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# Deflating Quadratic Matrix Polynomials with Structure Preserving Transformations

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## Abstract

Given a pair of distinct eigenvalues  $(\lambda_1, \lambda_2)$  of an  $n \times n$  quadratic matrix polynomial  $Q(\lambda)$  with nonsingular leading coefficient and their corresponding eigenvectors, we show how to transform  $Q(\lambda)$  into a quadratic of the form  $\begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}$  having the same eigenvalues as  $Q(\lambda)$ , with  $Q_d(\lambda)$  an  $(n-1) \times (n-1)$  quadratic matrix polynomial and  $q(\lambda)$  a scalar quadratic polynomial with roots  $\lambda_1$  and  $\lambda_2$ . This block diagonalization cannot be achieved by a similarity transformation applied directly to  $Q(\lambda)$  unless the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  are parallel. We identify conditions under which we can construct a family of  $2n \times 2n$  elementary similarity transformations that (a) are rank-two modifications of the identity matrix, (b) act on linearizations of  $Q(\lambda)$ , (c) preserve the block structure of a large class of block symmetric linearizations of  $Q(\lambda)$ , thereby defining new quadratic matrix polynomials  $Q_1(\lambda)$  that have the same eigenvalues as  $Q(\lambda)$ , (d) yield quadratics  $Q_1(\lambda)$  with the property that their eigenvectors associated with  $\lambda_1$  and  $\lambda_2$  are parallel and hence can subsequently be deflated by a similarity applied directly to  $Q_1(\lambda)$ . This is the first attempt at building elementary transformations that preserve the block structure of widely used linearizations and which have a specific action.

*Key words:* quadratic eigenvalue problem, linearization, structure preserving transformation, deflation

*2000 MSC:* 15A18, 65F15, 65F30

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## 1. Introduction

Consider the quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$ , where  $M, C, K \in \mathbb{R}^{n \times n}$  with  $M$  nonsingular, and the associated quadratic eigenvalue

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problem

$$Q(\lambda)x_R = 0, \quad x_L^*Q(\lambda) = 0, \quad (1)$$

where  $\lambda$  is an eigenvalue and  $x_R$  and  $x_L$  are corresponding right and left eigenvectors, respectively. Throughout, we use the subscript  $R$  to denote right eigenvectors or when referring to transformations applied to the right, and the subscript  $L$  for left eigenvectors and transformations applied to the left. We also denote by  $\Lambda(Q)$  the spectrum of  $Q$ .

Given two eigentriples  $(\lambda_j, x_{Rj}, x_{Lj})$ ,  $j = 1, 2$  satisfying appropriate conditions, we propose a deflation procedure that decouples  $Q(\lambda)$  into a quadratic  $Q_d(\lambda) = \lambda^2 M_d + \lambda C_d + K_d$  of dimension  $n - 1$  and a scalar quadratic  $q(\lambda) = \lambda^2 m + \lambda c + k = m(\lambda - \lambda_1)(\lambda - \lambda_2)$  such that

$$\Lambda(Q) = \Lambda(Q_d) \cup \{\lambda_1, \lambda_2\}$$

and there exist well-defined relations between the eigenvectors of  $Q(\lambda)$  and those of the decoupled quadratic

$$\tilde{Q}(\lambda) = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}. \quad (2)$$

This is termed “strong deflation” in the engineering community, as opposed to “weak deflation”, which is achieved by introducing zeros in the trailing rows or columns of the matrices.

Unlike for linear polynomials  $A - \lambda B$ , we cannot in general construct an  $n \times n$  equivalence transformation with nonsingular matrices  $P_L$  and  $P_R$  such  $P_L^T Q(\lambda) P_R = \tilde{Q}(\lambda)$ , where  $\tilde{Q}(\lambda)$  is the decoupled quadratic in (2) [17]. The standard way of treating quadratic matrix polynomials, both theoretically and numerically, is to convert them into equivalent linear matrix pencils of twice the dimension, a process called linearization [11]. For example, when  $M$  is nonsingular the block symmetric pencil

$$L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$

is a linearization of  $Q(\lambda)$  in the sense that  $L_2(\lambda)$  satisfies

$$E(\lambda)L_2(\lambda)F(\lambda) = \begin{bmatrix} Q(\lambda) & 0 \\ 0 & I_n \end{bmatrix}$$

for some unimodular  $E(\lambda)$  and  $F(\lambda)$ , where  $I_n$  is the  $n \times n$  identity matrix [11], [22]. This implies that  $c \cdot \det(L_2(\lambda)) = \det(Q(\lambda))$  for some nonzero constant  $c$ , so that  $L_2$  and  $Q$  have the same eigenvalues. Deflation procedures for matrix pencils ignore the block structure of linearizations such as  $L_2(\lambda)$ . They produce a deflated pencil that is not in general a linearization of a quadratic matrix polynomial [16].

Garvey, Friswell, and Prells [8] and later Chu and Xu [7] showed that for quadratics with symmetric coefficients and semisimple eigenvalues (i.e., each

eigenvalue  $\lambda$  appears only in  $1 \times 1$  Jordan blocks in a Jordan triple for  $Q$  [11]), there exists a real nonsingular matrix  $W \in \mathbb{R}^{2n \times 2n}$  such that

$$W^T L_2(\lambda) W = \lambda \begin{bmatrix} 0 & D_M \\ D_M & D_C \end{bmatrix} + \begin{bmatrix} -D_M & 0 \\ 0 & D_K \end{bmatrix} =: L_D(\lambda), \quad (3)$$

with  $D_M, D_C, D_K$  diagonal. The pencil  $L_D(\lambda)$  is a linearization of the diagonal quadratic  $Q_D(\lambda) = \lambda^2 D_M + \lambda D_C + D_K$ , which clearly has the same eigenvalues as  $Q(\lambda)$ . The proof of the diagonalization of the blocks of  $L_2(\lambda)$  in (3) is constructive and requires the knowledge of all the eigenvalues and eigenvectors of  $Q$ . Most importantly it shows that by increasing the dimension of the transformations from  $n \times n$  when working directly on  $Q$  to  $2n \times 2n$  by working on a pencil of twice the dimension of  $Q$ , total decoupling of the underlying second order system can be achieved. The congruence in (3) is an example of a *structure preserving transformation* (SPT). More generally, we say that a pair  $(W_L, W_R)$  of  $2n \times 2n$  real nonsingular matrices defines a structure preserving transformation for an  $n \times n$  quadratic matrix polynomial  $Q(\lambda) = \lambda^2 M + \lambda C + K$  with  $M$  nonsingular if

$$W_L^T \left( \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \right) W_R = \left( \begin{bmatrix} 0 & M_1 \\ M_1 & C_1 \end{bmatrix}, \begin{bmatrix} -M_1 & 0 \\ 0 & K_1 \end{bmatrix} \right), \quad (4)$$

where  $M_1, C_1$ , and  $K_1$  are  $n \times n$  matrices [21] that define a new quadratic  $Q_1(\lambda) = \lambda^2 M_1 + \lambda C_1 + K_1$  having the same eigenvalues as  $Q(\lambda)$ .

Because the problem is quadratic, we need to deflate two eigenvalues at a time. For a given pair of eigenvalues  $\lambda_1, \lambda_2$  and their associated left and right eigenvectors  $x_{Lj}, x_{Rj}$ ,  $j = 1, 2$ , we identify conditions under which there exist elementary SPTs  $(W_L, W_R)$  that are rank-two modifications of the  $2n \times 2n$  identity matrix and transform  $Q(\lambda)$  into a new quadratic  $Q_1(\lambda)$  for which  $\lambda_1$  and  $\lambda_2$  share the same left eigenvector  $z_L$  and same right eigenvector  $z_R$ , that is,

$$z_L^* Q_1(\lambda_j) = 0, \quad Q_1(\lambda_j) z_R = 0, \quad j = 1, 2. \quad (5)$$

In particular we find that  $\lambda_1$  and  $\lambda_2$  must be semisimple and distinct and that, if they are both real, they must also satisfy

$$\text{sign} \left( \frac{x_{L2}^T Q'(\lambda_2) x_{R2}}{x_{L1}^T Q'(\lambda_1) x_{R1}} \right) = \text{sign} \left( \frac{x_{L2}^T Q'(\lambda_1) x_{R1}}{x_{L1}^T Q'(\lambda_2) x_{R2}} \right),$$

which for symmetric quadratics  $Q$  means that  $\lambda_1$  and  $\lambda_2$  must have opposite type [3]. Under these conditions we characterize a family of elementary SPTs that transform  $Q(\lambda)$  with eigentriples  $(\lambda_j, x_{Rj}, x_{Lj})$  to a new quadratic  $Q_1(\lambda)$  with eigentriples  $(\lambda_j, z_R, z_L)$ ,  $j = 1, 2$ . Since our transformations are structure preserving we never work with the  $2n \times 2n$  matrices in (4). Indeed the matrix coefficients of  $Q_1(\lambda)$  turn out to be low rank modifications of  $M, C$  and  $K$  and are therefore not expensive to compute. When (5) holds we then show how to construct two nonsingular matrices  $G_L, G_R$  such that  $G_L^T Q_1(\lambda) G_R = \tilde{Q}(\lambda)$

with  $\tilde{Q}(\lambda)$  block diagonal as in (2), that is, the pair  $(G_L, G_R)$  deflates the two eigenvalues  $\lambda_1, \lambda_2$ .

This paper is organized as follows. After some preliminary results in section 2 on structure preserving transformations, we explain in section 3 how to deflate eigenvalues of symmetric quadratic matrix polynomials. We then extend in the following section the symmetric deflation procedure to quadratics with nonsymmetric coefficient matrices. We present in section 5 some numerical examples that illustrate our deflation procedure. To the best of our knowledge, this work is the first attempt at constructing a family of *nontrivial elementary SPTs that have a specific action of practical use: that of “mapping” two linearly independent eigenvectors to a set of linearly dependent eigenvectors.*

## 2. Structure preserving transformations

In this section we recall some necessary results from [9] and [21]. SPTs, defined in (4), have a number of important and useful properties that we begin by summarizing.

**Lemma 1.** [21] *Let  $(W_L, W_R)$  be an SPT transforming  $Q(\lambda) = \lambda^2 M + \lambda C + K$  with  $M$  nonsingular into  $\tilde{Q}(\lambda) = \lambda^2 \tilde{M} + \lambda \tilde{C} + \tilde{K}$ . Then*

- (i)  $Q(\lambda)$  and  $\tilde{Q}(\lambda)$  share the same eigenvalues.
- (ii)  $\tilde{M}$  is nonsingular.
- (iii) If  $(\lambda, x, y)$  is an eigentriple of  $Q(\lambda)$  then

$$W_R^{-1} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} \lambda \tilde{x} \\ \tilde{x} \end{bmatrix}, \quad W_L^{-1} \begin{bmatrix} \bar{\lambda} y \\ y \end{bmatrix} = \begin{bmatrix} \bar{\lambda} \tilde{y} \\ \tilde{y} \end{bmatrix},$$

for some nonzero  $\tilde{x}, \tilde{y} \in \mathbb{C}^n$  such that  $\tilde{Q}(\lambda)\tilde{x} = 0$  and  $\tilde{y}^* \tilde{Q}(\lambda) = 0$ .

- (iv) If  $L(\lambda)$  belongs to the vector space of pencils [14], [18]

$$\mathbb{DL}(Q) = \left\{ \lambda \begin{bmatrix} v_1 M & v_2 M \\ v_2 M & v_2 C - v_1 K \end{bmatrix} + \begin{bmatrix} v_1 C - v_2 M & v_1 K \\ v_1 K & v_2 K \end{bmatrix} : v \in \mathbb{R}^2 \right\},$$

with vector  $v$  then  $\tilde{L}(\lambda) = W_L^T L(\lambda) W_R \in \mathbb{DL}(\tilde{Q})$  with vector  $v$ . In other words, the SPT  $(W_L, W_R)$  preserves the block structure of  $\mathbb{DL}(Q)$ . Moreover if  $L(\lambda)$  is a linearization of  $Q$  then  $\tilde{L}(\lambda)$  is a linearization of  $\tilde{Q}(\lambda)$ .

- (v) If  $W_L = W_R$  and  $Q(\lambda)$  is symmetric (i.e.,  $M, C$  and  $K$  are symmetric) then  $\tilde{Q}(\lambda)$  is symmetric.

Matrix pairs  $(G_L, G_R)$  of the form

$$G_S = \begin{bmatrix} \tilde{G}_S & 0 \\ 0 & \tilde{G}_S \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad \det(\tilde{G}_S) \neq 0, \quad S = L, R$$

always define an SPT for any  $n \times n$  quadratic  $Q$ . They have the property that if  $(G_L, G_R)$  transforms  $Q(\lambda)$  into  $\tilde{Q}(\lambda)$  then  $\tilde{Q}(\lambda) = \tilde{G}_L^T Q(\lambda) \tilde{G}_R$ . The pair

$(G_L, G_R)$  is called a *class one elementary SPT* when  $\tilde{G}_S = I - m_S n_S^T$  for some nonzero vectors  $m_S, n_S \in \mathbb{R}^n$ ,  $S = L, R$  [9].

The key elementary SPT used in our deflation procedure has the form

$$T_S = \begin{bmatrix} I + a_S b_S^T & a_S d_S^T \\ a_S f_S^T & I + a_S h_S^T \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad (6)$$

where  $a_S, b_S, d_S, f_S, h_S \in \mathbb{R}^n$  with  $a_S, d_S, f_S$  nonzero. The matrix  $T_S$  differs from the identity matrix by a matrix of rank at most two and it is nonsingular if [5], [21]

$$\det(T_S) = (1 + a_S^T b_S)(1 + a_S^T h_S) - (a_S^T d_S)(a_S^T f_S) \neq 0.$$

With the notation

$$\alpha_M := a_L^T M a_R, \quad \alpha_C := a_L^T C a_R, \quad \alpha_K := a_L^T K a_R,$$

a pair  $(T_L, T_R)$  of nonsingular matrices with  $T_S$ ,  $S = L, R$ , as in (6) forms a class two elementary SPT if [9], [21]

$$\alpha_C = a_L^T C a_R \neq 0 \quad (7)$$

and

$$\frac{1}{2} \alpha_C f_L + \alpha_M b_L = -M a_R, \quad (8)$$

$$\alpha_K f_L + \frac{1}{2} \alpha_C (b_L + h_L) + \alpha_M d_L = -C a_R, \quad (9)$$

$$\alpha_K h_L + \frac{1}{2} \alpha_C d_L = -K a_R, \quad (10)$$

$$\frac{1}{2} \alpha_C f_R + \alpha_M b_R = -M^T a_L, \quad (11)$$

$$\alpha_K f_R + \frac{1}{2} \alpha_C (b_R + h_R) + \alpha_M d_R = -C^T a_L, \quad (12)$$

$$\alpha_K h_R + \frac{1}{2} \alpha_C d_R = -K^T a_L. \quad (13)$$

The constraints (8)–(13) force preservation of structure. Multiplying the constraints (8) and (10) on the left by  $a_L^T$  and the constraints (11) and (13) on the left by  $a_R^T$  allows us to rewrite the determinant of  $T_L$  and  $T_R$  as

$$\det(T_S) = \alpha_C^{-2} (1 + a_S^T b_S)(1 + a_S^T h_S)(\alpha_C^2 - 4\alpha_K \alpha_M), \quad S = L, R$$

which shows that

$$\alpha_C^2 - 4\alpha_K \alpha_M \neq 0 \quad (14)$$

is a necessary condition for  $(T_L, T_S)$  to be an SPT.

From (8)–(13) we have that if  $(T_L, T_R)$  transforms  $Q(\lambda)$  to  $\tilde{Q}(\lambda)$  then

$$\begin{aligned}\tilde{K} &= K - \alpha_K h_L h_R^T - \frac{1}{2} \alpha_C (h_L d_R^T + d_L h_R^T) - \alpha_M d_L d_R^T, \\ \tilde{C} &= C - \alpha_K (h_L f_R^T + f_L h_R^T) - \frac{1}{2} \alpha_C (h_L b_R^T + b_L h_R^T + d_L f_R^T + f_L d_R^T) \\ &\quad - \alpha_M (d_L b_R^T + b_L d_R^T), \\ \tilde{M} &= M - \alpha_K f_L f_R^T - \frac{1}{2} \alpha_C (b_L f_R^T + f_L b_R^T) - \alpha_M b_L b_R^T,\end{aligned}$$

which shows that  $\tilde{M}$ ,  $\tilde{C}$ , and  $\tilde{K}$  are low rank modifications of  $M$ ,  $C$ , and  $K$ .

Note that once the two vectors  $a_L$  and  $a_R$  are chosen such that (7) and (14) hold, the structure preserving constraints (8)–(13) are linear in the remaining unknown vectors. They can be rewritten in matrix form as

$$VA = B \quad \iff \quad V_L A = B_R, \quad V_R A = B_L, \quad (15)$$

where  $A \in \mathbb{R}^{4 \times 3}$  and  $B = \begin{bmatrix} B_R \\ B_L \end{bmatrix} \in \mathbb{R}^{2n \times 3}$  are given by

$$A = \begin{bmatrix} \alpha_M & \frac{1}{2} \alpha_C & 0 \\ 0 & \alpha_M & \frac{1}{2} \alpha_C \\ \frac{1}{2} \alpha_C & \alpha_K & 0 \\ 0 & \frac{1}{2} \alpha_C & \alpha_K \end{bmatrix}, \quad B = - \begin{bmatrix} M a_R & C a_R & K a_R \\ M^T a_L & C^T a_L & K^T a_L \end{bmatrix} \quad (16)$$

and  $V = \begin{bmatrix} V_L \\ V_R \end{bmatrix} \in \mathbb{R}^{2n \times 4}$  with  $V_S = [b_S \ d_S \ f_S \ h_S] \in \mathbb{R}^{n \times 4}$  for  $S = L, R$  contains the remaining unknown vectors. Some calculations show that

$$\det(A^T A) = \frac{1}{4} (\alpha_C^2 - 4\alpha_M \alpha_K)^2 (\alpha_C^2 + \alpha_M^2 + \alpha_K^2)$$

which is nonzero by (14), so that  $A$  has full rank and all solutions to (15) are given by

$$V = BA^+ + U(I - AA^+) \quad \iff \quad \begin{cases} V_L = B_R A^+ + U_L (I - AA^+), \\ V_R = B_L A^+ + U_R (I - AA^+), \end{cases}$$

for some arbitrary  $U = \begin{bmatrix} U_L \\ U_R \end{bmatrix} \in \mathbb{R}^{2n \times 4}$ . Here  $A^+$  is the pseudoinverse of  $A$ , which is given by  $A^+ = (A^T A)^{-1} A^T$  since  $A$  has full rank (see Stewart and Sun [20, Sec. 3.1]).

The transformation  $T_S$  used in our deflation procedure performs a specific action: that of mapping a quadratic matrix polynomial with two non parallel eigenvectors associated with a pair of eigenvalues to a quadratic whose eigenvectors associated to that pair of eigenvalues are now parallel. This results in an additional constraint of the form  $z_S^T V_S = w_S^T$  for some given  $z_S$  and  $w_S$  that the solutions  $V_L$  and  $V_R$  of (15) must satisfy. The next result will then be useful.

**Theorem 2.** *Let  $A \in \mathbb{R}^{r \times k}$ ,  $r \geq k$  have full rank,  $B \in \mathbb{R}^{n \times k}$ ,  $w \in \mathbb{R}^r$ , and nonzero  $z \in \mathbb{R}^n$  be given. The problem of finding  $V \in \mathbb{R}^{n \times r}$  such that*

$$VA = B, \quad z^T V = w^T, \quad (17)$$

has a solution if and only if  $w^T A = z^T B$ . In this case the general solution is

$$V = (I - zz^+)BA^+ + U(I - AA^+) + z(z^T z)^{-1}w^T, \quad (18)$$

where  $U \in \mathbb{R}^{n \times r}$  is any matrix such that  $z^T U = 0$ .

**Proof.** If  $V$  is a solution to (17) then  $z^T B = z^T V A = w^T A$ . Conversely, if  $z^T B = w^T A$  then since  $A^+ A = I$  multiplying  $V$  in (18) on the right by  $A$  yields  $V A = B$  and since  $z^T U = 0$  we have that  $z^T V = w^T$  so that  $V$  in (18) is a solution to (17). Now every solution  $V$  to (17) can be rewritten as

$$\begin{aligned} V &= (I - zz^+)VAA^+ - (I - zz^+)VAA^+ + V - zz^+V + zz^+V \\ &= (I - zz^+)VAA^+ + (I - zz^+)V(I - AA^+) + zz^+V \\ &= (I - zz^+)BA^+ + (I - zz^+)V(I - AA^+) + z(z^T z)^{-1}w^T, \end{aligned}$$

which is of the form (18) with  $U := (I - zz^+)V$  satisfying  $z^T U = 0$ .  $\square$

### 3. Deflation for symmetric quadratics

Symmetric quadratics have the property that if  $x$  is a right eigenvector associated with the eigenvalue  $\lambda$  then  $y = \bar{x}$  is the corresponding left eigenvector. So if we use congruence transformations to preserve the symmetry of the quadratic we just need to consider the deflation of eigenpairs rather than eigentriples. We denote by  $(\lambda_1, x_1)$  and  $(\lambda_2, x_2)$  the two eigenpairs to be deflated. First we show that when  $x_1$  and  $x_2$  are parallel there exists an  $n \times n$  congruence transformation which, when applied directly to  $Q$ , deflates  $\lambda_1$  and  $\lambda_2$ . When  $x_1$  and  $x_2$  are linearly independent, we show how to construct a class two elementary SPT that transforms  $Q$  to a new quadratic  $Q_1$  for which  $\lambda_1$  and  $\lambda_2$  share the same eigenvector. In other words, the SPT allows us to transform the original deflation problem into one we know how to handle.

#### 3.1. Linearly dependent eigenvectors

We first treat the case where the eigenvalues  $\lambda_1$  and  $\lambda_2$  have a common eigenvector  $z \in \mathbb{R}^n$ . The next lemma is crucial to proving the existence of a congruence transformation that deflates these two eigenvalues. Some relations in this lemma have already been observed by Chu, Hwang, and Lin [6].

**Lemma 3.** Consider the  $n \times n$  symmetric quadratic  $Q(\lambda) = \lambda^2 M + \lambda C + K$ .

- (i) If  $Q(\lambda_j)z = 0$ ,  $j = 1, 2$  with  $z \in \mathbb{R}^n \setminus \{0\}$  and  $\lambda_1 \neq \lambda_2$  then  $Cz = cMz$  and  $Kz = kMz$  with  $c = -(\lambda_1 + \lambda_2)$  and  $k = \lambda_1\lambda_2$ . Moreover, if  $\lambda_1$  and  $\lambda_2$  are semisimple then  $z^T Mz \neq 0$ .
- (ii) If  $Cz = cMz$  and  $Kz = kMz$  for some nonzero  $z \in \mathbb{R}^n$  and  $c, k \in \mathbb{C}$  then  $Q(\lambda_j)z = 0$ ,  $j = 1, 2$  with  $\lambda_{1,2} = -(c \pm \sqrt{c^2 - 4k})/2$ .



**Proof.** (i) It follows from  $\lambda_j^2 Mz + \lambda_j Cz + Kz = 0$ ,  $j = 1, 2$  that when  $\lambda_1 \neq \lambda_2$ ,  $Cz = -(\lambda_1 + \lambda_2)Mz = cMz$  and then  $Kz = -\lambda_1^2 Mz + \lambda_1(\lambda_1 + \lambda_2)Mz = \lambda_1 \lambda_2 Mz = kMz$ . If  $\lambda_1, \lambda_2$  are semisimple then  $0 \neq z^T Q'(\lambda_j)z$  [1, Theorem. 3.2] and  $z^T Q'(\lambda_j)z = (2\lambda_j + c)z^T Mz$  which imply that  $z^T Mz \neq 0$ . Note that here  $Q'(\lambda)$  is the first derivative of  $Q$  with respect to  $\lambda$ , that is,  $Q'(\lambda) = 2\lambda M + C$ .

(ii) If  $Cz = cMz$  and  $Kz = kMz$  then  $Q(\lambda_j)z = (\lambda_j^2 + \lambda_j c + k)Mz = 0$ ,  $j = 1, 2$ , from which the formula for  $\lambda_{1,2}$  follows.  $\square$

Assume there exists a nonsingular matrix  $G$  such that

$$Ge_n = z, \quad G^T(Mz) = me_n, \quad m = z^T Mz, \quad (19)$$

where  $e_n$  is the last column of the  $n \times n$  identity matrix. Since  $G$  and  $M$  are nonsingular we must have  $m \neq 0$ , or equivalently,  $z^T Mz \neq 0$  which by Lemma 3(i) holds when  $\lambda_1$  and  $\lambda_2$  are distinct and semisimple. Then we have that

$$G^T M G e_n = G^T M z = m e_n.$$

Now if  $\lambda_1$  and  $\lambda_2$  are distinct then by Lemma 3(i),  $Cz = cMz$  and  $Kz = kMz$ , so that

$$G^T(\lambda^2 M + \lambda C + K)G = \lambda^2 \begin{bmatrix} \tilde{M} & 0 \\ 0 & m \end{bmatrix} + \lambda \begin{bmatrix} \tilde{C} & 0 \\ 0 & mc \end{bmatrix} + \begin{bmatrix} \tilde{K} & 0 \\ 0 & mk \end{bmatrix}, \quad (20)$$

where  $c = -(\lambda_1 + \lambda_2)$  and  $k = \lambda_1 \lambda_2$ ; thus  $G$  deflates the two eigenvalues  $\lambda_1$  and  $\lambda_2$ . Note that if  $\lambda_1 = \lambda_2$  and,  $Cz$  and  $Kz$  are multiples of  $Mz$  then, as long as  $z^T Mz \neq 0$ ,  $G$  in (19) deflates  $\lambda_1$  and  $\lambda_2$  from  $Q$ . It is easily seen from (20) that in this case  $\lambda_1 (= \lambda_2)$  must be a defective eigenvalue with partial multiplicity 2.

We build the matrix  $G$  in two steps. First, we construct a Householder reflector  $H = I - 2vv^T/(v^T v)$  [12] such that

$$H(Mz) = \|Mz\|_2 e_n.$$

Second, we form  $L = I_n + rs^T$ , where  $s^T e_n = 1$  and  $r = \frac{\|Mz\|_2}{m} Hz - e_n$ , so that

$$Le_n = \frac{\|Mz\|_2}{m} Hz, \quad L^T e_n = e_n$$

since  $r^T e_n = \frac{\|Mz\|_2}{m} z^T H e_n - 1 = \frac{z^T Mz}{m} - 1 = 0$ . Hence

$$G = \frac{m}{\|Mz\|_2} HL \quad (21)$$

satisfies (19). It is shown in [10] that taking

$$s = e_n - \frac{1 + \sqrt{1 + r^T r}}{r^T r} r$$

minimizes the condition number  $\kappa(L)$  of  $L$  and that with this choice,

$$\kappa_2(G)^2 = \kappa_2(L)^2 = \frac{\sqrt{1 + \|r\|_2^2} + \|r\|_2}{\sqrt{1 + \|r\|_2^2} - \|r\|_2},$$

which is reasonably small as long as  $\|r\|_2$  is not much larger than 1. Using  $\|Mz\|_2 He_n = Mz$  and the definition of  $r$  we have that

$$\|r\|_2^2 = r^T r = (z^T M^2 z)(z^T z)/(z^T M z)^2 - 1$$

showing that  $\|r\|_2$  does not depend on the norm of  $z$  or  $M$ .

Note that  $G$  in (21) depends on  $2n$  parameters: the Householder vector  $v \in \mathbb{R}^n$  and  $r \in \mathbb{R}^n$  which is consistent with the  $2n$  constraints in (19).

### 3.2. Linearly independent eigenvectors

When  $x_1$  and  $x_2$  are linearly independent there is clearly no nonsingular transformation mapping the full rank matrix  $[x_1 \ x_2]$  to the rank-one matrix  $[e_n \ e_n]$ . The idea in this case is to build an SPT  $T$  that transforms  $Q(\lambda)$  with eigenpairs  $(\lambda_j, x_j)$ ,  $j = 1, 2$  to  $Q_1(\lambda)$  with eigenpairs  $(\lambda_j, z)$ ,  $j = 1, 2$  that can then be deflated using the procedure described in section 3.1. We only consider the case where  $\lambda_1 \neq \lambda_2$ . Indeed when the two eigenvalues are equal and  $x_1$  is not parallel to  $x_2$ ,  $\lambda_1$  and  $\lambda_2$  belong to two distinct Jordan blocks. In this case, the decoupling (20) cannot be achieved.

Since we aim to treat the deflation of real eigenpairs together with that of complex conjugate eigenpairs, we introduce the real matrices  $A \in \mathbb{R}^{2 \times 2}$  and  $X \in \mathbb{R}^{n \times 2}$  defined by

$$A = \begin{cases} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are real,} \\ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} & \text{if } \lambda_1 = \bar{\lambda}_2 = \alpha + i\beta \text{ with } \beta \neq 0, \end{cases} \quad (22)$$

and

$$X = \begin{cases} [x_1 \ x_2] & \text{for real eigenpairs,} \\ [u \ v] & \text{for complex eigenpairs with } x_1 = \bar{x}_2 = u + iv. \end{cases} \quad (23)$$

We want to construct a class two elementary SPT  $T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix}$  with  $a, b, d, f, h \in \mathbb{R}^n$  and a nonzero vector  $z \in \mathbb{R}^n$  such that

$$T^{-1} \begin{bmatrix} XA \\ X \end{bmatrix} = \begin{bmatrix} ze^T A \\ ze^T \end{bmatrix}, \quad (24)$$

where  $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . This constraint means that  $T^{-1} \begin{bmatrix} \lambda_j x_j \\ x_j \end{bmatrix} = \begin{bmatrix} \lambda_j \delta_j z \\ \delta_j z \end{bmatrix}$ , for some nonzero  $\delta_j$ ,  $j = 1, 2$ . Hence if  $T$  transforms  $Q(\lambda)$  to  $Q_1(\lambda)$  then by Lemma 1(iii),  $Q_1(\lambda_j)z = 0$ ,  $j = 1, 2$ . We rewrite (24) in terms of the  $6n$  unknown vectors  $a, b, d, f, h, z$  as

$$ze^T A + (b^T z)ae^T A + (d^T z)ae^T = XA, \quad (25)$$

$$ze^T + (f^T z)ae^T A + (h^T z)ae^T = X. \quad (26)$$

and solve (25)–(26) for  $a, z$  and the scalars  $b^T z, d^T z, f^T z, h^T z$  as follows.

Let nonzero  $p, q \in \mathbb{R}^2$  be such that

$$e^T p = 0, \quad e^T \Lambda p = 1, \quad e^T q = 1, \quad e^T \Lambda q = 0.$$

Since  $\lambda_1 \neq \lambda_2$ , it is easily seen that

$$p = \gamma(\lambda_1 - \lambda_2)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad q = \Lambda p - (\lambda_1 + \lambda_2)p, \quad \Lambda q = -\lambda_1 \lambda_2 p,$$

with  $\gamma = 1$  for real eigenpairs and  $\gamma = i$  for complex eigenpairs. Multiplying (26) on the right by  $p$  yields  $(f^T z)a = Xp$ . Since the columns of  $X$  are linearly independent, we have that  $f^T z \neq 0$ . Now without loss of generality, we normalize  $a$  such that  $a^T a = 1$ . It follows that

$$a = (f^T z)^{-1} Xp, \quad f^T z = \|Xp\|_2 \neq 0. \quad (27)$$

Multiplying (25) on the right by  $p$  yields  $z + (b^T z)a = X\Lambda p$ . If we choose to normalize  $z$  such that  $e_\ell^T z = 1$ , where we let  $\ell$  be such that  $|e_\ell^T a| = \|a\|_\infty$  then

$$b^T z = (e_\ell^T X\Lambda p - 1)/(e_\ell^T a), \quad z = X\Lambda p - (b^T z)a. \quad (28)$$

Multiplying (25)–(26) on the right by  $q$  and on the left by  $e_\ell^T$  gives

$$d^T z = (e_\ell^T X\Lambda q)/(e_\ell^T a), \quad h^T z = (e_\ell^T Xq - 1)/(e_\ell^T a). \quad (29)$$

What is now left is the construction of  $V := [b \ d \ f \ h]$  such that  $z^T V = w^T$ , where  $w^T = [b^T z \ d^T z \ f^T z \ h^T z]$ , and  $VA = B$ , since  $T$  is structure preserving (see section 2), where  $B = -[Ma \ Ca \ Ka]$  and  $A$  is as in (16) with  $\alpha_M = a^T Ma$ ,  $\alpha_C = a^T Ca \neq 0$  and  $\alpha_K = a^T Ka$ . We know from Theorem 2 that a solution  $V$  to  $VA = B$ ,  $z^T V = w^T$  exists if and only if

$$w^T A = z^T B. \quad (30)$$

The next lemma, crucial for the deflation process, provides a necessary and sufficient condition on the eigenpairs  $(\lambda_j, x_j)$ ,  $j = 1, 2$  for (30) to hold.

**Lemma 4.** *The relation  $w^T A = z^T B$  holds if and only if the eigenpairs  $(\lambda_1, x_1)$  and  $(\lambda_2, x_2)$  of  $Q(\lambda)$  satisfy*

$$x_1^T Q'(\lambda_1)x_1 = \epsilon x_2^T Q'(\lambda_2)x_2 \quad (31)$$

with  $\epsilon = -1$  for real eigenpairs and  $\epsilon = 1$  for complex conjugate eigenpairs.

**Proof.** Tedious calculations left to Appendix A show that the row vector  $g^T = w^T A - z^T B$  has the form

$$g^T = \gamma_g (x_1^T Q'(\lambda_1)x_1 - \epsilon x_2^T Q'(\lambda_2)x_2) [1 \ c \ k],$$

where  $\gamma_g$  is a nonzero scalar,  $c = -(\lambda_1 + \lambda_2)$ ,  $k = \lambda_1 \lambda_2$ ,  $\epsilon = -1$  for real eigenpairs and  $\epsilon = 1$  for complex eigenpairs.  $\square$

For real eigenpairs, the condition (31) implies that  $\lambda_1$  and  $\lambda_2$  must have opposite type, the type of a real eigenvalue  $\lambda$  of  $Q(\lambda)$  with associated eigenvector  $x$  being the sign of  $x^T Q'(\lambda)x = 2\lambda x^T Mx + x^T Cx$ . Note that this is to be expected from the theory of Hermitian matrix polynomials since for a symmetric quadratic with  $2r$  distinct real eigenvalues,  $r$  of them are of positive type and  $r$  of them are of negative type (see [11]). Hence when deflating two real eigenpairs, one must be of positive type and the other of negative type. Now under this condition, (31) is achieved with the scaling

$$x_1 \leftarrow x_1 / \sqrt{|x_1^T Q'(\lambda_1)x_1|}, \quad x_2 \leftarrow x_2 / \sqrt{|x_2^T Q'(\lambda_2)x_2|}$$

as long as both  $\lambda_1$  and  $\lambda_2$  are semisimple, so that  $x_j^T Q'(\lambda_j)x_j \neq 0$ ,  $j = 1, 2$ .

For complex conjugate eigenpairs, (31) is achieved with the scaling

$$x_1 \leftarrow x_1 / \sqrt{|x_1^T Q'(\lambda_1)x_1|}, \quad x_2 = \bar{x}_1$$

if  $x_1^T Q'(\lambda_1)x_1 \neq 0$  and no scaling otherwise. (Note here the use of “ $T$ ” rather than “ $*$ ”.)

With the above scaling, Lemma 4 together with Theorem 2 tells us that the equations  $VA = B$  and  $z^T V = w^T$  have the solutions

$$V = \left( I - \frac{zz^T}{z^T z} \right) BA^+ + U(I - AA^+) + \frac{z}{z^T z} w^T, \quad (32)$$

where  $U \in \mathbb{R}^{n \times 4}$  is any matrix such that  $z^T U = 0$ . It follows that (27)–(29) and (32) define a family of class two elementary SPTs  $T$  transforming  $Q(\lambda)$  with eigenpairs  $(\lambda_j, x_j)$  to  $Q_1(\lambda)$  with eigenpairs  $(\lambda_j, z)$ ,  $j = 1, 2$ . Identifying which solution minimizes the condition number  $\kappa_2(T) = \|T\|_2 \|T^{-1}\|_2$  remains an open problem.

#### 4. Deflation for nonsymmetric quadratics

The deflation procedure described in section 3 extends to the case where  $M, C$ , and  $K$  are nonsymmetric. We denote by  $(\lambda_j, x_{Rj}, x_{Lj})$ ,  $j = 1, 2$  the two eigentriples to be deflated from  $Q(\lambda)$  with  $(\lambda_2, x_{R2}, x_{L2}) = (\bar{\lambda}_1, \bar{x}_{R1}, \bar{x}_{L1})$  when  $\text{Im}(\lambda_1) \neq 0$ . In contrast with the symmetric deflation procedure we use equivalence transformations rather than congruence transformations since we do not need to preserve symmetry. Three situations must be considered.

##### 4.1. Parallel left eigenvectors and parallel right eigenvectors

Without loss of generality let us assume in this case that  $x_{L1} = x_{L2} \equiv z_L$  and  $x_{R1} = x_{R2} \equiv z_R$  with  $z_L, z_R \in \mathbb{R}^n$  so that

$$z_L^T Q_1(\lambda_j) = 0, \quad Q_1(\lambda_j) z_R = 0, \quad j = 1, 2. \quad (33)$$

As in Lemma 3 it is easily shown that if (33) holds with  $\lambda_1 \neq \lambda_2$  then

$$C_1 z_R = c M_1 z_R, \quad K_1 z_R = k M_1 z_R, \quad (34)$$

$$z_L^T C_1 = c z_L^T M_1, \quad z_L^T K_1 = k z_L^T M_1, \quad (35)$$

where  $c = -(\lambda_1 + \lambda_2)$  and  $k = \lambda_1 \lambda_2$ . Moreover if  $\lambda_1$  and  $\lambda_2$  are semisimple then  $z_L^T M z_R \neq 0$ . Suppose there exist nonsingular matrices  $G_L$  and  $G_R$  such that

$$G_L^T M z_R = m e_n, \quad G_L e_n = z_L, \quad (36)$$

$$G_R^T M^T z_L = m e_n, \quad G_R e_n = z_R, \quad (37)$$

where  $m = z_L^T M z_R$ . (Note that the left (right) transformation  $G_L$  ( $G_R$ ) depends on the right (left) eigenvector.) Since  $M$ ,  $G_L$ , and  $G_R$  are nonsingular we must have  $m \neq 0$  which is guaranteed when  $\lambda_1$  and  $\lambda_2$  are distinct and semisimple. With  $G_L$  and  $G_R$  satisfying (36) and (37) we have

$$G_L^T M G_R e_n = G_L^T M z_R = m e_n, \quad e_n^T G_L^T M G_R = z_L^T M G_R = m e_n^T$$

and on using (34)–(37) it follows that

$$G_L^T (M, C, K) G_R = \left( \begin{bmatrix} \tilde{M} & 0 \\ 0 & m \end{bmatrix}, \begin{bmatrix} \tilde{C} & 0 \\ 0 & mc \end{bmatrix}, \begin{bmatrix} \tilde{K} & 0 \\ 0 & mk \end{bmatrix} \right). \quad (38)$$

If we let  $u_L = M z_R$  and  $u_R = M^T z_L$ , the matrices  $G_L$  and  $G_R$  can be taken in the form

$$G_S = \frac{m}{\|u_S\|_2} H_S L_S, \quad S = L, R,$$

where  $H_S$  is a Householder reflector such that  $H_S u_S = \|u_S\|_2 e_n$  and  $L_S = I_n - r_S s_S^T$  with

$$r_S = \frac{\|u_S\|_2}{m} H_S z_S - e_n, \quad s_S = e_n - \frac{1 + \sqrt{1 + r_S^T r_S}}{r_S^T r_S} r_S$$

so that

$$L_S e_n = \frac{\|u_S\|_2}{m} H_S z_S, \quad L_S^T e_n = e_n.$$

Then it is easy to check that the pair  $(G_L, G_R)$  satisfies (34) and (35) and therefore deflates  $\lambda_1$  and  $\lambda_2$  from  $Q$ .

#### 4.2. Non parallel left eigenvectors and non parallel right eigenvectors

As for the symmetric case our aim is to build a class two elementary SPT  $(T_L, T_R)$ , with  $T_L$  not necessarily equal to  $T_R$ , that transforms  $Q(\lambda)$  to a new quadratic  $Q_1(\lambda)$  for which  $\lambda_1$  and  $\lambda_2$  share the same left eigenvector  $z_L$  and the same right eigenvector  $z_R$ . In order to apply the deflation process of section 4.1, we assume that  $\lambda_1$  and  $\lambda_2$  are semisimple and distinct. When  $\lambda_1 = \lambda_2$  with linearly independent eigenvectors then  $\lambda_1$  and  $\lambda_2$  belong to two distinct Jordan blocks and the decoupling (38) cannot be achieved.

Let  $T_S$  be such that

$$T_S^{-1} \begin{bmatrix} X_S A_S \\ X_S \end{bmatrix} = \begin{bmatrix} z_S e^T A_S \\ z_S e^T \end{bmatrix}, \quad (39)$$

with  $A_L = \Lambda^T$  and  $A_R = \Lambda$  where  $\Lambda$ ,  $X_L$  and  $X_R$  are formed as in (22) and (23), and  $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . If the pair  $(T_L, T_R)$  is structure preserving and transforms  $Q(\lambda)$  to  $Q_1(\lambda)$  then the constraint (39) for  $S = L$  and  $S = R$  together with Lemma 1(iv) implies that  $z_L^T Q_1(\lambda_j) = 0$  and  $Q_1(\lambda_j) z_R = 0$ ,  $j = 1, 2$ .

Now if we choose  $T_S$  to have the form (6) then with the following normalizations of  $a_S$  and  $z_S$ ,

$$a_S^T a_S = 1, \quad e_{\ell_S}^T z_S = 1, \quad |e_{\ell_S}^T a_S| = \|a_S\|_\infty, \quad (40)$$

we obtain in a similar way to the symmetric case described in section 3.2, that under the constraint (39),

$$\begin{aligned} f_S^T z_S &= \|X_S p_S\|_2 \neq 0, & a_S &= (f_S^T z_S)^{-1} X_S p_S, \\ b_S^T z_S &= (e_{\ell_S}^T X_S A_S p_S - 1)/(e_{\ell_S}^T a_S), & z_S &= X_S A_S p_S - (b_S^T z_S) a_S, \\ d_S^T z_S &= (e_{\ell_S}^T X_S A_S q_S)/(e_{\ell_S}^T a_S), & h_S^T z_S &= (e_{\ell_S}^T X_S q_S - 1)/(e_{\ell_S}^T a_S), \end{aligned} \quad (41)$$

where  $p_S, q_S \in \mathbb{R}^2$  are such that

$$e^T p_S = 0, \quad e^T A_S p_S = 1, \quad e^T q_S = 1, \quad e^T A_S q_S = 0.$$

Assuming that  $a_L^T C a_R \neq 0$ , the class two elementary SPT  $(T_L, T_R)$  is completely determined if we can find two matrices  $V_L, V_R \in \mathbb{R}^{n \times 4}$  of the form  $\begin{bmatrix} b_S & d_S & f_S & h_S \end{bmatrix}$  with  $S = L, R$  such that

$$V_L A = B_R, \quad z_L^T V_L = w_L^T, \quad (42)$$

$$V_R A = B_L, \quad z_R^T V_R = w_R^T, \quad (43)$$

where  $A \in \mathbb{R}^{4 \times 3}$  and  $B \in \mathbb{R}^{2n \times 3}$  are as in (16) and  $w_S^T = [b_S^T z_S \quad d_S^T z_S \quad f_S^T z_S \quad h_S^T z_S]$ ,  $S = L, R$ . From Theorem 2, a solution  $V_L$  to (42) and a solution  $V_R$  to (43) exist if and only if  $w_L^T A = Z_L^T B_R$  and  $w_R^T A = Z_R^T B_L$ .

**Lemma 5.** *The relations*

$$w_L^T A - Z_L^T B_R = 0, \quad w_R^T A - Z_R^T B_L = 0$$

hold if and only if the eigentriples  $(\lambda_1, x_{R1}, x_{L1})$  and  $(\lambda_2, x_{R2}, x_{L2})$  of  $Q(\lambda)$  satisfy

$$x_{L1}^T Q'(\lambda_1) x_{R1} = \epsilon x_{L2}^T Q'(\lambda_2) x_{R2}, \quad x_{L1}^T Q'(\lambda_2) x_{R2} = -x_{L2}^T Q'(\lambda_1) x_{R1} \quad (44)$$

with  $\epsilon = -1$  for real eigentriples and  $\epsilon = 1$  for complex conjugate eigentriples.

**Proof.** Let  $g_L^T = w_L^T A - Z_L^T B_R$  and  $g_R^T = w_R^T A - Z_R^T B_L$ . Calculations along the same lines as those presented in Appendix A for the symmetric case show that for real eigentriples,

$$\begin{aligned} g_L^T &= \gamma_L (\xi_1 + \xi_2 - \xi_3 - \xi_4) [1 \quad c \quad k], \\ g_R^T &= \gamma_R (\xi_1 + \xi_2 - \xi_5 - \xi_6) [1 \quad c \quad k], \end{aligned}$$

where  $\gamma_L$  and  $\gamma_R$  are nonzero scalars,  $c = -(\lambda_1 + \lambda_2)$ ,  $k = \lambda_1 \lambda_2$  and

$$\begin{aligned} \xi_1 &= x_{L1}^T Q'(\lambda_1) x_{R1}, & \xi_3 &= x_{L1}^T Q'(\lambda_1) x_{R2}, & \xi_5 &= x_{L1}^T Q'(\lambda_2) x_{R2}, \\ \xi_2 &= x_{L2}^T Q'(\lambda_2) x_{R2}, & \xi_4 &= x_{L2}^T Q'(\lambda_2) x_{R1}, & \xi_6 &= x_{L2}^T Q'(\lambda_1) x_{R1}. \end{aligned} \quad (45)$$

From  $x_{L1}^T Q(\lambda_j) x_{R2} = 0$ ,  $j = 1, 2$  we find that  $x_{L1}^T C x_{R2} = -(\lambda_1 + \lambda_2) x_{L1}^T M x_{R2}$ , from which it follows that  $x_{L1}^T Q'(\lambda_1) x_{R2} = -x_{L1}^T Q'(\lambda_2) x_{R2}$ , that is,  $\xi_3 = -\xi_5$ . In an analogous way we find that  $x_{L2}^T Q'(\lambda_1) x_{R1} = -x_{L2}^T Q'(\lambda_2) x_{R1}$ , that is,  $\xi_4 = -\xi_6$ . Hence,  $g_L = g_R = 0$  if and only if  $\xi_1 + \xi_2 = 0$  and  $\xi_5 + \xi_6 = 0$ .

For complex conjugate eigentriples, we find that

$$\begin{aligned} g_L^T &= \tilde{\gamma}_L (i\xi_7 + i\xi_8 + \xi_5 + \xi_6) [1 \quad c \quad k], \\ g_R^T &= \tilde{\gamma}_R (i\xi_1 + i\xi_2 + \xi_5 + \xi_6) [1 \quad c \quad k], \end{aligned}$$

where  $\tilde{\gamma}_L$  and  $\tilde{\gamma}_R$  are nonzero complex scalars,  $\xi_j$ ,  $j = 1, 2, 5, 6$  are defined in (45) and  $\xi_7 = x_{L1}^T Q'(\lambda_2) x_{R1}$ ,  $\xi_8 = x_{L2}^T Q'(\lambda_1) x_{R2}$ . Using  $x_{L1}^* Q(\lambda_j) x_{R2} = 0$ ,  $j = 1, 2$  it is easily shown that  $x_{L1}^* Q'(\lambda_1) x_{R2} = -x_{L1}^* Q'(\lambda_2) x_{R2}$  which, by taking the conjugate, becomes  $\xi_7 = -\xi_1$ . We show similarly that  $\xi_8 = -\xi_2$ . Hence,  $g_L = g_R = 0$  if and only if  $\xi_1 - \xi_2 = 0$  and  $\xi_5 + \xi_6 = 0$  which completes the proof.  $\square$

The assumption that  $\lambda_1$  and  $\lambda_2$  are semisimple implies that in (44) the terms on the left-hand side relation for real eigentriples and the terms on the right-hand side relation for complex conjugate eigentriples are nonzero. If  $x_{Lj}^T Q'(\lambda_j) x_{Rj} = 0$  or  $x_{Lj}^T Q'(\lambda_k) x_{Rk} = 0$ ,  $j \neq k$ , then a scaling similar to that described after Lemma 4 can be applied to ensure that (44) holds. When both  $x_{L1}^T Q'(\lambda_1) x_{R1}$  and  $x_{L1}^T Q'(\lambda_2) x_{R2}$  are nonzero, we let

$$\rho_1 = \frac{x_{L2}^T Q'(\lambda_2) x_{R2}}{x_{L1}^T Q'(\lambda_1) x_{R1}}, \quad \rho_2 = \frac{x_{L2}^T Q'(\lambda_1) x_{R1}}{x_{L1}^T Q'(\lambda_2) x_{R2}}.$$

Then for real eigentriples, the relations (44) hold after an appropriate scaling of the eigenvectors only if  $\text{sign}(\rho_1) = \text{sign}(\rho_2)$ , in which case we can apply the scaling

$$\begin{aligned} x_{L1} &\leftarrow |\rho_1|^{1/2} x_{L1}, & x_{R1} &\leftarrow |\rho_1|^{1/2} x_{R1}, \\ x_{L2} &\leftarrow |\rho_2|^{-1/2} x_{L2}, & x_{R2} &\leftarrow |\rho_2|^{1/2} x_{R2}. \end{aligned} \quad (46)$$

For complex eigentriples,  $(\lambda_1, x_{R1}, x_{L1}) = \overline{(\lambda_2, x_{R2}, x_{L2})} = (\lambda, x, y)$  and (44) holds when  $x$  and  $y$  are scaled such that  $y^T Q'(\lambda) x$  is real and  $y^* Q'(\lambda) x$  is purely imaginary.

When (44) holds, Lemma 5 and Theorem 2 tell us that the set of solutions to (42) and (43) is given by

$$\begin{aligned} V_L &= \left( I - \frac{z_L z_L^T}{z_L^T z_L} \right) B_R A^+ + U_L (I - A A^+) + \frac{z_L}{z_L^T z_L} w_L^T, \\ V_R &= \left( I - \frac{z_R z_R^T}{z_R^T z_R} \right) B_L A^+ + U_R (I - A A^+) + \frac{z_R}{z_R^T z_R} w_R^T, \end{aligned}$$

where  $U_L, U_R \in \mathbb{R}^{n \times m}$  are any matrices such that  $z_S^T U_S = 0$ ,  $S = L, R$ .

The matrices  $V_L$  and  $V_R$  together with  $a_L$  and  $a_R$  in (41) define an SPT  $(T_L, T_R)$  that transforms  $Q(\lambda)$  into  $Q_1(\lambda)$  such that (33) holds.

#### 4.3. Non parallel left (right) eigenvectors and parallel right (left) eigenvectors

When for example  $\text{rank}([x_{L1}, x_{L2}]) = 1$  and  $\text{rank}([x_{R1}, x_{R2}]) = 2$  we might want to look for an SPT of the form  $(I_{2n}, T_R)$  with  $T_R$  a class two elementary SPT, since the left eigenvectors are already parallel to each other. Unfortunately, the pair  $(I_{2n}, T_R)$  is not structure preserving. However we can still use the procedure described in section 4.2 to map  $(\lambda_j, x_{Rj}, x_{Lj})$  to  $(\lambda_j, z_R, z_L)$ ,  $j = 1, 2$  as long as we make sure that after the scaling (46), the vector  $X_{LpL}$  is nonzero so that  $a_L$  in (41) is defined. If  $X_{LpL} = 0$  then we replace  $x_{L1}$  by  $\mu x_{L1}$  and  $x_{R1}$  by  $\mu x_{R1}$ , where  $\mu = -1$  for real eigentriples and  $\mu = i$  for complex conjugate eigenpairs so that (46) still holds but  $X_{LpL}$  is nonzero.

## 5. Numerical experiments

We now describe some numerical experiments designed to give insight into our deflation procedure. It is not our aim to investigate the numerical stability properties of the procedure. This is a separate issue that will be addressed in a future paper. In all our experiments we take  $U = 0$  in (18). Our computations were done in MATLAB 7.6 (R2008a) for which  $u = 2^{-53} \approx 1.1 \times 10^{-16}$ .

Recall that  $(T_L, T_R)$  defines a class two elementary SPT that maps a quadratic matrix polynomial with two non parallel eigenvectors associated with a pair of eigenvalues to a quadratic whose eigenvectors associated to that pair of eigenvalues are now parallel, and that  $(G_L, G_R)$  defines a deflating transformation. We drop the subscripts  $R$  and  $L$  when the left and right transformations are equal. If  $Q(\lambda)$  is  $n \times n$ , the cost of deflating  $(\lambda_1, \lambda_2)$  is  $O(n^2)$  operations.

*Experiment 1.* Our first example is a  $2 \times 2$  symmetric quadratic  $Q(\lambda) = \lambda^2 M + \lambda C + K$  defined by

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad (47)$$

with  $A(Q) = \{-0.34 \pm 1.84i, 0.14 \pm 0.51i\}$  to two decimal places. Note that  $M^{-1}C$  does not commute with  $M^{-1}K$ , so  $Q(\lambda)$  is not proportionally damped. Therefore the system cannot be decoupled by a  $2 \times 2$  congruence transformation directly applied to  $Q(\lambda)$ .



Table 1: Relative magnitude of the off-diagonal elements of the deflated quadratic  $Q_2(\lambda) = \lambda^2 M_2 + \lambda C_2 + K_2$  in Experiment 2 and condition number of the transformations.

Deflated e'values	off( $M_2$ )	off( $C_2$ )	off( $K_2$ )	$\kappa_2(T_L)$	$\kappa_2(T_R)$	$\kappa_2(G_L)$	$\kappa_2(G_R)$
Real	3.0e-15	1.7e-13	1.6e-13	6.0e5	2.0e2	3.6e1	3.3e0
Complex	2.0e-16	1.4e-14	5.6e-14	1.8e3	4.5e1	1.0	1.1

Table 2: Condition numbers of the SPTs  $T$  and deflating transformations  $G$  for different pairs of eigenvalues for experiment 4.

	$(\lambda_1, \lambda_5)$	$(\lambda_1, \lambda_6)$	$(\lambda_1, \lambda_7)$	$(\lambda_1, \lambda_8)$
$\kappa_2(T)$	4.62e1	1.43e3	4.41e2	7.15e1
$\kappa_2(L)$	2.09e0	6.41e0	1.61e0	4.61e0

Given the pair of complex conjugate eigenvalues  $\lambda_{1,2} = -0.34 \pm 1.84i$  and their associated linearly independent eigenvectors our symmetric deflation procedure decouples  $Q(\lambda)$  into

$$\lambda^2 \begin{bmatrix} 5.6 & 2.0e-16 \\ 2.0e-16 & -1.4e-1 \end{bmatrix} + \lambda \begin{bmatrix} -1.6 & -9.4e-16 \\ -9.4e-16 & -9.3e-2 \end{bmatrix} + \begin{bmatrix} 1.6 & -9.8e-17 \\ -9.8e-17 & -4.8e-1 \end{bmatrix},$$

to two significant digits, with  $\kappa_2(T) = 7.9$  and  $\kappa_2(G) \approx 1$ . Thus we have accomplished (2) to within the working precision.

*Experiment 2.* Our second example is a  $2 \times 2$  quadratic matrix polynomial arising in the study of the dynamic behaviour of a bicycle [19]. The coefficient matrices are nonsymmetric. They can be generated using the NLEVP MATLAB toolbox [4] via `nlevp('bicycle')`. This quadratic has two real eigenvalues,  $\lambda_1 = -0.32$  and  $\lambda_2 \approx -14$  and two complex conjugate eigenvalues  $-0.78 \pm 4.5i$ . Table 1 shows that the left and right transformations corresponding to the deflation of the complex conjugate eigenvalues have a smaller condition number than those used for the deflation of the real eigenvalues. The large condition number of  $T_L$  in the real case affects the size of the off-diagonal elements of the deflated quadratic. Here  $\text{off}(E) = \|E - \text{diag}(E)\|_2 / \|E\|_2$ ,  $E = M_2, C_2, K_2$ .

*Experiment 3.* Our next example is a  $4 \times 4$  hyperbolic symmetric quadratic eigenvalue problem generated as in [13, Sec. 6]. The eigenvalues, real since the quadratic is hyperbolic, are uniformly distributed between 1 and 8. If we order them increasingly then  $\lambda_1, \dots, \lambda_4$  have negative type and  $\lambda_5, \dots, \lambda_8$  have positive type [2, Proof of Thm. 1]. Any pairs  $(\lambda_j, \lambda_k)$  with  $1 \leq j \leq 4$  and  $5 \leq k \leq 8$  can be deflated from the quadratic. Table 2 displays the condition numbers of the SPT  $T$  and deflating transformation  $G$  for different pairings. It shows that the choice of pairings affects the conditioning of the transformations.

*Experiment 4.* We now consider a symmetric quadratic eigenvalue problem coming from a model describing the motion of a beam simply supported at both

Table 3: Scaled residuals and condition numbers of the transformations used in Example 4.

$n$	$\text{res}(M)$	$\text{res}(C)$	$\text{res}(K)$	$\kappa_2(G_{acc})$	$\kappa_2(W)$
8	3.07e-15	4.63e-18	3.90e-16	1.69e1	1.52e1
16	5.52e-15	5.08e-17	3.59e-15	4.47e1	3.79e1
32	1.34e-13	3.15e-16	1.68e-14	9.57e1	7.84e1
64	3.22e-12	6.09e-15	3.56e-14	1.95e2	1.57e2

ends and damped at the midpoint. It can be generated with the NLEVP toolbox via `nlevp('damped_beam', ne1e)`, where `ne1e` is the number of finite elements. It is shown in [15, Thm. A1] that the damped problem  $Q(\lambda) = \lambda^2 M + \lambda C + K$  and the undamped problem  $Q_u(\lambda) = \lambda^2 M + K$  have  $n$  eigenvalues and  $n$  eigenvectors in common: those corresponding to the anti-symmetric modes. Because  $M$  and  $K$  are positive definite, the eigenvalues of  $Q_u(\lambda)$  are pure imaginary; they come in pairs  $(\lambda, \bar{\lambda})$ , each pair sharing the same eigenvector.

We computed the  $n$  eigenpairs corresponding to the antisymmetric modes of  $Q_u(\lambda)$  using the MATLAB function `eig` with the option `'chol'` and deflated all of them from  $Q(\lambda)$  using the procedure described in section 3.1. Let

$$\tilde{Q}(\lambda) = G_{acc}^T Q(\lambda) G_{acc} = \lambda^2 \tilde{M} + \lambda \tilde{C} + \tilde{K}$$

be the deflated quadratic, where  $G_{acc}$  is the matrix which accumulates the product of the  $n/2$  deflating transformations of the form (21) and  $\tilde{M}, \tilde{C}, \tilde{K}$  are block  $2 \times 2$  diagonal with  $(n/2) \times (n/2)$  blocks, the lower block being diagonal. Table 3 displays the scaled residuals  $\text{res}(M)$ ,  $\text{res}(C)$ , and  $\text{res}(K)$ , where

$$\text{res}(E) = \frac{\|G_{acc}^T E G_{acc} - \tilde{E}\|_2}{\|G_{acc}\|_2^2 \|E\|_2 + \|\tilde{E}\|_2},$$

and the 2-norm condition numbers  $\kappa_2(G_{acc})$  for different values of  $n = 2 \times \text{ne1e}$ .

The quadratic of the beam problem can be block diagonalized as (see [15, App. A1])

$$W^T Q(\lambda) W = \left[ \begin{array}{c|c} \lambda^2 M_1 + \lambda D_1 + K_1 & 0 \\ \hline 0 & \lambda^2 M_2 + K_2 \end{array} \right],$$

where  $W$  is orthogonal,  $M_2$  and  $K_2$  are both symmetric positive definite and  $\lambda^2 M_2 + K_2$  contains the anti-symmetric modes. The last column of Table 3 displays the condition number of the transformation  $W$  that block diagonalizes  $\lambda^2 M_2 + K_2$ . As a comparison, we note that  $\kappa_2(G_{acc})$  is not much larger than  $\kappa_2(W)$ .

#### A. Technical results for the proof of Lemma 4

We start by recalling the notation. Let  $(\lambda_1, x_1)$  and  $(\lambda_2, x_2)$  be two eigenpairs of a symmetric quadratic  $Q(\lambda) = \lambda^2 M + \lambda C + K$  such that  $\lambda_1 \neq \lambda_2$ . For real

eigenpairs let  $\Lambda = \text{diag}(\lambda_1, \lambda_2)$  and let  $X = [x_1 \ x_2]$ . For complex conjugate eigenpairs let  $\Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$  and  $X = [u \ v]$ , where  $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta$ ,  $\beta \neq 0$  and  $x_1 = \bar{x}_2 = u + iv$ . Let

$$p = \gamma(\lambda_1 - \lambda_2)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad q = \Lambda p - (\lambda_1 + \lambda_2)p$$

with  $\gamma = 1$  for real eigenpairs and  $\gamma = i$  for complex eigenpairs and let

$$\begin{aligned} f^T z &= \|Xp\|_2 \neq 0, & a &= (f^T z)^{-1} Xp, \\ b^T z &= (e_\ell^T X \Lambda p - 1)/(e_\ell^T a), & z &= X \Lambda p - (b^T z)a, \\ d^T z &= (e_\ell^T X \Lambda q)/(e_\ell^T a), & h^T z &= (e_\ell^T X q - 1)/(e_\ell^T a), \end{aligned}$$

where  $\ell$  is such that  $a_\ell = e_\ell^T a \neq 0$ . Define

$$A = \begin{bmatrix} \alpha_M & \frac{1}{2}\alpha_C & 0 \\ 0 & \alpha_M & \frac{1}{2}\alpha_C \\ \frac{1}{2}\alpha_C & \alpha_K & 0 \\ 0 & \frac{1}{2}\alpha_C & \alpha_K \end{bmatrix}, \quad B = -[Ma \quad Ca \quad Ka], \\ V = [b \quad d \quad f \quad h], \\ w^T = [b^T z \quad d^T z \quad f^T z \quad h^T z],$$

where  $\alpha_M = a^T M a$ ,  $\alpha_C = a^T C a$  and  $\alpha_K = a^T K a$ . The next lemma contains useful relations.

**Lemma 6.** *The following relations hold.*

$$x_1^T C x_2 = c x_1^T M x_2, \quad (48)$$

$$x_1^T K x_2 = k x_1^T M x_2, \quad (49)$$

$$d^T z = -k f^T z, \quad (50)$$

$$h^T z - b^T z = c f^T z, \quad (51)$$

where  $c = -(\lambda_1 + \lambda_2)$  and  $k = \lambda_1 \lambda_2$ . Also for any symmetric matrix  $E$  we have

$$a^T E a = \alpha_E = (f^T z)^{-2} p^T X^T E X p, \quad (52)$$

$$z^T E a = (f^T z)^{-1} p^T \Lambda^T X^T E X p - (b^T z)(f^T z)^{-2} p^T X^T E X p, \quad (53)$$

with

$$p^T X^T E X p = \begin{cases} \mu(x_1^T E x_1 + x_2^T E x_2 - 2x_1^T E x_2) & \text{for real eigenpairs,} \\ \frac{\mu}{4}(ix_1^T E x_1 - ix_2^T E x_2 + 2x_1^T E x_2) & \text{otherwise,} \end{cases} \quad (54)$$

$$p^T \Lambda^T X^T E X p = \begin{cases} \mu(\lambda_1 x_1^T E x_1 + \lambda_2 x_2^T E x_2 + cx_2^T E x_1) & \text{for real eigenpairs,} \\ \frac{\mu}{4}(i\lambda_1 x_1^T E x_1 - i\lambda_2 x_2^T E x_2 - cx_2^T E x_1) & \text{otherwise,} \end{cases} \quad (55)$$

where  $\mu = (\lambda_1 - \lambda_2)^{-2} \neq 0$  is defined since  $\lambda_1 \neq \lambda_2$ .

**Proof.** The relations (48) and (49) follow from  $x_1^T Q(\lambda_1)x_2 = x_2^T Q(\lambda_1)x_1 = 0$  and  $x_1^T Q(\lambda_2)x_2 = 0$ . The relations (50)–(53) follow from the definition of  $p$ ,  $q$ ,  $a$  and  $z$  and (54)–(55) follow from the definition of  $\Lambda$  and  $X$  and  $p$ .  $\square$

With these relations in hand we can now prove the formula for  $g^T = w^T A - z^T B$  in Lemma 4. From the definition of  $A$ ,  $B$ ,  $w$  and  $z$  we find that

$$g = \begin{bmatrix} (b^T z)\alpha_M + \frac{1}{2}(f^T z)\alpha_C + z^T M a \\ \frac{1}{2}(b^T z)\alpha_C + (d^T z)\alpha_M + (f^T z)\alpha_k + \frac{1}{2}(h^T z)\alpha_C + z^T C a \\ \frac{1}{2}(d^T z)\alpha_C + \alpha_K h^T z + z^T K a \end{bmatrix}.$$

Using (52) with  $E = M$  and  $E = C$  and (53) with  $E = M$  we obtain that the first component of  $g$  satisfies

$$2(f^T z)g_1 = p^T X^T C X p + 2p^T \Lambda^T X^T M X p. \quad (56)$$

In a similar way we find that the other components of  $g$  satisfy

$$\begin{aligned} 2(f^T z)g_2 &= cp^T X C X p - 2kp^T X M X p + 2p^T \Lambda^T X^T C X p + 2p^T X K X p, \\ 2(f^T z)g_3 &= -kp^T X^T C X p + 2cp^T X^T K X p + 2p^T \Lambda^T X^T K X p. \end{aligned}$$

Using (54) and (55) with  $E = M$ ,  $C$  and  $K$  and the relations (48)–(51) we find that for real eigenpairs,

$$2(f^T z)g^T = \mu(x_1^T Q'(\lambda_1)x_1 + x_2^T Q'(\lambda_2)x_2) [1 \quad c \quad k]$$

and that for complex conjugate eigenpairs,

$$2(f^T z)g^T = \frac{i}{4}\mu(x_1^T Q'(\lambda_1)x_1 - x_2^T Q'(\lambda_2)x_2) [1 \quad c \quad k].$$

## References

- [1] A. L. Andrew, K. E. Chu, and P. Lancaster. Derivatives of eigenvalues and eigenvectors of matrix functions. *SIAM J. Matrix Anal. Appl.*, 14(4): 903–926, 1993.
- [2] L. Barkwell and P. Lancaster. Overdamped and gyroscopic vibrating systems. *Trans. AME: J. Applied Mechanics*, 59:176–181, 1992.
- [3] L. Barkwell, P. Lancaster, and A. Markus. Gyroscopically stabilized systems: A class of quadratic eigenvalue problems with real spectrum. *Canad. J. Math.*, 44(1):42–53, 1992.
- [4] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, and F. Tisseur. NLEVP: A collection of nonlinear eigenvalue problems, 2008. <http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html>.
- [5] K. W. Brodlie. Corrections to matrices expressed in non-symmetric product form. *J. Inst. Math. Appl.*, 14:141–144, 1974.
- [6] M. T. Chu, T.-M. Hwang, and W.-W. Lin. A novel deflation technique for solving quadratic eigenvalue problems. NCTS Preprints in Mathematics 2005-3-002, National Center for Theoretical Sciences, National Tsing Hua University, Taiwan, 2005.

- [7] M. T. Chu and S.-F. Xu. Spectral decomposition of real symmetric quadratic  $\lambda$ -matrices and its applications. *Math. Comp.*, 78:293–313, 2009.
- [8] S. D. Garvey, M. I. Friswell, and U. Prells. Co-ordinate transformations for second order systems. part I: General transformations. *Journal of Sound and Vibration*, 258(5):885–909, 2002.
- [9] S. D. Garvey, M. I. Friswell, and U. Prells. Co-ordinate transformations for second order systems. part II: Elementary structure-preserving transformations. *Journal of Sound and Vibration*, 258(5):911–930, 2002.
- [10] S. D. Garvey, F. Tisseur, M. I. Friswell, and J. E. T. Penny. Simultaneous tridiagonalization of two symmetric matrices. *Int. J. Numer. Meth. Engng.*, 57(12):1643–1660, 2003.
- [11] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982. ISBN 0-12-287160-X. xiv+409 pp.
- [12] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, Baltimore, MD, USA, third edition, 1996. ISBN 0-8018-5413-X (hardback), 0-8018-5414-8 (paperback). xxvii+694 pp.
- [13] C.-H. Guo, N. J. Higham, and F. Tisseur. Detecting and solving hyperbolic quadratic eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 30(4):1593–1613, 2009.
- [14] N. J. Higham, D. S. Mackey, N. Mackey, and F. Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29(1):143–159, 2006.
- [15] N. J. Higham, D. S. Mackey, F. Tisseur, and S. D. Garvey. Scaling, sensitivity and stability in the numerical solution of quadratic eigenvalue problems. *Internat. J. Numer. Methods Eng.*, 73(3):344–360, 2008.
- [16] V. N. Kublanovskaya, V. B. Mikhailov, and V. B. Khazanov. Eigenvalue problem for an irregular  $\lambda$ -matrix. *Journal of Mathematical Sciences*, 13(2):251–260, 1980.
- [17] P. Lancaster and I. Zaballa. Diagonalizable quadratic eigenvalue problems. *To appear in Mechanical Systems and Signal Processing*, 2008.
- [18] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.
- [19] J. P. Meijaard, J. M. Papadopoulos, A. Ruina, and A. L. Schwab. Linearized dynamics equations for the balance and steer of a bicycle: A benchmark and review. *Proc. Roy. Soc. London Ser. A*, 463(2084):1955–1982, 2007.

- [20] G. W. Stewart and J. Sun. *Matrix Perturbation Theory*. Academic Press, London, 1990. ISBN 0-12-670230-6. xv+365 pp.
- [21] F. Tisseur and N. J. Higham. On structure preserving transformations for quadratic matrix polynomials. MIMS EPrint, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2009. In preparation.
- [22] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. *SIAM Rev.*, 43(2):235–286, 2001.