v_i))_φ = (D_tγ_φ(v_i), DD_tγ_φ(v_i), D_tγ_φ(v_i)) := (γ'_φ(t), γ''_φ(t)). $$

So γ must satisfy the (local) equation

$$ γ''_φ(t) = Γ_φ(γ_φ(t))(γ'_φ(t), γ'_φ(t)). $$

Consequently the following theorem holds for Banach modelled manifolds.

**Theorem 7.1.** Let ζ be the spray assigned to ∇. There is a one-to-one correspondence between geodesics of ∇ and geodesic sprays of ζ.

Here we try to generalize this to the case of Fréchet manifolds where difficulties arise due to intrinsic problems of the model spaces of these manifolds and mainly due to the inability to solve general differential equations (see [3], [17] and [27]). We show that if one focuses on the category of projective limit manifolds, then similar results can be obtained.

**Theorem 7.2.** Let $M = \lim M_i$ and $ζ = \lim ζ_i$ be a spray on $M$ with k-Lipschitz local components. Let $x_0 \in M$ and $y_0 \in T_{x_0}M$. If for a chart $(U, ϕ)$ around $x_0$, $M_ϕ = \sup\{p_i(x_0)^2 + p_i(ϕ_ϕ(x_0))[y_0, y_0]^2\}^{1/2} \in \mathbb{N} < ∞$, then there exists a locally unique geodesic spray $γ : (-ε, ε) → M$ such that $γ(0) = x_0$, $T_1γ(0) = y_0$ and ε > 0 is independent of the index i.

**Proof.** Let $ζ : TM → TT'M$ be a spray. Consider $\{ζ_i\}_{i \in \mathbb{N}}$, $x_0 = (x_{0i})_{i \in \mathbb{N}} \in \lim M_i$ and $y_0 = (y_{0i})_{i \in \mathbb{N}} \in T_{x_0}M_i$. For every $i \in \mathbb{N}$, $ζ_i$ is a spray on $M_i$. Since $M_i$ is a Banach manifold, by the existence theorem for ordinary differential equations, there exists $γ_i : (-ε_i, ε_i) → M_i$ with

$$ γ''_φ(t) = Γ_φ(γ_φ(t))[γ'_φ(t), γ'_φ(t)], $$

satisfying $γ_i(0) = x_{0i}$ and $T_1γ_i(0) = y_{0i}$. For $j ≥ i$, we claim that $φ_{jφi} = γ_j = γ_i$ and consequently $\{γ_i\}_{i \in \mathbb{N}}$ forms a projective system of curves on $\{M_i\}_{i \in \mathbb{N}}$ with the limit $γ = \lim γ_i$. Note that

$$ (φ_i ∘ φ_{jφi} ∘ γ_{jφi})''(t) = (ρ_{ji} ∘ φ_j ∘ γ_{jφi})''(t) = ρ_{ji}(φ_j ∘ γ_{jφi})''(t) = ρ_{ji}Γ_{jφi}(γ_{jφi}(t)) $$

$$ (γ''_{jφi}(t), γ''_{jφi}(t)) = Γ_{jφi}((ρ_{ji} ∘ φ_j ∘ γ_{jφi})''(t), (ρ_{ji} ∘ φ_j ∘ γ_{jφi})''(t), (ρ_{ji} ∘ φ_j ∘ γ_{jφi})''(t)) $$

Moreover $(φ_{jφi} ∘ γ_i)(0) = φ_{jφi}(x_{0j}) = x_{0i}$ and $T_t(φ_{jφi} ∘ γ_j)(0) = y_{0i}$. By uniqueness of solutions for ordinary differential equations on Banach spaces (manifolds) we have $φ_{jφi} ∘ γ_j = γ_i$ and consequently $γ = \lim γ_i$ exists. Furthermore

$$ T_{γ_i(v_i)}T_γ(v_i) = \{T_{γ_i(v_i)}T_tγ_i(v_i)\}_{i \in \mathbb{N}} = \{ζ_i(T_tγ_i(v_i))\}_{i \in \mathbb{N}} = ζ(T_tγ(v_t)). $$
According to Theorem 9.1 $\varepsilon$ does not converge to 0 and consequently there exists $\varepsilon > 0$ such that $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ is a local geodesic spray with respect to $\zeta$.

Let $\beta : (-\varepsilon_1, \varepsilon_1) \rightarrow M$ be another curve such that $T_{t,\beta(v_i)} T_t \beta(v_i) = \zeta(T_t \beta(v_i))$, satisfying in the above boundary conditions. For every $i \in \mathbb{N}$, $\beta_i = \psi_i \circ \beta$ satisfies in equation (2) with $\beta_i(0) = x_0$, and $T_t \beta_i(0) = y_0$. Hence $\beta_i = \gamma_i$ and consequently $\beta = \lim_\varepsilon \beta_i = \lim_\varepsilon \gamma_i = \gamma$ on the intersection of their domains.

Finally in a similar way one can prove the theorem for geodesics with respect to the connection $\nabla$. As a conclusion we can state the following corollary.

**Corollary 7.3.** For a projective limit manifold $M = \lim M_i$ there is a one-to-one correspondence between (linear) connections and sprays. Moreover, the geodesics of $\nabla$ are geodesic sprays of $\zeta$.

### 8 Parallel translation

Vilms [32] defines a connection on $M$ as a vector bundle morphism $\nabla : T(TM) \rightarrow TM$. So $\nabla$ is fully determined by its local components, called Christoffel symbols, denoted by $\{\Gamma_{\alpha}\}_{\alpha \in I}$ corresponding to an atlas of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of $M$. Then, $\Gamma_{\varphi} : \varphi U \rightarrow L^2(\mathbb{E}, \mathbb{E})$, and for two charts $(U, \varphi)$ and $(V, \psi)$ at $m \in M, e_1, e_2 \in \mathbb{E}, u = \varphi(m), v = \psi(m), F = \psi \circ \varphi^{-1}$ we have

$$\Gamma_{\varphi}(v)((DF(u), e_1, DF(u), e_2) = DF(u), \Gamma_{\varphi}(u)(e_1, e_2) + D^2 F(e_1, e_2).$$

Clearly, our definition coincides with the above; we next consider parallel transport of vectors along a curve.

**Theorem 8.1.** Given $\nabla : T(TM) \rightarrow TM$ a connection on $(TM, M, \pi)$, take a smooth curve $c : (a, b) \rightarrow M$ with $0 \in (a, b), c(0) = x$. Then, there is a neighbourhood $U$ of $T_x M \times \{0\} \subseteq T_x M \times (a, b)$ and a smooth mapping $\tilde{c} : U \rightarrow TM$ such that:

(i) $\pi(\tilde{c}(u_x, t)) = c(t)$ and $\tilde{c}(u_x, 0) = u_x$,

(ii) $\nabla(\frac{d}{dt}(\tilde{c})(u_x, t)) = 0$.

**Proof.** For every $(U, \varphi)$ chart of $M$, $\nabla((\frac{d}{dt}(\tilde{c})(u_x, t)) = 0$, locally gives

$$-\Gamma_{\alpha}(c(t))(\frac{d}{dt}(\varphi(c(t), \gamma(y, t))) + \frac{d}{dt}\gamma(y, t)) = 0,$$

where $T\varphi(\tilde{c}(t, T\varphi^{-1}(x, y), t)) = (c(t), \gamma(y, t))$ (i.e. $\gamma : E \times (a, b) \rightarrow S$). For $M$ a Banach manifold, by the existence theorem for differential equations, $\tilde{c}$ always exists.

Using our method one can prove a similar theorem for parallel transport along curves in the category of projective limit manifolds. The equivalence of linear connections with sprays means that parallel transport is equivalently determined by a spray [22].

**Example 8.2. Geodesics on the diffeomorphism group of the circle**

The main reference for this example is Constantin and Kolev [9]. Let $D = Diff(S^1)^+$ be the group of all smooth orientation-preserving diffeomorphisms of the circle $S^1$. We can endow $D$ with a smooth manifold structure based on the Fréchet space $C^\infty(S^1)$. 

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Moreover a right invariant weak Riemannian metric on $D$ is defined. Note that $C^\infty(S^1) = \bigcap_{n \geq 2k+1} H^n(S^1)$ where $H^n(S^1)$, $n \geq 0$ is the space $L^2(S^1)$ of all square integrable functions $f$ with the distributional derivatives up to order $n$, $\partial_x^i$ with $i = 0, 1, ..., n$, in $L^2(S^1)$. $H^n(S^1)$, $n \geq 0$ is a Hilbert space with the norm
\[ \|f\|_n^2 = \sum_{i=0}^n \int_S (\partial_x^i f)^2(x)dx. \]
The main difference of this example for our method lies in the existence of a metric and this allows us to prove the length minimizing property of geodesics.

We move this problem to the projective limit framework. In this special case like [25] the connecting morphisms of the model space are inclusions
\[ \rho_{n+1,n} : H^{n+1} \to H^n, \quad f \mapsto f. \]
The meaning of this morphism is clear, namely if $f \in H^{n+1}$ with the norm on $H^{n+1}$ then $\sum_{i=0}^n \int_S (\partial_x^i f)^2(x)dx < \infty$. Clearly $\sum_{i=0}^n \int_S (\partial_x^i f)^2(x)dx < \infty$, so the function $f$ belongs precisely to $H^n$. Consequently if $f \in \bigcap_{n \geq 2k+1} H^n(S^1)$ then $(f) \in \lim H^n(S^1)$ and, conversely, $C^\infty(S^1) = \bigcap_{n \geq 2k+1} H^n(S^1) = \lim_{n \geq 2k+1} H^n(S^1)$.

For $\varphi_0 \in D$ define $U_0 = \{ \varphi \in D : \| \varphi - \varphi_0 \|_{C^\infty(S^1)} < 1/2 \}$ and $u : u_0 \to C^\infty(S^1)$ with $u(x) = \frac{1}{2\pi} \ln(\varphi_0(x)\varphi(x))$, $x \in S$. Then $(U_0, \psi_0)$ is a local chart with $\psi_0(\varphi) = u$ and change of charts given by $\psi_2 \circ \psi_1^{-1} = u_1 + \frac{1}{2\pi} \ln(\varphi_2 / \varphi_1)$. Note that $\psi_2 \circ \psi_1^{-1} : \psi_1(u_1) \subseteq C^\infty(S^1) \to \psi_2(u_2) \subseteq C^\infty(S^1)$ can be recognized as the projective limit on Hilbert components, say $(\psi_2 \circ \psi_1^{-1})_t : H^t(S^1) \to H^t(S^1)$, $(\psi_2 \circ \psi_1^{-1}) = \lim (\psi_2 \circ \psi_1^{-1})$. These maps are called $k$-Lipschitz and so $(\psi_2 \circ \psi_1^{-1})$. This structure endows $D$ with a smooth manifold structure based on the Fréchet space $C^\infty(S^1)$.

Let $k \geq 0$, for $n \geq 0$ define the linear seminorms $A_k : H^{n+2k}(S^1) \to H^n(S^1)$ with $A_k = 1 - \frac{d^2}{dx^2} + ... + (-1)^k \frac{d^{2k}}{dx^{2k}}$. This enables us to define the bilinear operator $B_k : C^\times \times C^\times \to C^\times$ with $B_k(u, v) = A_k^{-1}(2u_2 A_k(u) + v A_k(u))$, $u, v \in C^\times$. Note that $B = \lim_{n \to \infty} B_k$ where $B_k^n : H^n(S^1) \times H^n(S^1) \to H^{n-2k}(S^1)$. As stated in [9] Theorem 1, there exists a unique linear (Riemannian) connection $\nabla^k$ on $D$.

If $\varphi : J \to D$ is a $C^2$-curve satisfying the autoparallel equation with respect to the linear connection $\nabla^k$, then
\[ u_t = B_k(u, u), \quad t \in J \]
where $u = \varphi_t \circ \varphi^{-1} \in T_{id}D \equiv C^\times(S^1)$. The term autoparallel rather than geodesic is better since there is no underlying Riemannian metric. However the utilization of a weak Riemannian metric is an issue that remains open. Since $B_k = \lim B_k^n$, is the projective system of bilinear maps (as Christoffel symbols) we can endow $D$ with the linear connection $\nabla^k = \lim \nabla^k$. Given an initial value we obtain the unique autoparallel $\varphi : J \to D$ obtained as the projective limit on Hilbert components. The problem is much easier than the general case. Specifically, let the solution on the $H^n(S^1)$, $n \geq 2k+1$ have the manifold domain $[0, T_n]$ with $T_n > 0$. If $T_n < T_{2k+1}$ then $T_n = T_{2k+1}$ for all $n \geq 2k + 1$ i.e. the solution $\varphi_n$ on $H^n(S^1)$, $n \geq 2k$ defined on $[0, T_{2k+1})$ for every $n$. 


Note that in the general case there is no way to model the diffeomorphism group of a manifold $M$ on a Banach space. However, there is the possibility to view $\text{Diff}(M)$ as a projective limit of Hilbert manifolds \([20]\). Moreover, the existence of the geodesics in the general case of $\text{Diff}(M)$ is an open question, so using the proposed technique with an appropriate choice of imposed metric may yield some results.

### 9 Appendix: Existence and uniqueness theorem for second order ordinary differential equations on Fréchet spaces

Start with the assumptions of \([20]\). Namely, let $F$ be a Fréchet space and \(\{p_i\}_{i\in\mathbb{N}}\) be a countable family of seminorms which determine the topology of $F$.

**Theorem 9.1.** Let $F$ be a Fréchet space and $\Phi : \mathbb{R} \times F \times F \to F$ a projective limit $k$-Lipschitz mapping. For the second order differential equation
\[
x'' = \Phi(t, x, x')
\]
with the initial condition $(t_0, x_0, y_0)$, if there exists a constant $\tau \in \mathbb{R}^+$ such that
\[
M = \sup \{p_i(y_0)^2 + p_i(\Phi(t, x_0, y_0))^2 \}^{1/2}; \quad i \in \mathbb{N}, t \in [t_0 - \tau, t_0 + \tau]\n\]
and $a = \min\{\tau, \frac{1}{M+k}\}$, then (2) has a unique solution on $I = [t_0 - a, t_0 + a]$.

**Proof.** If we set $x' = y$, $x'' = y' = \Phi(t, x, y)$. Denoting $z = (x, y)$ one takes:
\[
z' = (x, y)' = (x', y') = (y, \Phi(t, x, y)) = \tilde{\Phi}(t, z)
\]
where $\tilde{\Phi} : \mathbb{R} \times F \times F \times F \to F \times F$, is also a $k$-Lipschitz mapping. Since
\[
\left( p_i(y_0)^2 + p_i(\Phi(t, x_0, y_0))^2 \right)^{1/2} = p_i(y_0, \Phi(t, x_0, y_0)) = p_i(\tilde{\Phi}(t, z_0));
\]
and
\[
M = \sup \{p_i(\tilde{\Phi}(t, z_0)) = \left( p_i(y_0)^2 + p_i(\Phi(t, x_0, y_0))^2 \right)^{1/2}; \quad i \in \mathbb{N}, t \in [t_0 - a, t_0 + a]\} < \infty
\]
by Theorem 3 in \([20]\), (4) has a unique solution on $I = [t_0 - a, t_0 + a]$ such that $a = \min\{\tau, \frac{1}{M+k}\}$. Hence there exists also a solution for (3) say $z : I \to F \times F$. If $z = (z_1, z_2)$ then, $z_1$ and $z_2$ are unique solution for $x' = y$ and $y' = \Phi(t, x, y)$ respectively on $I$. Consequently $z_1'' = \Phi(t, z_1, z_1')$ i.e.
\[
z_1'' = \Phi(t, z_1, z_1') \text{ on } I.
\]
Note that the interval $I$ is independent of the index $i$. For each $i \in \mathbb{N}$ from the equation
\[
x_i'' = \Phi_i(t, x_i, x_i')
\]
with the initial condition $(t_0, x_0, y_0)$ we have the unique solution $x_i$. On the other hand for $i \leq j$, $f_{ji} \circ x_j$ is also a solution of (4) with $f_{ji} \circ x_j(t_0) = x_{0i}$ and $(f_{ji} \circ x_j)'(t_0) = y_0$. Hence $f_{ji} \circ x_j = x_i$ for $i \leq j$, i.e. $x = \lim_{t \to \infty} x_i$ can be defined. Moreover
\[
x'' = (x''_i)_{i \in \mathbb{N}} = (\Phi_i(t, x_i, x_i'))_{i \in \mathbb{N}} = \Phi(t, x, x'),
\]
i.e. $\lim_{t \to \infty} x_i$ is a solution for (2). The uniqueness of $x$ follows from the uniqueness of solution for Banach components.

\qed
References


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