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TORIC GENERA

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ABSTRACT. Our primary aim is to develop a theory of equivariant genera for stably complex manifolds equipped with compatible actions of a torus T^k. In the case of omnioriented quasitoric manifolds, we present computations that depend only on their defining combinatorial data; these draw inspiration from analogous calculations in toric geometry, which seek to express arithmetic, elliptic, and associated genera of toric varieties in terms only of their fans. Our theory focuses on the universal toric genus Φ, which was introduced independently by Krichever and Löffler in 1974, albeit from radically different viewpoints. In fact Φ is a version of tom Dieck's bundling transformation of 1970, defined on T^k-equivariant complex cobordism classes and taking values in the complex cobordism algebra $\Omega_U^*(BT_+^k)$ of the classifying space. We proceed by combining the analytic, the formal group theoretic, and the homotopical approaches to genera, and refer to the index theoretic approach as a recurring source of insight and motivation. The resultant flexibility allows us to identify several distinct genera within our framework, and to introduce parametrised versions that apply to bundles equipped with a stably complex structure on the tangents along their fibres. In the presence of isolated fixed points, we obtain universal localisation formulae, whose applications include the identification of Krichever's generalised elliptic genus as universal amongst genera that are rigid on SU-manifolds. We follow the traditions of toric geometry by working with a variety of illustrative examples wherever possible. For background and prerequisites we attempt to reconcile the literature of east and west, which developed independently for several decades after the 1960s.

1. INTRODUCTION

The study of equivariant cobordism theory was begun by Conner and Floyd during the early 1960s, in the context of smooth manifolds with periodic diffeomorphisms [30]. They extended their ideas to stably almost complex manifolds soon afterwards [31], and the subject has continued to flourish ever since. Beyond 1965, however, two schools of research emerged, whose interaction was curtailed by difficulties of communication, and whose literature is only now being reconciled.

In Moscow, Novikov [69], [70] brought the theory of formal group laws to bear on the subject, and provided a beautiful formula for invariants of the complex cobordism class of an almost complex $\mathbb{Z}/p$-manifold with isolated fixed points. He applied his formula to deduce relations in the complex cobordism algebra $\Omega_U^*(B\mathbb{Z}/p)$ of the classifying space of $\mathbb{Z}/p$, which he named the *Conner-Floyd relations*. These were developed further by Kasparov [53], Mischenko [67], Buchstaber and Novikov [19], Gusein-Zade and Krichever [46], and Panov [73], [74], and extended to circle actions by Gusein-Zade [44], [45]. Generalisation to arbitrary Lie groups quickly followed, and led to *Krichever's formula* [57, (2.7)] for the universal toric genus

$\Phi(N)$ of an almost complex manifold. Subsequently [59], Krichever applied complex analytic techniques to the study of formal power series in the cobordism algebra $\Omega_U^*(BT^k)$, and deduced important rigidity properties for equivariant generalised elliptic genera.

Related results appeared simultaneously in the west, notably by tom Dieck [37] in 1970, who developed ideas of Boardman [10] and Conner [29] in describing a *bundling transformation* for any Lie group G. These constructions were also inspired by the K-theoretic approach of Atiyah, Bott, Segal, and Singer [3], [5], [6] to the index theory of elliptic operators, and by applications of Atiyah and Hirzebruch [4] to the signature and $\widehat{A}$-genus. Quillen immediately interpreted tom Dieck's work in a more geometrical context [77], and computations of various G-equivariant complex cobordism rings were begun soon afterwards. Significant advances were made by Kosniowski [55], Landweber [60], and Stong [84] for $\mathbb{Z}/p$, and by Kosniowski and Yahia [56] for the circle, amongst many others. In 1974, Löffler [62] focused on the toric case of tom Dieck's transformation, and adapted earlier results of Conner and Floyd on stably complex T^k-manifolds to give a formula for $\Phi(N)$; henceforth, we call this *Löffler's formula*. More recent publications of Comezaña [33] and Sinha [83] included informative summaries of progress during the intervening years, and showed that much remained to be done, even for abelian G. In 2005 Hanke [47] confirmed a conjecture of Sinha for the case T^k, and revisited tom Dieck's constructions. Surprisingly, there appears to be no reference in these works to the Conner-Floyd relations or to Krichever's formula (or even Löffler's formula!) although several results are closely related, and lead to overlapping information.

A third strand of research was stimulated in Japan by the work of Hattori [48] and Kawakubo [54], but was always well integrated with western literature.

One of our aims is to discuss Φ from the perspective of both schools, and to clarify relationships between their language and results. In so doing, we have attempted to combine aspects of their notation and terminology with more recent conventions of equivariant topology. In certain cases we have been guided by the articles in [66], which provide a detailed and coherent overview, and by the choices made in [25], where some of our results are summarised. We have also sought consistency with [24], where the special case of the circle is outlined.

As explained by Comezaña [33], G-equivariant complex bordism and cobordism occur in several forms, which have occasionally been confused in the literature. The first and second forms are overtly geometric, and concern smooth G-manifolds whose equivariant stably complex structures may be normal or tangential; the two possibilities lead to distinct theories, in contrast to the non-equivariant case. The third form is homotopical, and concerns the G-equivariant Thom spectrum MU_G, whose constituent spaces are Thom complexes over Grassmannians of complex representations. Finally, there is the Borel form, defined as the non-equivariant complex bordism and cobordism of an appropriate Borel construction. We describe and distinguish between these forms as necessary below. They are closely inter-connected, and tom Dieck's bundling transformation may be interpreted as a completion procedure on any of the first three, taking values in the fourth.

Following Löffler, and motivated by the emergence of toric topology, we restrict attention to the standard k–dimensional torus T^k, where T is the multiplicative group of unimodular complex numbers, oriented anti-clockwise. We take advantage of tom Dieck's approach to introduce a generalised genus Φ_X, which depends on a compact T^k-manifold X as parameter space. Interesting new examples ensue, that we shall investigate in more detail elsewhere; here we focus mainly on the situation when X is a point, for which Φ_X reduces to Φ. It takes values in the algebra $\Omega_*^U[[u_1, \ldots, u_k]]$ of formal power series over the non-equivariant complex cobordism ring, and may be evaluated on an arbitrary stably complex T^k-manifold N of normal

or tangential form. The coefficients of $\Phi(N)$ are also universal invariants of N, whose representing manifolds were described by Löffler [62]; their origins lie in work of Conner and Floyd [30]. These manifolds continue to attract attention [83], and we express them here as total spaces of bundles over products of the bounded flag manifolds B_n of [81] and [22]. The B_n are stably complex boundaries, and our first contribution is to explain the consequences of this fact for the rigidity properties of equivariant genera.

When N has isolated fixed points x, standard localisation techniques allow us to rewrite the coefficients of $\Phi(N)$ as functions of the *signs* $\varsigma(x)$ and *weights* $w_j(x)$, for $1 \leqslant j \leqslant k$. Our second contribution is to extend Krichever's formula to this context; his original version made no mention of the $\varsigma(x)$, and holds only for almost complex N (in which case all signs are necessarily $+1$). By focusing on the constant term of $\Phi(N)$, we also obtain expressions for the non-equivariant complex cobordism class $[N]$ in terms of local data. This method has already been successfully applied to evaluate the Chern numbers of complex homogeneous spaces in an arbitrary complex-oriented homology theory [26].

Ultimately, we specialise to omnioriented quasitoric manifolds M^{2n}, which admit a T^n-action with isolated fixed points and a compatible tangential stably complex structure. Any such M is determined by its *combinatorial data* [21], consisting of the face poset of a simple polytope P and an integral dicharacteristic matrix Λ. Our third contribution is to express the signs and weights of each fixed point in terms of (P, Λ), and so reduce the computation of $\Phi(M)$ to a purely combinatorial affair. By way of illustration, we discuss families of examples whose associated polyhedra are simplices or hypercubes; these were not available in combinatorial form before the advent of toric topology.

Philosophically, four approaches to genera have emerged since Hirzebruch's original formulation, and we hope that our work offers some pointers towards their unification. The analytic and function theoretic viewpoint has prospered in Russian literature, whereas the homotopical approach has featured mainly in the west; the machinery of formal group laws, however, has been well-oiled by both schools, and the rôle of index theory has been enhanced by many authors since it was originally suggested by Gelfand [40]. Ironically, the dichotomy between the first and second is illustrated by comparing computations of the Chern-Dold character [15] with applications of the Boardman homomorphism and Hurewicz genus [79]. In essence, these bear witness to the remarkable influence of Novikov and Adams respectively.

Our work impinges least on the index-theoretic approach, to which we appeal mainly for motivation and completeness. Nevertheless, in Section 3 we find it instructive to translate Krichever's description of classical rigidity into the universal framework. The relationship between indices and genera was first established by the well-known Hirzebruch-Riemann-Roch theorem [49], which expresses the index of the twisted Dolbeault complex on a complex manifold M as a characteristic number $\langle ch\, \xi\, td\,(M), \sigma_M^H \rangle$, where ξ is the twisting bundle; in particular, the untwisted case yields the Todd genus. The cohomological form [7] of the Atiyah-Singer index theorem [6] leads to explicit descriptions of several other Hirzebruch genera as indices. For example, the $\widehat{A}$-genus is the index of the Dirac operator on Spin manifolds, and the signature is the index of the signature operator on oriented manifolds.

Before we begin it is convenient to establish the following notation and conventions, of which we make regular use.

All our topological spaces are compactly generated and weakly Hausdorff [86], and underlie the model category of T^k-spaces and T^k-maps. It is often important to take account of basepoints, in which case we insist that they be fixed by T^k; for example, X_+ denotes the union of a T^k-space X and a disjoint fixed point $*$.

For any $n \geqslant 0$, we refer to the action of T^{n+1} by coordinatewise multiplication on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ as the *standard action*, and its restriction to the diagonal circle $T_\delta < T^n$ as the *diagonal action*. The orbit space S^{2n+1}/T_δ is therefore $\mathbb{C}P^n$; it is a toric variety with respect to the action of the quotient n-torus T^{n+1}/T_δ, whose fan is normal to the standard n-simplex $\Delta^n \subset \mathbb{R}^n$. The k-fold product

$$(1.1) \qquad \Pi S(q) := \prod^{k} S^{2q+1} \subset \mathbb{C}^{k(q+1)}$$

of $(2q+1)$-spheres admits a diagonal action of T^k, whose orbit space $\Pi P(q)$ is the corresponding k-fold product of $\mathbb{C}P^q$s. The induced action of the quotient kq-torus $T^{k(q+1)}/T^k$ turns $\Pi P(q)$ into a product of toric varieties, whose fan is normal to the k-fold product of simplices $\Pi \Delta^q \subset \mathbb{R}^{kq}$.

Given a free T^k-space X and an arbitrary T^k-space Y, we write the quotient of $X \times Y$ by the T^k-action

$$t \cdot (x, y) = (t^{-1}x, ty)$$

as $X \times_{T^k} Y$; it is the total space of a bundle over X/T^k, with fibre Y. If X is the universal contractible T^k-space ET^k, then $ET^k \times_{T^k} Y$ is the *Borel construction* on Y, otherwise known as the *homotopy quotient* of the action. The corresponding bundle

$$Y \longrightarrow ET^k \times_{T^k} Y \longrightarrow BT^k$$

therefore lies over the classifying space BT^k.

We let V denote a generic $|V|$-dimensional unitary representation space for T^k, which may also be interpreted as a $|V|$-dimensional T^k-bundle over a point. Its unit sphere $S(V) \subset V$ and one-point compactification $V \subset S^V$ both inherit canonical T^k-actions. We may also write V for the product T^k-bundle $X \times V \to X$ over an arbitrary T^k-space X, under the diagonal action on the total space.

For any commutative ring spectrum E, we adopt the convention that the homology and cohomology groups $E_*(X)$ and $E^*(X)$ are *reduced* for all spaces X. So their unreduced counterparts are given by $E_*(X_+)$ and $E^*(X_+)$. The *coefficient ring* E_* is the homotopy ring $\pi_*(E)$, and is therefore given by $E_*(S^0)$ or $E^{-*}(S^0)$; we identify the homological and cohomological versions without further comment, and interpret $E_*(X_+)$ and $E^*(X_+)$ as E_*-modules and E_*-algebras respectively. The corresponding conventions apply equally well to the equivariant cobordism spectra MU_{T^k}, and the T^k-spaces X with which we deal.

Our ring spectra are also *complex oriented*, by means of a class x^E in $E^2(\mathbb{C}P^\infty)$ that restricts to a generator of $E^2(\mathbb{C}P^1)$. It follows that $E^*(\mathbb{C}P^\infty_+)$ is isomorphic to $E_*[[x^E]]$ as E_*-algebras, and that E-theory Chern classes exist for complex vector bundles; in particular, x^E is the first Chern class $c_1^E(\zeta)$ of the *conjugate* Hopf line bundle $\zeta := \bar\eta$ over $\mathbb{C}P^\infty$. The universal example is the complex cobordism spectrum MU, whose homotopy ring is the complex cobordism ring Ω_*^U. If we identify the Thom space of ζ with $\mathbb{C}P^\infty$, then x^E may be interpreted as a Thom class $t^E(\zeta)$, and extended to a universal Thom class $t^E \colon MU \to E$.

Such spectra E may be *doubly oriented* by choosing distinct orientations x_1 and x_2. These are necessarily linked by mutually inverse power series

$$(1.2) \qquad x_2 = \sum_{j \geqslant 0} b_j^E x_1^{j+1} \qquad \text{and} \qquad x_1 = \sum_{j \geqslant 0} m_j^E x_2^{j+1}$$

in $E^2(\mathbb{C}P^\infty)$, where b_j^E and m_j^E lie in E_{2j} for $j \geqslant 1$; in particular, $b_0^E = m_0^E = 1$. We abbreviate (1.2) whenever possible, by writing the series as $b^E(x_1)$ and $m^E(x_2)$ respectively; thus $b^E(m^E(x_2)) = x_2$ and $m^E(b^E(x_1)) = x_1$. It follows that the Thom classes $t_1(\zeta)$ and $t_2(\zeta)$ differ by a unit in $E^0(\mathbb{C}P^\infty_+)$, which is determined by

(1.2). More precisely, the equations

$$t_2(\zeta) \;=\; t_1(\zeta) \cdot b_+^E(x_1) \qquad \text{and} \qquad t_1(\zeta) \;=\; t_2(\zeta) \cdot m_+^E(x_2)$$

hold in $E^2(\mathbb{C}P_+^\infty)$, where $b_+^E(x_1) = b^E(x_1)/x_1$ and $m_+^E(x_2) = m^E(x_2)/x_2$.

A smash product $E \wedge F$ of complex oriented spectra is doubly oriented by $x^E \wedge 1$ and $1 \wedge x^F$ in $(E \wedge F)^2(\mathbb{C}P^\infty)$; we abbreviate these to x^E and x^F respectively.

Example 1.3. The motivating example is that of $E = H$, the integral Eilenberg-Mac Lane spectrum, and $F = MU$. Then $b_j := b_j^{H \wedge MU}$ is the standard indecomposable generator of $H_{2j}(MU)$, and t^H is the Thom class in $H^0(MU)$. The inclusion $h^{MU} \colon MU \to H \wedge MU$ may also be interpreted as a Thom class, and induces the Hurewicz homomorphism $h_*^{MU} \colon \Omega_*^U \to H_*(MU)$.

We refer readers to Adams [1] for further details, and to [22] for applications of the universal doubly oriented example $MU \wedge MU$.

Several colleagues have provided valuable input and assistance during the preparation of this work, and it is a pleasure to acknowledge their contribution. Particular thanks are due to Andrew Baker, who drew our attention to the relevance of the work of Dold [39], and to Burt Totaro, whose comments on an earlier version led directly to Theorem 3.24.

2. The universal toric genus

In this section we introduce tom Dieck's bundling transformation α in the case of the torus T^k, and explain how to convert it into our genus Φ. We discuss the geometric and homotopical forms of T^k-equivariant complex cobordism theory on which Φ is defined, and the Borel form which constitutes its target. In particular, we revisit formulae of Löffler and Krichever for its evaluation. We refer readers to the papers of Comezaña [33], Comezaña and May [34], Costenoble [35], and Greenlees and May [42] for a comprehensive introduction to equivariant bordism and cobordism theory.

We begin by recalling the Thom T^k-spectrum MU_{T^k}, whose spaces are indexed by the complex representations of T^k. Each $MU_{T^k}(V)$ is the Thom T^k-space of the universal $|V|$-dimensional complex T^k-vector bundle $\gamma_{|V|}$, and each spectrum map $S^{W \ominus V} \wedge MU_{T^k}(V) \to MU_{T^k}(W)$ is induced by the inclusion $V < W$ of a T^k-submodule. The *homotopical form* $MU_{T^k}^*(X_+)$ of equivariant complex cobordism theory is defined for any T^k-space X by stabilising the pointed T^k-homotopy sets $[S^V \wedge X_+, MU_{T^k}(W)]_{T^k}$ as usual.

Applying the Borel construction to $\gamma_{|V|}$ yields a complex $|V|$-plane bundle, whose Thom space is homeomorphic to $ET_+^k \wedge_{T^k} MU_{T^k}(V)$ for every complex representation V. The resulting classifying maps induce maps of pointed homotopy sets $[S^V \wedge X_+, MU_{T^k}(W)]_{T^k} \to [(ET_+^k \wedge S^V \wedge X_+)/T^k, MU(|W|)]$, whose target stabilises to the non-equivariant cobordism algebra $\Omega_U^{*+2|V|}((S^V \wedge ET_+^k \wedge X_+)/T^k)$. Applying the Thom isomorphism yields the transformation

$$\alpha \colon MU_{T^k}^*(X_+) \longrightarrow \Omega_U^*((ET^k \times_{T^k} X)_+),$$

which is multiplicative and preserves Thom classes [37, Proposition 1.2].

In the language of [34], we may summarise the construction of α using the homomorphism $MU_{T^k}^*(X_+) \to MU_{T^k}^*((ET^k \times X)_+)$ induced by the T^k- projection $ET^k \times X \to X$; since MU_{T^k} is *split* and T^k acts freely on $ET^k \times X$, the target may be replaced by $\Omega_U^*((ET^k \times_{T^k} X)_+)$. Moreover, α is an isomorphism whenever X is compact and T^k acts freely.

Löffler's *completion theorem* [66] states that α is precisely the homomorphism of completion with respect to the augmentation ideal in $MU_{T^k}^*(X_+)$.

Our primary purpose is to compute α on geometrically defined cobordism classes, so it is natural to seek an equivariant version of Quillen's formulation [77] of the

complex cobordism functor. However, his approach relies on normal complex structures, whereas many of our examples present themselves most readily in terms of tangential information. In the non-equivariant situation, the two forms of data are, of course, interchangeable; but the same does not hold equivariantly. This fact was often ignored in early literature, and appears only to have been made explicit in 1995, by Comezaña [33, §3]. As we shall see below, tangential structures may be converted to normal, but the procedure is not reversible.

Over any smooth compact manifold X, we formulate our constructions in terms of smooth bundles $E \xrightarrow{\pi} X$ with compact d-dimensional fibre F, following Dold [39].

Definitions 2.1. The bundle π is *stably tangentially complex* when the bundle $\tau_F(E)$ of tangents along the fibre is equipped with a stably complex structure $c_\tau(\pi)$. Two such bundles π and π' are *equivalent* when $c_\tau(\pi)$ and $c_\tau(\pi')$ are homotopic upon the addition of appropriate trivial summands; and their equivalence classes are *cobordant* when there exists a third bundle ρ, whose boundary $\partial L \xrightarrow{\rho} X$ may be identified with $E_1 \sqcup E_2 \to X$ in the standard fashion.

If π is T^k-equivariant, then it is stably tangentially complex *as a T^k-bundle* when $c_\tau(\pi)$ is also T^k-equivariant. The notions of *equivariant equivalence* and *equivariant cobordism* apply to such bundles accordingly.

We interpret $c_\tau(\pi)$ as a real isomorphism

$$(2.2) \qquad\qquad \tau_F(E) \oplus \mathbb{R}^{2l-d} \longrightarrow \xi,$$

for some complex l-plane bundle ξ over E; so the composition

$$(2.3) \qquad r(t)\colon \xi \xrightarrow{c_\tau^{-1}(\pi)} \tau_F(E) \oplus \mathbb{R}^{2l-d} \xrightarrow{d(t\cdot)\oplus I} \tau_F(E) \oplus \mathbb{R}^{2l-d} \xrightarrow{c_\tau(\pi)} \xi$$

is a complex transformation for any $t \in T^k$, where $d(t\cdot)$ is the differential of the action by t. In other words, (2.3) determines a representation $r\colon T^k \to \mathrm{Hom}_\mathbb{C}(\xi, \xi)$.

If $X_+ = S^0$, then X is the one point-space $*$, and we may identify both F and E with some d-dimensional smooth T^k-manifold M, and $\tau_F(E)$ with its tangent bundle $\tau(M)$. So $c_\tau(\pi)$ reduces to a stably tangentially complex T^k-equivariant structure c_τ on M, and its cobordism class belongs to the geometric bordism group $\Omega_d^{U:T^k}$ of [33, §3]. We therefore denote the set of equivariant cobordism classes of stably tangentially complex T^k-bundles over X by $\Omega_{U:T^k}^{-d}(X_+)$. It is an abelian group under disjoint union of total spaces, and $\Omega_{U:T^k}^*(X_+)$ is a graded $\Omega_*^{U:T^k}$-module under cartesian product. Furthermore, $\Omega_{U:T^k}^*(-)$ is functorial with respect to pullback along smooth T^k-maps $Y \to X$.

Proposition 2.4. *Given any smooth compact T^k-manifold X, there is a canonical homomorphism*

$$\nu\colon \Omega_{U:T^k}^{-d}(X_+) \longrightarrow MU_{T^k}^{-d}(X_+)$$

for every $d \geqslant 0$.

Proof. Let π in $\Omega_{U:T^k}^{-d}(X_+)$ denote the cobordism class of a stably tangentially complex T^k-bundle $F \to E \xrightarrow{\pi} X$. Following the construction of the transfer [9], choose a T^k-equivariant embedding $i\colon E \to V$ into a unitary T^k-representation space V [14], and consider the embedding $(\pi, i)\colon E \to X \times V$; it is T^k-equivariant with respect to the diagonal action on $X \times V$, and its normal bundle admits an equivariant isomorphism $c\colon \tau_F(E) \oplus \nu(\pi, i) \to V$ of bundles over E. Now combine c with $c_\tau(\pi)$ of (2.2) to obtain an equivariant isomorphism

$$(2.5) \qquad\qquad W \oplus \nu(\pi, i) \longrightarrow \xi^\perp \oplus V \oplus \mathbb{R}^{2l-d},$$

where $W := \xi^\perp \oplus \xi$ is a unitary T^k-decomposition for some representation space W.

If d is even, (2.5) determines a complex T^k-structure on an equivariant stabilisation of $\nu(\pi, i)$; if d is odd, a further summand $\mathbb{R}$ must be added. For notational

convenience, assume the former, and write $d = 2n$. Then Thomify the classifying map for $\xi^\perp \oplus V \oplus \mathbb{C}^{l-n}$ to give a T^k-equivariant map

$$S^W \wedge M(\nu(\pi, i)) \longrightarrow M(\gamma_{|V|+|W|-n}),$$

and compose with the Pontryagin-Thom construction on $\nu(\pi, i)$ to obtain

$$f(\pi) \colon S^{V \oplus W} \wedge X_+ \longrightarrow M(\gamma_{|V|+|W|-n}).$$

If π and π' are equivalent, then $f(\pi)$ and $f(\pi')$ differ only by suspension; if they are cobordant, then $f(\pi)$ and $f(\pi')$ are stably T^k-homotopic. So define $\nu(\pi)$ to be the T^k-homotopy class of $f(\pi)$, as an element of $MU_{T^k}^{-d}(X_+)$.

The linearity of ν follows immediately from the fact that addition in $\Omega_{U:T^k}^{-d}(X_+)$ is induced by disjoint union. □

The proof of Proposition 2.4 also shows that ν factors through the geometric cobordism group of stably normally complex T^k-manifolds over X.

Definition 2.6. For any smooth compact T^k-manifold X, the *universal toric genus* is the homomorphism

$$\Phi_X \colon \Omega_{U:T^k}^*(X_+) \longrightarrow \Omega_U^*((ET^k \times_{T^k} X)_+)$$

given by the composition $\alpha \cdot \nu$.

Restricting attention to the case $X_+ = S^0$ defines

$$\Phi \colon \Omega_*^{U:T^k} \longrightarrow \Omega_*^U[[u_1, \ldots, u_k]],$$

where the target is isomorphic to $\Omega_U^*(BT_+^k)$, and u_j is the cobordism Chern class $c_1^{MU}(\zeta_j)$ of the conjugate Hopf bundle over the jth factor of BT^k, for $1 \leqslant j \leqslant k$. This version of Φ appears in [24] for $k = 1$, and is essentially the homomorphism introduced independently by Löffler [62] and Krichever [58]. It is defined on triples (M, a, c_τ), where a denotes the T^k-action; whenever possible, we omit one or both of a, c_τ from the notation.

We follow Krichever by interpreting Φ as an equivariant genus, in the sense that it is a multiplicative cobordism invariant of stably complex T^k-manifolds, and takes values in a graded ring with explicit generators and relations. As such it is an equivariant extension of Hirzebruch's original notion of genus [49], and is closely related to the theory of formal group laws.

When $X_+ = S^0$, Hanke [47] and Löffler [62, (3.1) Satz] prove that ν and α are monic; therefore so is Φ. On the other hand, there are two important reasons why ν cannot be epic. Firstly, it is defined on stably tangential structures by converting them into stably normal information; this procedure cannot be reversed equivariantly, because the former are stabilised only by trivial representations of T^k, whereas the latter are stabilised by arbitrary representations V. Secondly, homotopical cobordism groups are periodic, and each representation W gives rise to an invertible *Euler class* $e(W)$ in $MU_{-|W|}^{T^k}$ whenever the fixed point set $\mathrm{Fix}(W)$ is non-empty; this phenomenon exemplifies the failure of equivariant transversality. Since tom Dieck also proves that α is not epic, the same is true of Φ.

It is now convenient to describe Φ_X purely in terms of stably complex structures, by using geometrical models for the target algebra $\Omega_U^*((ET^k \times_{T^k} X)_+)$.

Let π in $\Omega_{U:T^k}^{-d}(X_+)$ be represented by $\pi \colon E \to X$, as in the proof of Proposition 2.4, and consider the smooth fibre bundle

$$(2.7) \qquad F \longrightarrow ET^k \times_{T^k} E \xrightarrow{1 \times_{T^k} \pi} ET^k \times_{T^k} X$$

obtained by applying the Borel construction. The bundle of tangents along the fibre is the Borelification $1 \times_{T^k} \tau_F(E)$, and so inherits a stably complex structure $1 \times_{T^k} c_\tau(\pi)$. Moreover, $1 \times_{T^k} c_\tau(\pi)$ and $1 \times_{T^k} c_\tau(\pi')$ are equivalent whenever π and π' are equivalent representatives for π, and cobordant whenever π and π' are

cobordant, albeit over the infinite dimensional manifold $ET^k \times_{T^k} X$. In this sense, (2.7) represents $\Phi_X(\pi)$. For a more conventional description, we convert (2.7) to a Quillen cobordism class by mimicking the proof of Proposition 2.4 as follows.

The Borelification h of the embedding (π, i) is also an embedding, whose normal bundle admits the stably complex structure that is complementary to $1 \times_{T^k} c_\tau(\pi)$. So the factorisation

$$(2.8) \qquad ET^k \times_{T^k} E \xrightarrow{\ h\ } ET^k \times_{T^k} (X \times V) \xrightarrow{\ r\ } ET^k \times_{T^k} X$$

determines a *complex orientation* for $1 \times_{T^k} \pi$, as defined in [77, (1.1)]. The map $1 \times_{T^k} \pi$ has dimension d, and its construction preserves cobordism classes in the appropriate sense; applying the Pontryagin-Thom construction confirms that it represents $\Phi_X(\pi)$ in $\Omega_U^{-d}((ET^k \times_{T^k} X)_+)$. Following [77, (1.4)], we may therefore express $\Phi_X(\pi)$ as $(1 \times_{T^k} \pi)_* 1$, where

$$(1 \times_{T^k} \pi)_* \colon \Omega_U^*(ET^k \times_{T^k} E) \longrightarrow \Omega_U^{*-d}(ET^k \times_{T^k} X),$$

denotes the *Gysin homomorphism*. For notational simplicity, we abbreviate $\Phi_X(\pi)$ to $\Phi(\pi)$ from this point onward.

In all relevant examples, the infinite dimensionality of $ET^k \times_{T^k} X$ presents no problem because ET^k may be approximated by the compact manifolds $\Pi S(q)$ of (1.1), and $\lim^1$ arguments applied. In certain circumstances, infinite dimensional manifolds may actually be incorporated into the definitions, as proposed in [8].

To be more explicit, we use the model $(\mathbb{C}P^\infty)^k$ for BT^k. The first Chern class in $\Omega_U^2(\mathbb{C}P^q)$ is represented geometrically by an inclusion $\mathbb{C}P^{q-1} \to \mathbb{C}P^q$, whose normal bundle is ζ. As q increases, these classes form an inverse system, whose limit defines $u := x^{MU}$ in $\Omega_U^2(\mathbb{C}P^\infty)$; it is represented geometrically by the inclusion of a hyperplane. An additive basis for $\Omega_*^U[[u_1, \ldots, u_k]]$ in dimension $2|\omega|$ is given by the monomials $u^\omega = u_1^{\omega_1} \cdots u_k^{\omega_k}$, where ω ranges over nonnegative integral vectors $(\omega_1, \ldots, \omega_k)$, and $|\omega| = \sum_j \omega_j$. Every such monomial is represented geometrically by a k-fold product of complex subspaces of codimension $(\omega_1, \ldots, \omega_k)$ in $(\mathbb{C}P^\infty)^k$, with normal bundle $\omega_1 \zeta_1 \oplus \cdots \oplus \omega_k \zeta_k$.

Now restrict attention to the case $X = *$. Given the equivariant cobordism class of (M^{2n}, a, c_τ), we may approximate (2.7) and (2.8) over $\Pi P(q)$ by stably tangential and stably normal complex structures on the $2(kq+n)$-dimensional manifold $W_q = \Pi S(q) \times_{T^k} M$. The corresponding complex orientation is described by

$$(2.9) \qquad W_q \xrightarrow{\ i\ } \Pi S(q) \times_{T^k} V \xrightarrow{\ r\ } \Pi P(q),$$

where i denotes the Borelification of a T^k-equivariant embedding $M \to V$, and r is the complex vector bundle induced by projection. The normal bundle $\nu(i)$ is invested with the complex structure complementary to $1 \times_{T^k} c_\tau$, which determines a complex cobordism class in $U^{-2n}(\Pi P(q))$. As q increases, these classes form an inverse system, whose limit is $\Phi(M, c_\tau)$ in $U^{-2n}(BT^k)$; it is represented geometrically by the complex orientation $ET^k \times_{T^k} M \to ET^k \times_{T^k} V \to BT^k$, which factorises the projection $1 \times_{T^k} \pi$. In particular, $\Phi(M, c_\tau)$ is the Gysin image $(1 \times_{T^k} \pi)_* 1$.

If we write

$$(2.10) \qquad \Phi(M, c_\tau) = \sum_\omega g_\omega(M)\, u^\omega$$

in $\Omega_*^U[[u_1, \ldots, u_k]]$, then the coefficients $g_\omega(M)$ lie in $\Omega_{2(|\omega|+n)}^U$, and their representatives may be interpreted as universal operations on the cobordism class of M. If c_τ is converted to stably *normal* data, then Löffler [62, (3.2) Satz] has described these operations using constructions of Conner and Floyd and tom Dieck, as follows.

Let $S^3 \subset \mathbb{C}^2$ be a T-space, via $t \cdot (z_1, z_2) = (t z_1, t^{-1} z_2)$, and let T operate on M by restricting the T^k-action to the jth coordinate circle, for $1 \leqslant j \leqslant k$. So

$\Gamma_j(M) := M \times_T S^3$ is a stably normally complex T^k-manifold under the action

$$(t_1, \ldots, t_k) \cdot [m, (z_1, z_2)] \;=\; [(t_1, \ldots, t_k) \cdot m, (t_j z_1, z_2)],$$

and the operations Γ_j may be composed.

Proposition 2.11. *The coefficient $g_\omega(M)$ of (2.10) is represented by the stably complex manifold $\Gamma^\omega(M) := \Gamma_1^{\omega_1} \ldots \Gamma_k^{\omega_k}(M)$; in particular, $g_0(M) = [M]$ in Ω_{2n}^U.*

We shall rewrite Löffler's formula in terms of stably tangential data in §3, and describe its application to rigidity phenomena.

There is an elegant alternative expression for $\Phi(M, c_\tau)$ when the fixed points x are isolated, involving their *weight vectors* $w_j(x) \in \mathbb{Z}^k$, for $1 \leqslant j \leqslant n$. Each such vector determines a line bundle

$$\zeta^{w_j(x)} \;:=\; \zeta_1^{w_{j,1}(x)} \otimes \cdots \otimes \zeta_k^{w_{j,k}(x)}$$

over BT^k, whose first Chern class is a formal power series

$$(2.12) \qquad [w_j(x)](u) \;:=\; \sum_\omega a_\omega [w_{j,1}(x)](u_1)^{\omega_1} \cdots [w_{j,k}(x)](u_k)^{\omega_k}$$

in $\Omega_U^2(BT^k)$. Here $[m](u_j)$ denotes the power series $c_1^{MU}(\zeta_j^m)$ in $\Omega_U^2(\mathbb{C}P^\infty)$ for any integer m, and the a_ω are the coefficients of $c_1^{MU}(\zeta_1 \otimes \cdots \otimes \zeta_k)$. The $[m](u_j)$ form the *power system* or *m-series* of the universal formal group law $F(u_1, u_2)$, and the a_ω are the coefficients of its iteration in $\Omega_{2|\omega|}^U$; see [18], [19] for further details. Modulo decomposables we have that

$$(2.13) \qquad [w_j(x)](u_1, \ldots, u_k) \;\equiv\; w_{j,1} u_1 + \cdots + w_{j,k} u_k,$$

and it is convenient to rewrite the right hand side as a scalar product $w_j(x) \cdot u$.

Krichever [57, (2.7)] obtained the following localisation formula.

Proposition 2.14. *If the structure c_τ is almost complex and the set of fixed points $x \in M$ is finite, then the equation*

$$(2.15) \qquad \Phi(M, c_\tau) \;=\; \sum_{Fix(M)} \prod_{j=1}^n \frac{1}{[w_j(x)](u)}$$

is satisfied in $\Omega_U^{-2n}(BT^k)$.

All terms of negative degree must cancel in the right-hand side of (2.15), imposing strong restrictions on the normal data of the fixed point set; these are analogues of Novikov's original Conner-Floyd relations. We shall extend Krichever's formula to stably complex structures, and discuss applications to non-equivariant cobordism.

3. Genera and rigidity

In this section we consider multiplicative cobordism invariants of stably complex manifolds, and define equivariant extensions to $\Omega_*^{U:T^k}$ using Φ. We recover a generalisation of Löffler's formula from the tangential viewpoint, explain its relationship with bounded flag manifolds, and discuss consequences for a universal concept of rigidity. We draw inspiration from Hirzebruch's original theory of genera, but give equal weight to Novikov's interpretation in terms of formal group laws and Adams's translation into complex-oriented cohomology theory.

3.1. **Background and examples.** Hirzebruch's [49] studies formal power series over a commutative ring R with identity. For every $Q(x) \equiv 1 \bmod x$ in $R[[x]]$, he defines a multiplicative homomorphism $\ell_Q \colon \Omega_*^U \to R$, otherwise known as the genus associated to Q. The image of ℓ_Q is a graded subring $R' \leqslant R$, whose precise description constitutes an *integrality theorem* for ℓ_Q. Hirzebruch also shows that his procedure is reversible. For every homomorphism $\ell \colon \Omega_*^U \to R$ into a torsion-free ring, he defines a formal power series $Q_\ell(x) \equiv 1 \bmod x$ in $R \otimes \mathbb{Q}[[x]]$. In this context, the integrality theorem corresponds to describing the subring $R'' \subseteq R \otimes \mathbb{Q}$ generated by the coefficients of Q_ℓ.

The interpretation of genera in terms of formal group laws F was pioneered by Novikov [69], [70], and is surveyed in [18]. His crucial insight identifies the formal power series $x/Q(x)$ with the exponential series $f_F(x)$ over $R \otimes \mathbb{Q}$, and leads directly to the analytic viewpoint of $f_F(x)$ as a function over $R \otimes \mathbb{C}$. The associated genus ℓ_F actually classifies F, because Ω_*^U may be identified with the Lazard ring, following Quillen [76]. These ideas underlie many subsequent results of the Moscow school, including generalisations of the classical elliptic genus by Krichever [59] and Buchstaber [16].

Our third, and least familiar, approach to genera involves complex oriented spectra D and E, and may initially appear more complicated. Nevertheless, it encodes additional homotopy theoretic information, as was recognised by developers such as Adams [1] and Boardman [10]. Many cobordism related calculations in western literature have been influenced by their point of view, including those of [22] and [79]. We use the notation of Section 1, especially Example 1.3.

Definition 3.1. A *homotopical genus* (t^D, v) with *defining spectrum D* and *evaluating spectrum E* consists of a Thom class t^D, and homotopy commutative diagram

$$
\begin{array}{ccc}
MU & \xrightarrow{\ t^D\ } & D \\
{\scriptstyle h^{MU}}\downarrow & & \downarrow{\scriptstyle v} \\
H \wedge MU & \xrightarrow{\ s^E\ } & E
\end{array}
$$

of complex orientable spectra, where D_* and E_* are concentrated in even dimensions and free of additive torsion, and $v_* \colon D_* \to E_*$ is monic. The *associated orientations* are the classes $s_*^E x^H$ and $v_* x^D$ in $E^2(\mathbb{C}P^\infty)$, abbreviated to x^H and x^E respectively.

It follows from Definition (3.1) that x^E extends to the Thom class $t^E := v_* t^D$. For fixed t^D, the *initial v* is $h^D \colon D \to H \wedge D$, with $s^{H \wedge D} = 1 \wedge t^D$, and the *final v* is rationalisation $r_D \colon D \to D\mathbb{Q}$, with $s^{D\mathbb{Q}} = r_H \wedge t^D$. If D is an H-module spectrum or is rational, then we may take v to be the identity 1^D.

The formal group law $F = F_{(t^D, v)}$ associated to a homotopical genus is classified by the homomorphism $t_*^D \colon \Omega_*^U \to D_*$ of graded rings. Its exponential and logarithm are the formal power series $b^E(x^H)$ and $m^E(x^E)$ of (1.2), which link the associated orientations in $E^2(\mathbb{C}P^\infty)$; then $F(u_1, u_2)$ is given by $b^E(m^E(u_1) + m^E(u_2))$ over E_*. The bordism classes of the stably complex Milnor hypersurfaces $H^j(1) \subset \times^{j+1} S^2$ are rational polynomial generators for Ω_*^U, as are those of the projective spaces $\mathbb{C}P^j$, for $j \geqslant 0$. So the homomorphism t_*^E is evaluated by either of the sequences

$$
(3.2) \qquad t_*^E(H^j(1)) \ = \ (j+1)!\, b_j^E \quad \text{or} \quad t_*^E(\mathbb{C}P^j) \ = \ (j+1)\, m_j^E
$$

in E_*, following [82, (2.11)] and [69, Appendix 1] respectively. Since v_* is monic, t_*^D may be recovered from (3.2).

Not all Hirzebruch genera are homotopical, but the following procedure usually creates a workable link. For any ring homomorphism $\ell \colon \Omega_*^U \to R$, first identify a complex oriented spectrum D such that some natural homomorphism $D_* \to R$ approximates the image of ℓ, and the corresponding Thom class $t^D \colon MU \to D$ induces ℓ on homotopy rings. Then identify a map $v \colon D \to E$ of complex oriented

ring spectra such that $E_* \leqslant D_* \otimes \mathbb{Q}$ approximates the graded subring generated by the coefficients of the exponential series $f_\ell(x)$, and $t^E = v \cdot t^D$ factors through h^{MU}. Should D_* already be isomorphic to an appropriately graded version of R, we indicate the forgetful homomorphism by $R_* \to R$.

The existence of E amounts to a homotopical integrality theorem, and $f_\ell(x^H)$ is synonymous with $b^E(x^H)$ in $E_*[[x^H]]$. The initial example (t^D, h^D) sheds light on the integrality properties of arbitrary Hirzebruch genera, homotopical or not, by suggesting the option of linearising the formal group law F_ℓ over the extension

$$(3.3) \qquad R \leqslant R \otimes_{\Omega_*^U} H_*(MU) \leqslant R \otimes \mathbb{Q}.$$

An alternative implementation of (3.3) is to work exclusively with the ring of *Hurwitz series* over R [28], following the lead of umbral calculus [82]. From the analytic and formal group theoretic viewpoints, F_ℓ is usually linearised over $R \otimes \mathbb{Q}$, and homotopy theoretic information is then lost.

We emphasise that the art of calculation with homotopical genera may still rely upon the Hirzebruch's original methods, as well as Krichever's analytic approach. By way of illustration, we rederive Hirzebruch's famous formula (3.8) for computing t_*^E on an arbitrary stably complex manifold (M^{2n}, c_τ). Sometimes, it proves more convenient to work with the complementary normal structure c_ν, which arises naturally from the Pontryagin-Thom construction. The virtual bundles τ and ν are related by $\perp^* \nu = \tau$, where $\perp \colon BU \to BU$ is the involution that classifies the complement of the universal virtual bundle. It induces an automorphism $\perp^*$ of $H \wedge MU$, which is specified uniquely by its action on $(H \wedge MU)^2(\mathbb{C}P^\infty)$; we write $\perp^* x^{MU}$ as $a(x^H) := x^\perp$, and note that it is determined by either of the identities

$$(3.4) \qquad a(x^H) = (x^H)^2 / b(x^H) \qquad \text{or} \qquad a_+(x^H) = 1/b_+(x^H).$$

So $x^\perp$ is also a complex orientation, with $a_1 = -b_1$ and $a_2 = b_1^2 - b_2$, for example.

The complex orientability of M ensures the existence of fundamental classes σ_M^H and σ_M^E in $E_{2n}(M)$. The coefficients $b_j := b_j^E$ of $b^E(x^H)$ generate monomials $b^\omega := b_1^{\omega_1} \ldots b_n^{\omega_n}$ in $E_{2|\omega|}$, which may not be independent. On the other hand, the orientation class x^H defines basis elements c_ω^H in $E^{2|\omega|}(BU)$, of which c_j^H restricts to the jth Chern class in $H^{2j}(BU)$, for every $j \geqslant 0$.

Proposition 3.5. *The homomorphism t_*^E satisfies*

$$(3.6) \qquad t_*^E(M, c_\nu) = \sum_\omega \left\langle b^\omega c_\omega^H(\nu), \sigma_M^H \right\rangle$$

in E_{2n}, for any stably complex manifold (M, c_ν).

Proof. The case $(1^{MU}, h^{MU})$ is universal. In $(H \wedge MU)_{2n}$, the formula

$$h_*^{MU}(M, c_\nu) = \sum_\omega \left\langle b^\omega c_\omega^H(\nu), \sigma_M^H \right\rangle$$

holds because b^ω and c_ω^H define dual bases; so (3.6) follows by applying s_*^E. $\qquad \square$

Corollary 3.7. *The value of $t_*^E(M, c_\nu)$ may be rewritten as*

$$(3.8) \qquad \left\langle \prod_i b_+(x_i^H)(\nu), \sigma_M^H \right\rangle = \left\langle \prod_i a_+(x_i^H)(\tau), \sigma_M^H \right\rangle.$$

Proof. These formulae are to be interpreted by writing c_ω^H as the appropriate symmetric function in the variables $x_1^H, x_2^H, \ldots$. So

$$\prod_i \left(1 + b_1 x_i^H + b_2 (x_i^H)^2 + \ldots \right) = \sum_\omega b^\omega c_\omega^H,$$

and (3.8) follows immediately from (3.6) and (3.4). $\qquad \square$

We now describe six homotopical genera (t^D, v), and evaluate each t^E_* explicitly; (1), (2), (3) and (6) are well-known, but (4) and (5) may be less familiar.

Examples 3.9.

(1) The *augmentation genus ag* and *universal genus ug* are the extreme cases, given by $(t^H, 1^H)$ and $(1^{MU}, h^{MU})$ respectively. Thus $ag(M) = 0$ and $ug(M) = [M]$ for any stably complex M of dimension > 0.

(2) The *Hurewicz genus hr* of Example 1.3 is given by $(h^{MU}, 1^{H \wedge MU})$, where $(H \wedge MU)_* \cong \mathbb{Z}[b_1, b_2, \ldots]$. Then

$$x^{MU} = b(x^H) \quad \text{and} \quad x^H = m(x^{MU})$$

in $(H \wedge MU)^2(\mathbb{C}P^\infty)$, so $hr(H^j(1)) = (j+1)!\, b_j$ and $hr(\mathbb{C}P^j) = (j+1)m_j$ for $j \geqslant 0$.

(3) The *Todd genus td* of [32] is given by (t^K, h^K), where $K_* \cong \mathbb{Z}[z, z^{-1}]$, and $(H \wedge K)_* \cong \mathbb{Q}[z, z^{-1}]$. Here

$$x^K = (\exp zx^H - 1)/z \quad \text{and} \quad x^H = (\ln(1 + zx^K))/z$$

in $(H \wedge K)^2(\mathbb{C}P^\infty)$, so $td(H^j(1)) = z^j$ and $td(\mathbb{C}P^j) = (-z)^j$.

(4) The c_n *genus cg* is given by $(t^C, 1^C)$, where C denotes $H \wedge X(2)$ for the Thom spectrum $X(2)$ of [78, §9.1] over ΩS^3; thus $C_* \cong \mathbb{Z}[v]$. Here

$$x^C = x^H/(1 + vx^H) \quad \text{and} \quad x^H = x^C/(1 - vx^C)$$

in $C^2(\mathbb{C}P^\infty)$, so $cg(H^j(1)) = (j+1)!\,(-v)^j$ and $cg(\mathbb{C}P^j) = (j+1)v^j$.

(5) The *Abel genus ab* and *2-parameter Todd genus t2* are given by $(t^A, 1^L)$ and $(t^{K2}, 1^L)$ respectively, where L denotes $H \wedge K \wedge K$; thus $L_* \cong \mathbb{Q}[y^{\pm 1}, z^{\pm 1}]$. Here

$$x^A = (\exp yx^H - \exp zx^H)/(y - z) \quad \text{and} \quad x^{K2} = (y - z)x^A/(y \exp zx^H - z \exp yx^H)$$

in $L^2(\mathbb{C}P^\infty)$, so $ab(H^j(1)) = \sum_{i=0}^{j} y^i z^{j-i}$ and $t2(H^j(1)) = \sum_\rho y^{r(\rho)} z^{f(\rho)}$, where each permutation $\rho \in S_{j+1}$ has $r(\rho)$ rises and $f(\rho)$ falls [41, §2.4.21]; similarly, $t2(\mathbb{C}P^j) = \sum_{i=0}^{j} (-1)^j y^i z^{j-i}$. If $y = 0$ then ab, $t2$, and td agree, and if $y = z = -v$ then $t2$ is cg; if $y = z$ then $x^A = x^H \exp yx^H$. The Abel genus appears in [17].

(6) The *signature sg* is given by $(t^S, h^{KO[1/2]})$, where $t^S \colon MU \to KO[1/2]$ is the Thom class of [49], $KO[1/2]_* \cong \mathbb{Z}[1/2][z^{\pm 2}]$, and $(H \wedge KO[1/2])_* \cong \mathbb{Q}[z^{\pm 2}]$. Then

$$x^S = (\tanh zx^H)/z \quad \text{and} \quad x^H = (\tanh^{-1} zx^H)/z$$

in $(H \wedge KO[1/2])^2(\mathbb{C}P^\infty)$, so $sg(\mathbb{C}P^{2j}) = z^{2j}$. In fact sg is $t2$ in case $y = -z$.

Remarks 3.10.

(1) Alternative formulae arise by composing with $\perp^*$ of (3.4), as in Corollary 3.7. The universal such example is the *tangential genus tg*, induced by the Thom class $t^\perp \colon MU \to H \wedge MU$ and determined by $tg(H^j(1)) = (j+1)!\, a_j$.

(2) The c_n genus and signature are given globally by $cg(M^{2n}, c_\tau) = \langle c_n^H(\tau), \sigma_M^H \rangle v^n$ and $sg(M^{2n}, c_\tau) = sg(M)z^n$. If c_τ is almost complex, then cg takes the value $\chi(M)v^n$, and is independent of the structure; sg is independent of c_τ in all cases.

(3) The genus ab may also be described homotopically as (t^{Ab}, h^{Ab}), where Ab denotes the complex oriented theory constructed in [27].

3.2. **Equivariant extensions.** Every genus $\ell \colon \Omega^U_* \to R_*$ has a T^k-*equivariant extension*

$$(3.11) \qquad \ell^{T^k} \colon \Omega^{U:T^k}_* \longrightarrow R_*[[u_1, \ldots, u_k]],$$

defined as the composition $\ell \cdot \Phi$. In this context, (2.10) yields the expression

$$(3.12) \qquad \ell^{T^k}(M, c_\tau) = \ell(M) + \sum_{|\omega| > 0} \ell(g_\omega(M))\, u^\omega.$$

In particular, the T^k-equivariant extension of the universal Examples 3.9(1) is Φ; hence the name *universal toric genus*.

More generally, we consider elements π of $\Omega_{U:T^k}^{-2n}(X_+)$, but restrict attention to those X for which $\Omega_U^*(ET^k \times_{T^k} X)$ is a finitely generated free $\Omega_U^*(BT^k)$-module. Then there exist generators $v_0, v_1, \ldots v_p$ in $\Omega_U^*(ET^k \times_{T^k} X)$, where $v_0 = 1$ and $\dim v_j = 2d_j$, which restrict to a basis for $\Omega_U^*(X)$ over Ω_*^U. The elements $v_j u^\omega$ form an Ω_*^U-basis for $\Omega_U^*(ET^k \times_{T^k} X)$; we write their duals as $a_{j,\omega}$, and choose singular stably complex manifolds $f_{j,\omega} \colon A_{j,\omega} \to ET^k \times_{T^k} X$ as their representatives. We may then generalise (2.10) to

$$(3.13) \qquad \Phi(\pi) \;=\; \sum_{j,\omega} g_{j,\omega}(\pi)\, v_j u^\omega$$

in $\Omega_U^{-2n}((ET^k \times_{T^k} X)_+)$, where the coefficients $g_{j,\omega}(\pi)$ lie in $\Omega_{2(n+d_j+|\omega|)}^U$. Moreover, $\sum_j g_{j,0}(\pi)\, v_j$ represents the non-equivariant cobordism class of the complex oriented map π in $\Omega_U^{-2n}(X_+)$.

The extension

$$(3.14) \qquad \ell^{T^k}(\pi) \colon \Omega_*^{U:T^k}(X_+) \longrightarrow R_*[[u_1, \ldots, u_k]]\langle v_0, v_1, \ldots, v_p \rangle$$

is the composition $\ell \cdot \Phi(\pi)$, and is evaluated by applying ℓ to each $g_{j,\omega}(\pi)$. Of course, (3.14) reduces to (3.11) when $X = *$ and $p = 0$. Alternatively, we may focus on homotopical genera t_*^E, and define the extension $(t_X^E)_*$ as the composition $t_*^E \cdot \Phi(\pi) \colon \Omega_{U:T^k}^*(X_+) \to E^*((ET^k \times_{T^k} X)_+)$ for *any* compact T^k-manifold X.

As an aid to evaluating equivariant extensions, we identify the $g_{j,\omega}(\pi)$ of (3.13).

Theorem 3.15. *For any smooth compact T^k-manifold X as above, the complex cobordism class $g_{j,\omega}(\pi)$ is represented on the $2(n+d_j+|\omega|)$-dimensional total space of a fibre bundle $F \to G_{j,\omega}(\pi) \to A_{j,\omega}$, whose stably complex structure is induced by those on $\tau(A_{j,\omega})$ and $\tau_F(E)$; in particular, $g_{0,0}(\pi) = [F]$ in Ω_{2n}^U.*

Proof. Formula (3.13) identifies $g_{j,\omega}(\pi)$ as the Kronecker product $\langle \Phi(\pi), a_{j,\omega} \rangle$. In terms of (2.9), it is represented on the pullback of the diagram

$$A_{j,\omega} \xrightarrow{f_{j,\omega}} \Pi S(q) \times_{T^k} X \xleftarrow{1 \times_{T^k} \pi} \Pi S(q) \times_{T^k} E$$

for suitably large q, and therefore on the pullback of the diagram

$$A_{j,\omega} \xrightarrow{f_{j,\omega}} ET^k \times_{T^k} X \xleftarrow{1 \times_{T^k} \pi} ET^k \times_{T^k} E.$$

of colimits. So the representing manifold is the total space of the pullback bundle $F \to G_{j,\omega}(\pi) \xrightarrow{p} A_{j,\omega}$. Moreover, both $\tau_F(E)$ and $\tau(A_{j,\omega})$ have stably complex structures, so the structure on the pullback is induced by the canonical isomorphism

$$(3.16) \qquad \tau(G_{j,\omega}(\pi)) \;\cong\; \tau_F(G_{j,\omega}(\pi)) \oplus p^*\tau(A_{j,\omega})$$

of real vector bundles. $\qquad\square$

In order to work with the case $X = *$, we must choose geometric representatives for the basis elements $b_\omega := b_{\omega_1} \otimes \cdots \otimes b_{\omega_k}$ of $\Omega_*^U(BT^k)$, dual to the monomials u^ω in $\Omega_U^*(BT^k)$. For this purpose, we recall the *bounded flag manifolds* B_j.

The subspace $(S^3)^j \subset \mathbb{C}^{2j}$ consists of all vectors satisfying $|z_i|^2 + |z_{i+j}|^2 = 1$ for $1 \leqslant i \leqslant j$, and is acted on freely by T^j according to

$$(3.17) \qquad t \cdot (z_1, \ldots, z_{2j}) \;=\; (t_1^{-1} z_1, t_1^{-1} t_2^{-1} z_2, \ldots, t_{j-1}^{-1} t_j^{-1} z_j, t_1 z_{j+1}, \ldots, t_j z_{2j})$$

for all $t = (t_1, \ldots, t_j)$. The quotient manifold $B_j := (S^3)^j / T^j$ is a j-fold iterated 2-sphere bundle over $B_0 = *$, and for $1 \leqslant i \leqslant j$ admits complex line bundles

$$\psi_i \colon (S^3)^j \times_{T^j} \mathbb{C} \longrightarrow B_j$$

via the action $t \cdot z = t_i z$ for $z \in \mathbb{C}$. The B_j are called bounded flag manifolds in [22], because their points may be represented by certain complete flags in $\mathbb{C}^{j+1}$.

For any $j > 0$, (3.17) determines an explicit isomorphism

$$\tau(B_j) \oplus \mathbb{C}^j \cong \psi_1 \oplus \psi_1\psi_2 \oplus \cdots \oplus \psi_{j-1}\psi_j \oplus \bar{\psi}_1 \oplus \cdots \oplus \bar{\psi}_j$$

of real $4j$-plane bundles, which defines a stably complex structure c_j^∂ on B_j. This extends over the associated 3-disc bundle, and so ensures that $[B_j, c_j^\partial] = 0$ in Ω_{2j}^U. The B_j may also be interpreted as *Bott towers* in the sense of [43], and therefore as complex algebraic varieties; however, the corresponding stably complex structures have nonzero Chern numbers, and are inequivalent to c_j^∂. We write the cartesian product $B_{\omega_1} \times \cdots \times B_{\omega_k}$ as B_ω, with the bounding stably complex structure c_ω^∂.

Proposition 3.18. *The basis element $b_\omega \in \Omega_{2|\omega|}^U(BT^k)$ is represented geometrically by the classifying map*

$$\psi_\omega \colon B_\omega \longrightarrow BT^k$$

for the external product $\psi_{\omega_1} \times \cdots \times \psi_{\omega_k}$ of circle bundles.

Proof. This follows directly from the case $k = 1$, proven in [81, Proposition 2.2]. $\square$

We may now return to Proposition 2.11, and the classes $g_\omega(M)$ of (2.10).

Definitions 3.19. Let T^ω be the torus $T^{\omega_1} \times \ldots \times T^{\omega_k}$ and $(S^3)^\omega$ the product $(S^3)^{\omega_1} \times \ldots \times (S^3)^{\omega_k}$, on which T^ω acts coordinatewise. The manifold $G_\omega(M)$ is the orbit space $(S^3)^\omega \times_{T^\omega} M$, where T^ω acts on M via the representation

$$(3.20) \qquad (t_{1,1}, \ldots, t_{1,\omega_1}; \ldots; t_{k,1}, \ldots, t_{k,\omega_k}) \longmapsto (t_{1,\omega_1}^{-1}, \ldots, t_{k,\omega_k}^{-1}).$$

The stably complex structure c_ω on $G_\omega(M)$ is induced by the tangential structures c_τ and c_ω^∂ on the base and fibre of the bundle $M \to G_\omega(M) \to B_\omega$.

Corollary 3.21. *The manifold $G_\omega(M)$ represents $g_\omega(M)$ in $\Omega_{2(|\omega|+n)}^U$.*

Proof. Apply Theorem 3.15 in the case $X = *$. Then π reduces to the identity map on M, and $G_{0,\omega}(\pi)$ reduces to $G_\omega(M)$; furthermore the stably complex structures (3.16) and c_ω of Definition 3.19 are equivalent. $\square$

Note that Löffler's $\Gamma^\omega(M)$ is T^k-equivariantly diffeomorphic to $G_\omega(M)$ for all ω.

3.3. Multiplicativity and rigidity.

Historically, the first rigidity results are due to Atiyah and Hirzebruch [4], and relate to the T-equivariant χ_y-genus and $\widehat{A}$-genus; their origins lie in the Atiyah-Bott fixed point formula [3], which also acted as a catalyst for the development of equivariant index theory.

Krichever [57], [59] considers rational valued genera l, and equivariant extensions $l' \colon \Omega_*^{U:T^k} \to K^0(BT_+^k) \otimes \mathbb{Q}$ that arise from l^{T^k} via the Chern-Dold character. If l may be realised on a stably complex manifold as the index of an elliptic complex, then the value of l' on any stably complex T^k-manifold M is a representation of T^k, which determines $l'(M)$ in the representation ring $RU(T^k) \otimes \mathbb{Q}$, and hence in its completion $K^0(BT_+^k) \otimes \mathbb{Q}$. Krichever's interpretation of rigidity is to require that $l'(M)$ should lie in the coefficient ring $K_0 \otimes \mathbb{Q}$ for every M. In the case of an index, this amounts to insisting that the corresponding T^k-representation is always trivial, and therefore conforms to Atiyah and Hirzebruch's original notion [4].

The following definition extends that of [50, Chapter 4] for the oriented case. It applies to fibre bundles of the form $M \to E \times_G M \xrightarrow{\pi} B$, where M and B are closed, connected, and stably tangentially complex, G is a compact Lie group of positive rank whose action preserves the stably complex structure on M, and $E \to B$ is a principal G-bundle. In these circumstances, π is stably tangentially complex, and $N := E \times_G M$ inherits a canonical stably complex structure.

Definitions 3.22. A genus $\ell \colon \Omega_*^U \to R_*$ is *multiplicative with respect to* the stably complex manifold M whenever $\ell(N) = \ell(M)\ell(B)$ for any such bundle π; if this holds for every M, then ℓ is *fibre multiplicative*.

The genus ℓ is T^k-*rigid on* M whenever $\ell^{T^k}\colon \Omega_*^{U:T^k} \longrightarrow R_*[[u_1, \ldots, u_k]]$ satisfies $\ell^{T^k}(M, c_\tau) = \ell(M)$; if this holds for every M, then ℓ is T^k-*rigid*.

We often indicate the rigidity of ℓ by referring to the formal power series $\ell^{T^k}(M)$ as *constant*, in which case $\ell(G_\omega(M)) = 0$ for every $|\omega| > 0$, by Corollary 3.21.

In fact ℓ is T^k-rigid if and only if its rationalisation $\ell_\mathbb{Q}$ is rigid in Krichever's original sense. By way of justification, we consider the universal example ug of Example 3.9(1); so $ug^{T^k} = \Phi$ by construction, and $ug_\mathbb{Q}$ coincides with the Hurewicz genus $hr_\mathbb{Q}\colon \Omega_*^U \to H_*(MU; \mathbb{Q}) \cong \Omega_*^U \otimes \mathbb{Q}$. The commutative square

$$(3.23) \qquad \begin{array}{ccc} MU & \xrightarrow{\ k^{MU}\ } & K \wedge MU \\[4pt] {\scriptstyle h^{MU}}\Big\downarrow & & \Big\downarrow{\scriptstyle h^K \wedge 1} \\[4pt] H \wedge MU & \xrightarrow{\ 1 \wedge k^{MU}\ } & H \wedge K \wedge MU \end{array}$$

defines a homotopical genus $(k^{MU}, h^K \wedge 1)$, where k^{MU} and $h^K\colon K \to H \wedge K$ induce the K-theoretic Hurewicz homomorphism and the Chern character respectively. The genus k^{MU} is used to develop integrality results by tom Dieck [37, §6].

Applying (3.23) to BT^k yields two distinct factorisations of the homomorphism

$$\Omega_*^U[[u_1, \ldots, u_k]] \longrightarrow H_*(MU; \mathbb{Q}[z, z^{-1}])[[u_1, \ldots, u_k]],$$

where z denotes the image of the Bott periodicity element in $H_2(K) \cong \mathbb{Q}$. For any stably complex T^k-manifold M, the lower-left factorisation maps $\Phi(M)$ to $\sum_{|\omega| \geqslant 0} hr_\mathbb{Q}(g_\omega(M)) u^\omega$. On the other hand, $k_*^{MU}\Phi(M)$ lies in $K_*(MU)[[u_1, \ldots, u_k]]$, where we may effect Krichever's change of orientation by following (1.2), and rewriting u_i as $\sum_j b_j^{K \wedge MU} x_i^K$ for $1 \leqslant i \leqslant k$. Applying the Chern character completes the right-upper factorisation, and identifies $\Phi(M)_\mathbb{Q}$ with Krichever's formal power series $ug_\mathbb{Q}'(x_1^K, \ldots, x_k^K)$ in $\Omega_*^U \otimes \mathbb{Q}[z, z^{-1}][[x_1^K, \ldots, x_k^K]]$.

But the change of orientation is invertible, so $\Phi(M)_\mathbb{Q}$ is a constant function of the u_i if and only if $ug_\mathbb{Q}'(x_1^K, \ldots, x_k^K)$ is a constant function of the x_i^K, as claimed.

Krichever's choice of orientation is fundamental to his proofs of rigidity, which bring techniques of complex analysis to bear on Laurent series such as those of Proposition 2.14. We follow his example to a limited extent in Section 6, where we interpret certain $\ell^{T^k}(M)$ as formal powers series over $\mathbb{C}$, and study their properties using classical functions of a complex variable. This viewpoint offers a powerful computational tool, even in the homotopical context.

For rational genera in the oriented category, Ochanine [72, Proposition 1] proves that fibre multiplicativity and rigidity are equivalent. In the toric case, we have the following stably complex analogue, whose conclusions are integral. It refers to bundles $E \times_G M \xrightarrow{\pi} B$ of the form required by Definition 3.22, where G has maximal torus T^k with $k \geqslant 1$.

Theorem 3.24. *If the genus ℓ is T^k-rigid on M, then it is mutiplicative with respect to M for bundles whose structure group G has the property that $\Omega_U^*(BG)$ is torsion-free; on the other hand, if ℓ is multiplicative with respect to a stably tangentially complex T^k-manifold M, then it is T^k-rigid on M.*

Proof. Let ℓ be T^k-rigid, and consider the pullback squares

$$(3.25) \qquad \begin{array}{ccccc} E \times_G M & \xrightarrow{\ f'\ } & EG \times_G M & \xleftarrow{\ i'\ } & ET^k \times_{T^k} M \\[4pt] {\scriptstyle \pi}\Big\downarrow & & {\scriptstyle \pi^G}\Big\downarrow & & {\scriptstyle \pi^{T^k}}\Big\downarrow \\[4pt] B^{2b} & \xrightarrow{\ f\ } & BG & \xleftarrow{\ i\ } & BT^k, \end{array}$$

where π^G is universal, i is induced by inclusion, f classifies π, and $E \times_G M = N$. If 1 is the unit in $\Omega^0((EG \times_G M)_+)$, then $\pi_*^G 1 = [M] \cdot 1 + \beta$, where β lies in the

reduced group $\Omega^{-2n}(BG)$ and satisfies $\pi_*1 = [M]\cdot1 + f^*\beta$ in $\Omega^{-2n}(B_+)$. Applying the Gysin homomorphism associated to the augmentation map $\epsilon^B\colon B \to *$ yields

$$(3.26) \qquad\qquad [N] \;=\; \epsilon^B_*\pi_*1 \;=\; [M][B] + \epsilon^B_*f^*\beta$$

in $\Omega_{2(n+b)}$; so $\ell(N) = \ell(M)\ell(B) + \ell(\epsilon^B_*f^*\beta)$. Moreover, $i^*\beta = \sum_{|\omega|>0}g_\omega(M)u^\omega$ in $\Omega^{-2n}(BT)$, so $\ell(i^*\beta) = 0$. The assumptions on G ensure that i^* is monic, and hence that $\ell(\beta) = 0$ in $\Omega^*_U(BG)\otimes_\ell R_*$. Multiplicativity then follows from (3.26).

Conversely, suppose that ℓ is multiplicative with respect to M, and consider the manifold $G_\omega(M)$ of Corollary 3.21. By Definition 3.19, it is the total space of the bundle $((S^3)^\omega \times_{T^\omega} T^k) \times_{T^k} M \to B_\omega$, which has structure group T^k; therefore $\ell(G_\omega(M)) = 0$, because B_ω bounds for every $|\omega| > 0$. So ℓ is T^k-rigid on M. $\qquad\square$

Remark 3.27. We may define ℓ to be *G-rigid* when $\ell(\beta) = 0$, as in the proof of Theorem 3.24. By choosing a suitable subcircle $T < T^k$, it follows that T-rigidity implies G-rigidity for any G such that $\Omega^*_U(BG)$ is torsion-free.

Example 3.28. The signature sg of Examples 3.9(6) is fibre multiplicative over any simply connected base [50], and so is rigid.

4. Isolated fixed points

In this section and the next, we focus on stably tangentially complex T^k-manifolds (M^{2n}, a, c_τ) for which the fixed points x are isolated; in other words, the fixed point set $\mathrm{Fix}(M)$ is finite. We proceed by adapting Quillen's methods to describe $\Phi(\pi)$ for any stably tangentially complex T^k-bundle $\pi\colon E \to X$, and deducing a localisation formula for $\Phi(M)$ in terms of fixed point data. This extends Krichever's formula (2.15). We give several illustrative examples, and describe the consequences for certain non-equivariant genera and their T^k-equivariant extensions. A condensed version of parts of this section appears in [24] for the case $k = 1$.

For any fixed point x, the representation $r_x\colon T^k \to GL(l, \mathbb{C})$ associated to (2.3) decomposes the fibre ξ_x as $\mathbb{C}^n \oplus \mathbb{C}^{l-n}$, where r_x has no trivial summands on $\mathbb{C}^n$, and is trivial on $\mathbb{C}^{l-n}$. Also, $c_{\tau,x}$ induces an orientation of the tangent space $\tau_x(M)$.

Definition 4.1. For any $x \in \mathrm{Fix}(M)$, the *sign* $\varsigma(x)$ is $+1$ if the isomorphism

$$\tau_x(M) \xrightarrow{\;i\;} \tau_x(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{\;c_{\tau,x}\;} \xi_x \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{\;p\;} \mathbb{C}^n$$

respects the canonical orientations, and -1 if it does not; here i and p are the inclusion of, and projection onto the first summand, respectively.

So $\varsigma(x)$ compares the orientations induced by r_x and $c_{\tau,x}$ on $\tau_x(M)$, and if M is almost complex then $\varsigma(x) = 1$ for every $x \in \mathrm{Fix}(M)$. The non-trivial summand of r_x decomposes into 1-dimensional representations as

$$(4.2) \qquad\qquad r_{x,1} \oplus \ldots \oplus r_{x,n},$$

and we write the integral *weight vector* of $r_{x,j}$ as $w_j(x) := (w_{j,1}(x), \ldots, w_{j,k}(x))$, for $1 \leqslant j \leqslant n$. We refer to the collection of signs $\varsigma(x)$ and weight vectors $w_j(x)$ as the *fixed point data* for (M, c_τ).

Localisation theorems for stably complex T^k-manifolds in equivariant generalised cohomology theories appear in tom Dieck [37], Quillen [77, Proposition 3.8], Krichever [57, Theorem 1.1], Kawakubo [54], and elsewhere. We prove our Corollary 4.9 by interpreting their results in the case of isolated fixed points, and identifying the signs explicitly. To prepare, we recall the most important details of [77, Proposition 3.8], in the context of (2.7) and (2.8).

Given any representative $\pi\colon E \to X$ for π in $\Omega^{-2n}_{U:T^k}(X_+)$, we approximate $\Phi(\pi)$ by the projection map $\pi(q) := 1 \times_{T^k} \pi$ of the smooth fibre bundle

$$F \longrightarrow \Pi S(q) \times_{T^k} E \longrightarrow \Pi S(q) \times_{T^k} X$$

for suitably large q. So $\pi(q)$ is complex oriented by the corresponding approximation to (2.8), whose associated embedding $h(q)\colon \Pi S(q) \times_{T^k} E \to \Pi S(q) \times_{T^k} (X \times V)$ is the Borelification of (π, i). Then there exists a commutative square

(4.3)
$$
\begin{array}{ccc}
\Pi P(q) \times \mathrm{Fix}(E) & \xrightarrow{\ 1 \times_{T^k} r_E\ } & \Pi S(q) \times_{T^k} E \\
{\scriptstyle \pi'(q)} \big\downarrow & & \big\downarrow {\scriptstyle \pi(q)} \\
\Pi P(q) \times \mathrm{Fix}(X) & \xrightarrow{\ 1 \times_{T^k} r_X\ } & \Pi S(q) \times_{T^k} X\,,
\end{array}
$$

where r_E and r_X are the inclusions of the fixed point submanifolds, and $\pi'(q)$ is the restriction of $1 \times \pi$.

Let $\mu'(q)$ and $\mu(V;q)$ denote the respective decompositions of the complex T^k-bundles $(1 \times_{T^k} r_E)^* \nu(h(q))$ and $\Pi S(q) \times_{T^k} V \to \Pi P(q)$ into eigenbundles, given by non-trivial 1-dimensional representations of T^k. They have respective Euler classes $e(\mu'(q))$ in $\Omega_U^{2(|V|-n)}(\Pi P(q) \times \mathrm{Fix}(E))$ and $e(\mu(V;q))$ in $\Omega_U^{2|V|}(\Pi P(q))$. Following [77, Remark 3.9], we then define

(4.4)
$$
e(\mu_{\pi(q)}) \;:=\; e(\mu'(q))/e(\mu(V,q)).
$$

This expression may be interpreted as a $-2n$ dimensional element of the localised ring $\Omega_U^*(\Pi P(q) \times \mathrm{Fix}(E))[e(\mu(V,q))^{-1}]$, or as a formal quotient whose denominator cancels wherever it appears in our formulae. In either event, $e(\mu_{\pi(q)})$ is determined by the stably complex structure $c_\tau(\pi)$ on $\tau_F(E)$, but is independent of the choice of complementary complex orientation.

Quillen shows that, for any $z \in \Omega_U^*(\Pi S(q) \times_{T^k} E)$, the equation

(4.5)
$$
(1 \times_{T^k} r_X)^* \pi(q)_* z \;=\; \pi'(q)_* \big(e(\mu_{\pi(q)}) \cdot (1 \times_{T^k} r_E)^* z\big)
$$

holds in $\Omega_U^*(\Pi P(q) \times \mathrm{Fix}(X))$. In evaluating such terms we may take advantage of the Künneth formula, because $\Omega_U^*(\Pi P(q))$ is free over Ω_*^U.

When the fixed points of $\pi\colon E \to X$ are isolated, this analysis confirms that the notion of fixed point data extends to $x \in \mathrm{Fix}(E)$; the signs $\varsigma(x)$ and weight vectors $w_j(x)$ are determined by the stably complex T^k-structure on $\tau_F(E)$, for $1 \leqslant j \leqslant n$. Every such x lies in a fibre $F_y := \pi^{-1}(y)$, for some fixed point $y \in \mathrm{Fix}(X)$.

Theorem 4.6. *For any stably tangentially complex T^k-bundle $F \to E \xrightarrow{\pi} X$ with isolated fixed points, the equation*

(4.7)
$$
\Phi(\pi)\big|_{\mathrm{Fix}(X)} \;=\; \sum_{\mathrm{Fix}(X)} \sum_{\mathrm{Fix}(F_y)} \varsigma(x) \prod_{j=1}^n \frac{1}{[w_j(x)](u)}
$$

is satisfied in $\bigoplus_{\mathrm{Fix}(X)} \Omega_U^{-2n}(BT_+^k \times y)$, where y and x range over $\mathrm{Fix}(X)$ and $\mathrm{Fix}(F_y)$ respectively .

Proof. Set $z = 1$ in (4.5). By definition, $\pi(q)_*(1)$ is represented by the complex oriented map $\pi(q)$ in $\Omega_U^{-2n}(\Pi S(q) \times_{T^k} X)$, and approximates $\Phi(\pi)$; denote it by $\Phi(\pi;q)$. Its restriction to $\Pi P(q) \times y$ for any fixed point y is represented by the induced projection $\pi_y(q)\colon \Pi S(q) \times_{T^k} F_y \to \Pi P(q) \times y$, which is complex oriented via the restriction

$$
1 \times_{T^k} i_y \colon \Pi S(q) \times_{T^k} F_y \longrightarrow \Pi S(q) \times_{T^k} V_y
$$

of the Borelification of (π, i) to F_y .

The commutative square (4.3) breaks up into components of the form

$$
\begin{array}{ccc}
\Pi P(q) \times \mathrm{Fix}(F_y) & \xrightarrow{\ 1 \times_{T^k} r_{F_y}\ } & \Pi S(q) \times_{T^k} F_y \\
{\scriptstyle \pi'_y(q)} \big\downarrow & & \big\downarrow {\scriptstyle \pi_y(q)} \\
\Pi P(q) \times y & \xrightarrow{\quad 1 \quad} & \Pi P(q) \times y
\end{array} \qquad ,
$$

on which (4.5) reduces to

$$(4.8) \qquad \Phi(\pi;q)|_y \;=\; \pi'_y(q)_* \big(e(\mu_{\pi_y(q)}) \big)$$

in $\Omega_U^{-2n}(\mathit{\Pi}P(q) \times y)$. But $\mathrm{Fix}(F_y)$ is finite, so the right hand side may be replaced by $\sum_x e(\mu(x,y))$, as x ranges over $\mathrm{Fix}(F_y)$; here $e(\mu(x,y))$ denotes the quotient $e(\mu'_y(q)|_x)/e(\mu(V_y,q)|_x)$ of (4.4), where $\mu'_y(q)|_x$ and $\mu(V_y,q)|_x$ are the respective decompositions of the T^k-bundles $1 \times_{T^k} r^*_{F_y} \nu(i_y)$ and $\mathit{\Pi}S(q) \times_{T^k} V_y \to \mathit{\Pi}P(q)$ into sums of complex line bundles, upon restriction to the fixed point x.

The restriction of the equivariant isomorphism $\tau(V)|_{F_y} \cong \tau(F_y) \oplus \nu(i_y)$ to x respects these decompositions, and reduces to $\mu(V_y,q)|_x \cong \tau_x(F_y) \oplus \mu'_y(q)|_x$. Taking Euler classes and applying Definition 4.1 and (4.2) then gives

$$e\big(\mu(V_y,q)|_x\big) \;=\; \varsigma(x) \cdot e(r_{x,1} \oplus \ldots \oplus r_{x,n}) \cdot e(\mu'_y(q)|_x)$$

in $\Omega_U^{2|V|}(\mathit{\Pi}P(q) \times y)$. So $e(\mu(x,y)) \cdot e(r_{x,1} \oplus \ldots \oplus r_{x,n}) = \varsigma(x)$, and (4.8) becomes

$$\Phi(\pi;q)|_y \;=\; \sum_x \varsigma(x) \frac{1}{e(r_{x,1} \oplus \ldots \oplus r_{x,n})}.$$

To complete the proof, recall from (2.12) that

$$e(r_{x,j}) \;=\; [w_j(x)](u)$$

in $\Omega_U^{-2n}(\mathit{\Pi}(q))$, for $1 \leqslant j \leqslant n$; then let $q \to \infty$, and sum over $y \in \mathrm{Fix}(X)$. $\qquad\square$

Corollary 4.9. *For any stably tangentially complex M^{2n} with isolated fixed points, the equation*

$$(4.10) \qquad \Phi(M) \;=\; \sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{[w_j(x)](u)}$$

is satisfied in $\Omega_U^{-2n}(BT^k_+)$.

Proof. Apply Theorem 4.6 to the case $E = M$ and $X = *$. $\qquad\square$

Remark 4.11. If the structure c_τ is almost complex, then $\varsigma(x) = 1$ for all fixed points x, and (4.10) reduces to Krichever's formula [59, (2.7)]).

The left-hand side of (4.10) lies in $\Omega_*^U[[u_1, \ldots, u_k]]$, whereas the right-hand side appears to belong to an appropriate localisation. It follows that all terms of negative degree must cancel, thereby imposing substantial restrictions on the fixed point data. These may be made explicit by rewriting (2.13) as

$$(4.12) \qquad [w_j(x)](tu) \;\equiv\; (w_{j,1}u_1 + \cdots + w_{j,k}u_k)t \mod (t^2)$$

in $\Omega_*^U[[u_1, \ldots, u_k, t]]$, and then defining the formal power series $\sum_l cf_l t^l$ to be

$$(4.13) \qquad t^n \Phi(M)(tu) \;=\; \sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{t}{[w_j(x)](tu)}$$

over the localisation of $\Omega_*^U[[u_1, \ldots, u_k]]$.

Proposition 4.14. *The coefficients cf_l are zero for $0 \leqslant l < n$, and satisfy*

$$cf_{n+m} \;=\; \sum_{|\omega|=m} g_\omega(M) u^\omega$$

for $m \geqslant 0$; in particular, $cf_n = [M]$.

Proof. Combine the definitions of cf_l in (4.13) and g_ω in (2.10). $\qquad\square$

Remarks 4.15.

(1) The equations $cf_l = 0$ for $0 \leqslant l < n$ are the T^k-analogues of the *Conner–Floyd relations* for $\mathbb{Z}/p$-actions [69, Appendix 4]; the extra equation $cf_n = [M]$ provides an expression for the cobordism class of M in terms of fixed point data. This is important because every element of Ω_*^U may be represented by a stably tangentially complex T^k-manifold with isolated fixed points [21].

(2) There are parametrised versions of the Conner-Floyd relations associated to any $\Phi(\pi)$, which arise from the individual fibres F_y. As we show in Example 6.3, they provide necessary conditions for the existence of stably tangentially complex T^k-equivariant bundles.

Theorem 4.6 applies to fibre bundles of the form

$$(4.16) \qquad\qquad H/T^k \longrightarrow G/T^k \xrightarrow{\pi} G/H$$

for any compact connected Lie subgroup $H < G$ of maximal rank k, by extrapolating the methods of [26]. Corollary 4.9 applies to homogeneous spaces G/H with nonzero Euler characteristic, to toric and quasitoric manifolds [20], and to many other families of examples.

Examples 4.17.

(1) For $U(1) \times U(2) < U(3)$, (4.16) is the T^3-bundle $\mathbb{C}P^1 \to U(3)/T^3 \xrightarrow{\pi} \mathbb{C}P^2$; it is the projectivisation of $\eta^\perp$, and all three manifolds are complex. Moreover, $U(3)/T^3$ has 6 fixed points and $\mathbb{C}P^2$ has 3, so π represents an element of $\Omega_{U:T^3}^{-2}(\mathbb{C}P^2)$, and $\Phi(\pi)$ lies in $\Omega_U^{-2}((ET^3 \times_{T^3} \mathbb{C}P^2)_+)$.

The 6 sets of weight vectors are singletons, given by the action of the Weyl group Σ_3 on the roots of $U(3)$, and split into 3 pairs, indexed by the cosets of $\Sigma_2 < \Sigma_3$. These pairs are $\pm(-1, 1, 0)$, $\pm(-1, 0, 1)$, and $\pm(0, -1, 1)$ respectively, corresponding to tangents along the fibres. So (4.7) evaluates $\Phi(\pi)|_{\mathrm{Fix}(\mathbb{C}P^2)}$ as

$$\left(\frac{1}{F(\bar{u}_1, u_2)} + \frac{1}{F(u_1, \bar{u}_2)} \right) + \left(\frac{1}{F(\bar{u}_1', u_3')} + \frac{1}{F(u_1', \bar{u}_3')} \right) + \left(\frac{1}{F(\bar{u}_2'', u_3'')} + \frac{1}{F(u_2'', \bar{u}_3'')} \right)$$

in $\Omega_U^{-2}(BT_+^3) \oplus \Omega_U^{-2}(BT_+^3)' \oplus \Omega_U^{-2}(BT_+^3)''$, where $F(u_1, u_2)$ denotes the universal formal group law $[(1, 1)](u_1, u_2)$, and $\bar{u}_j$ denotes $[-1]u_j$ for any j.

(2) The exceptional Lie group G_2 contains $SU(3)$ as a subgroup of maximal rank, and induces a canonical T^2-invariant almost complex structure on the quotient space S^6. This has two fixed points, with weight vectors $(1, 0)$, $(0, 1)$, $(-1, -1)$ and $(-1, 0)$, $(0, -1)$, $(1, 1)$ respectively. So Corollary 4.9 evaluates $\Phi(S^6, c_\tau)$ as

$$\frac{1}{u_1 u_2 F(\bar{u}_1, \bar{u}_2)} + \frac{1}{\bar{u}_1 \bar{u}_2 F(u_1, u_2)}$$

in $\Omega_U^{-6}(BT^2)$. The coefficient cf_0 of Proposition 4.14 is $(1-1)/u_1 u_2(u_1 + u_2)$, and is visibly zero; the cases cf_1 and cf_2 are more intricate.

Given any homotopical genus (t^D, v), we may adapt both Theorem 4.6 and Corollary 4.9 to express the genus $t_*^D \Phi$ in terms of fixed point data and the D-theory Euler class $e^D := t_*^D e$. Evaluating $t_*^E \Phi$ leads to major simplifications, because $F_{(t^D, v)}$ may be linearised over E_*; the tangential form is also useful.

Proposition 4.18. *For any stably tangentially complex T^k-bundle $F \to E \xrightarrow{\pi} X$ with isolated fixed points, $t_*^E \Phi(\pi)\big|_{\mathrm{Fix}(X)}$ takes the value*

$$(4.19) \qquad \sum_{\mathrm{Fix}(E)} \varsigma(x) \prod_{j=1}^n \frac{1}{b^E(w_j(x) \cdot u)} \;=\; \sum_{\mathrm{Fix}(E)} \varsigma(x) \prod_{j=1}^n \frac{(a^E)_+(w_j(x) \cdot u)}{w_j(x) \cdot u}$$

in $\bigoplus_{\mathrm{Fix}(X)} E^{-2n}(BT_+^k \times y)$.

Proof. Apply the Thom class t^E to Theorem 4.6, using the relation $e^E = b^E(e^H)$ in $E^2(\mathbb{C}P^\infty)$. Then the proof is completed by the linearising equations

$$e^E(r_{x,j}) = b^E(e^H(r_{x,j})) = b^E(w_j(x) \cdot u),$$

together with the identity $1/b_+^E(x^H) = a_+^E(x^H)$ in $E^0(\mathbb{C}P_+^\infty)$.   □

Corollary 4.20. *For any stably tangentially complex M^{2n} with isolated fixed points, the equation*

$$t_*^E \Phi(M) \;=\; \sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^n \frac{1}{b^E(w_j(x) \cdot u)}$$

is satisfied in $E^{-2n}(BT_+^k)$.

Remarks 4.21.

(1) The corresponding analogue of (4.13) is the series $\sum_l cf_l^E t^l$, defined by

$$(4.22) \qquad \sum_{\mathrm{Fix}(E)} \varsigma(x) \prod_{j=1}^n \frac{t}{b^E(tw_j(x) \cdot u)} \;=\; \sum_{\mathrm{Fix}(E)} \varsigma(x) \prod_{j=1}^n \frac{a_+^E(tw_j(x) \cdot u)}{w_j(x) \cdot u}$$

over the localisation of $E_*[[u_1, \ldots, u_k]]$. So cf_n^E is the first nonzero coefficient, and computes the nonequivariant genus $t_*^E(M)$. Similarly, the constant term of (4.19) is $\sum_{\mathrm{Fix}(X)} t_*^E(F_y)$, and the principal part is 0.

(2) The Hurewicz genus $hr(\Phi(\pi))$ of Examples 3.9(2) is used implicitly in many of the calculations of [26], and lies in $\bigoplus_{\mathrm{Fix}(X)}(H \wedge MU)^{-2n}(BT_+^k \times y)$.

Example 4.23. The augmentation genus $ag(\Phi(\pi))$ of Example 3.9 (1) may be computed directly from Theorem 4.6. It lies in $\bigoplus_{\mathrm{Fix}(X)} H^{-2n}(BT_+^k \times y)$, and is therefore zero for every π of positive fibre dimension. It follows that

$$(4.24) \qquad \sum_{\mathrm{Fix}(E)} \varsigma(x) \prod_{j=1}^n \frac{1}{w_j(x) \cdot u} \;=\; 0$$

for all M^{2n}, because $e^H(r_{x,j}) = w_j(x) \cdot u$ in $H^2(BT^k)$. In case $\pi \colon \mathbb{C}P^n \to *$ and T^{n+1} acts on $\mathbb{C}P^n$ homogeneous coordinatewise, every fixed point $x_0, \ldots, x_n$ has a single nonzero coordinate. So the weight vector $w_j(x_k)$ is $e_j - e_k$ for $0 \leqslant j \neq k \leqslant n$, and every $\varsigma(x_k)$ is positive; thus (4.24) reduces to

$$(4.25) \qquad \sum_{k=0}^n \prod_{0 \leqslant j \neq k \leqslant n} \frac{1}{u_j - u_k} \;=\; 0.$$

This is a classical identity, and is closely related to formulae in [50][§4.5].

Examples 4.26.

(1) The formula associated to Remarks 4.21(2) lies behind [26, Theorem 9]. The canonical T^n-action on the complex flag manifold $U(n)/T^n$ has $n!$ fixed points; these include the coset of the identity, whose weight vectors are the roots of $U(n)$, given by $e_i - e_j$ for $1 \leqslant i < j \leqslant n$. The action of the Weyl group Σ_n permutes the coordinates, and yields the weight vectors of the other fixed points. All have positive sign. Substituting this fixed point data into (4.22) gives

$$hr(U(n)/T^n) \;=\; \sum_{\Sigma_n} \mathrm{sign}(\rho) P_{\rho\delta}(a_1, \ldots, a_m),$$

in $(H \wedge MU)_{n(n-1)}$, where $\delta = (n-1, n-2, \ldots, 1, 0)$, $m = n(n-1)/2$, and ρ ranges over Σ_n; the $P_\omega(a_1, a_2, \ldots)$ are polynomials in $(H \wedge MU)_{2|\omega|}$, defined by

$$\prod_{1 \leqslant i < j \leqslant n} a_+(t(u_i - u_j)) \;=\; 1 + \sum_{|\omega| > 0} P_\omega(a_1, a_2, \ldots) t^{|\omega|} u^\omega.$$

(2) Similar methods apply to (S^6, c_τ) of Example 4.17(2), and show that

$$hr(S^6, c_\tau) = 2(a_1^3 - 3a_1 a_2 + 3a_3)$$

in $H_6(MU)$, as in [26]. This result may also be read off from calculations in [80].

5. QUASITORIC MANIFOLDS

As shown in [23], Davis and Januszkiewicz's *quasitoric manifolds* [36] provide a rich supply of stably tangentially complex T^n-manifolds (M^{2n}, a, c_τ) with isolated fixed points. In this section we review the basic theory, and highlight the presentation of an *omnioriented* quasitoric manifold in purely combinatorial terms. Our main goal is to compute the fixed point data directly from this presentation, and therefore to evaluate the genus $\Phi(M)$ combinatorially by applying the formulae of Section 4. We also describe the simplifications that arise in case M is a non-singular projective *toric variety*. Our main source of background information is [21], to which we refer readers for additional details.

An n-dimensional convex polytope P is the bounded intersection of m irredundant halfspaces in $\mathbb{R}^n$. The bounding hyperplanes $H_1, \ldots, H_m$ meet P in its *facets* $F_1, \ldots, F_m$, and P is *simple* when every face of codimension k may be expressed as $F_{j_1} \cap \cdots \cap F_{j_k}$, for $1 \leqslant k \leqslant n$. In this case, the facets of P are *finely ordered* by insisting that $F_1 \cap \cdots \cap F_n$ defines the *initial vertex* v_1.

Two polytopes are *combinatorially equivalent* whenever their face posets are isomorphic; the corresponding equivalence classes are known as *combinatorial polytopes*. At first sight, several of our constructions depend upon an explicit realisation of P as a convex subset of $\mathbb{R}^n$. In fact, they deliver equivalent output for combinatorially equivalent input, so it suffices to interpret P as combinatorial whilst continuing to work with a geometric representative when convenient. For further details on this point, we refer to [20] and [11, Corollary 4.7], for example. The facets of a combinatorial simple polytope P may also be finely ordered, and an *orientation* is then an equivalence class of permutations of $F_1, \ldots, F_n$.

There is a canonical affine transformation $i_P \colon P \to \mathbb{R}^m_\geqslant$ into the positive orthant of $\mathbb{R}^m$, which maps a point of P to its m-vector of distances from the hyperplanes $H_1, \ldots, H_m$. It is an embedding of manifolds with corners, and is specified by $i_P(x) = A_P x + b_P$, where A_P is the $m \times n$ matrix of inward pointing unit normals to $F_1, \ldots, F_m$. The *moment-angle manifold* $\mathcal{Z}_P$ is defined as the pullback

$$(5.1) \qquad \begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{\ i_Z\ } & \mathbb{C}^m \\ {\scriptstyle \mu_P} \downarrow & & \downarrow {\scriptstyle \mu} \\ P & \xrightarrow{\ i_P\ } & \mathbb{R}^m_\geqslant \end{array} \quad ,$$

where $\mu(z_1, \ldots, z_m)$ is given by $(|z_1|^2, \ldots, |z_m|^2)$. The vertical maps are projections onto the T^m-orbit spaces, and i_Z is a T^m-equivariant embedding. It follows from Diagram (5.1) that $\mathcal{Z}_P$ may be expressed as a complete intersection of $m - n$ real quadratic hypersurfaces in $\mathbb{C}^m$ [21], so it is smooth and equivariantly framed. We therefore acquire a T^m-equivariant isomorphism

$$(5.2) \qquad \tau(\mathcal{Z}_P) \oplus \nu(i_Z) \xrightarrow{\ \cong\ } \mathcal{Z}_P \times \mathbb{C}^m,$$

where $\tau(\mathcal{Z}_P)$ is the tangent bundle of $\mathcal{Z}_P$ and $\nu(i_Z)$ is the normal bundle of i_Z. Construction (5.1) was anticipated by work of López de Medrano [63].

It is important to describe the action of T^m on $\mathcal{Z}_P$ in more detail. For any vector z in $\mathbb{C}^m$, let $I(z) \subseteq [m]$ denote the set of subscripts j for which the coordinate z_j is zero; and for any point p in P, let $I(p) \subseteq [m]$ denote the set of subscripts for which p lies in F_j. The isotropy subgroup of z under the standard action on $\mathbb{C}^m$ is the *coordinate subgroup* $T_{I(z)} \leqslant T^m$, consisting of those elements $(t_1, \ldots, t_m)$ for which

$t_j = 1$ unless j lies in $I(z)$. We deduce that the isotropy subgroup of z in $\mathcal{Z}_P$ is $T_z = T_{I(\mu_P(z))} \leqslant T^m$, using the definitions of i_P and μ.

Definition 5.3. A *combinatorial quasitoric pair* (P, Λ) consists of an oriented combinatorial simple polytope P with finely ordered facets, and an integral $n \times m$ matrix Λ, whose columns λ_i satisfy

(1) $(\lambda_1 \ \ldots \ \lambda_n)$ is the identity submatrix I_n
(2) $\det(\lambda_{j_1} \ \ldots \ \lambda_{j_n}) = \pm 1$ whenever $F_{j_1} \cap \cdots \cap F_{j_n}$ is a vertex of P.

A matrix obeying (1) and (2) is called *refined*, and (2) is *Condition* $(*)$ of [36].

Given a combinatorial quasitoric pair, we use the matrix Λ to construct a quasitoric manifold $M = M^{2n}(P, \Lambda)$ from $\mathcal{Z}_P$. The first step is to interpret the matrix as an epimorphism $\Lambda \colon T^m \to T^n$, whose kernel we write as $K = K(\Lambda)$. The latter is isomorphic to T^{m-n} by virtue of Condition $(*)$. Furthermore, Λ restricts monomorphically to any of the isotropy subgroups $T_z < T^m$, for $z \in \mathcal{Z}_P$; therefore $K \cap T_z$ is trivial, and $K < T^m$ acts freely on $\mathcal{Z}_P$. This action is also smooth, so the orbit space $M = M(P, \Lambda) := \mathcal{Z}_P/K$ is a $2n$–dimensional smooth manifold, equipped with a smooth action a of the quotient n-torus $T^n \cong T^m/K$. By construction, μ_P induces a projection $\pi \colon M \to P$, whose fibres are the orbits of a. Davis and Januszkiewicz [36] also show that a is *locally standard*, in the sense that it is locally isomorphic to the coordinatewise action of T^n on $\mathbb{C}^n$.

In fact the pair (P, Λ) invests M with additional structure, obtained by factoring out the decomposition (5.2) by the action of K. We obtain an isomorphism

$$(5.4) \qquad \tau(M) \oplus (\xi/K) \oplus (\nu(i_Z)/K) \xrightarrow{\cong} \mathcal{Z}_P \times_K \mathbb{C}^m$$

of real $2m$–plane bundles, where ξ denotes the $(m-n)$–plane bundle of tangents along the fibres of the principal K-bundle $\mathcal{Z}_P \to M$. The right hand side of (5.4) is isomorphic to a sum $\bigoplus_{i=1}^{m} \rho_i$ of complex line bundles, where ρ_i is defined by the projection

$$\mathcal{Z}_P \times_K \mathbb{C}_i \longrightarrow M$$

associated to the action of K on the ith coordinate subspace. Both ξ and $\nu(i_Z)$ are equivariantly trivial over $\mathcal{Z}_P$, so ξ/K and $\nu(i_Z)/K$ are also trivial; explicit framings are given in [21, Proposition 4.5]. Hence (5.4) reduces to an isomorphism

$$(5.5) \qquad \tau(M) \oplus \mathbb{R}^{2(m-n)} \xrightarrow{\cong} \rho_1 \oplus \ldots \oplus \rho_m \, ,$$

which specifies a canonical T^n-equivariant stably tangentially complex structure $c_\tau = c_\tau(P, \Lambda)$ on M. The underlying smooth structure is that described above.

In order to make an unambiguous choice of isomorphism (5.5), we must also fix an orientation of M. This is determined by substituting the given orientation of P into the decomposition

$$(5.6) \qquad \tau_{(p,1)}(M) \cong \tau_p(P) \oplus \tau_1(T^n)$$

of the tangent space at $(p, 1) \in M$, where p lies in the interior of P and $1 \in T^n$ is the identity element. Together with the orientations underlying the complex structures on the bundles ρ_i, we obtain a sequence of $m + 1$ orientations; following [21], we call this the *omniorientation of M induced by* (P, Λ).

As real bundles, the ρ_i have an alternative description in terms of the *facial* (or *characteristic*) *submanifolds* $M_i \subset M$, defined by $\pi^{-1}(F_i)$ for $1 \leqslant i \leqslant m$. With respect to the action a, the isotropy subgroup of M_i is the subcircle of T^n defined by the column λ_i. Every M_i embeds with codimension 2, and has an orientable normal bundle ν_i that is isomorphic to the restriction of ρ_i; furthermore, ρ_i is trivial over $M \setminus M_i$. So an omniorientation may be interpreted as a sequence of orientations for the manifolds $M, M_1, \ldots, M_m$.

Definition 5.7. The stably tangentially complex T^n-manifold $M(P, \Lambda)$ is the omnioriented quasitoric manifold *corresponding to the pair* (P, Λ). Both c_τ and its underlying smooth structure are *canonical*; if the first Chern class of c_τ vanishes, then (P, Λ) and its induced omniorientation are *special*.

Remarks 5.8.

(1) Although the smooth T^m-structure on $\mathcal{Z}_P$ is unique, we do not know whether the same is true for the canonical smooth structure on M.

(2) Reversing the orientation of P acts by negating the complex structure c_τ, whereas negating the column λ_i acts by conjugating the complex line bundle ρ_i, for $1 \leqslant i \leqslant m$. None of these operations affects the validity of Condition $(*)$ for Λ.

Example 5.9. By applying an appropriate affine transformation to any simple polytope $P \subset \mathbb{R}^n$, it is possible to locate its initial vertex at the origin, and realise the inward pointing normal vectors to $F_1, \ldots, F_n$ as the standard basis. A small perturbation of the defining inequalities then ensures that the normals to the remaining $m - n$ facets are integral, without changing the combinatorial type of P. Any such sequence of normal vectors form the columns of an integral $n \times m$ matrix $N(P)$, whose transpose reduces to A_P of (5.1) after normalising the rows. For certain *unsupportive* P (such as the dual of a cyclic polytope with many vertices [36, Nonexamples 1.22]), $N(P)$ can never satisfy Condition $(*)$. On the other hand, if P is *supportive* then $(P, N(P))$ is a combinatorial quasitoric pair, and the corresponding quasitoric manifold is a *non-singular projective toric variety* [21].

The most straightforward example is provided by the n-simplex $P = \Delta$, whose standard embedding in $\mathbb{R}^n$ gives rise to the $n \times (n + 1)$ matrix $N(\Delta) = (I_n : -1)$, where -1 denotes the column vector $(-1, \ldots, -1)^t$. The corresponding quasitoric manifold $M(\Delta, N(\Delta))$ is the complex projective space $\mathbb{C}P^n$, equipped with the stabilisation of the standard complex structure $\tau(\mathbb{C}P^n) \oplus \mathbb{C} \cong \oplus^{n+1}\zeta$, and the standard T^n-action $t \cdot [z_0, \ldots, z_n] = [z_0, t_1 z_1, \ldots, t_n z_n]$.

We now reverse the above procedures, and start with an omnioriented quasitoric manifold (M^{2n}, a). By definition, there is a projection $\pi \colon M \to P$ onto a simple n-dimensional polytope $P = P(M)$, which is oriented by the underlying orientation of M. We continue to denote the facets of P by $F_1, \ldots, F_m$, so that the facial submanifolds $M_i \subset M$ are given by $\pi^{-1}(F_i)$ for $1 \leqslant i \leqslant m$, and the corresponding normal bundles ν_i are oriented by the omniorientation of M. The isotropy subcircles $T(M_i) < T^n$ are therefore specified unambiguously, and determine m primitive vectors in $\mathbb{Z}^n$; these form the columns of an $n \times m$ matrix Λ', which automatically satisfies Condition $(*)$. However, condition (1) of Definition (5.3) can only hold if the F_i are finely ordered, in which case the matrix $\Lambda(M) := L\Lambda'$ is refined, for an appropriate choice of L in $GL(n, \mathbb{Z})$. So $(P(M), \Lambda(M))$ is a quasitoric combinatorial pair, which we call the *combinatorial data* associated to M.

The omnioriented quasitoric manifold of the pair $(P(M), \Lambda(M))$ is *equivalent* to (M, a) in the sense of [36], being T^n-equivariantly homeomorphic up to an automorphism of T^n. The automorphism is specified by the choice of L above. On the other hand, the combinatorial data associated to $M(P, \Lambda)$ is (P, Λ) itself, unless an alternative fine ordering is chosen. In that case, the new ordering differs from the old by a permutation σ, with the property that $F_{\sigma^{-1}(1)} \cap \cdots \cap F_{\sigma^{-1}(n)}$ is non-empty. If we interpret σ as an $m \times m$ permutation matrix Σ, then the resulting combinatorial data is $(P', L\Lambda\Sigma)$, which we deem to be *coincident with* (P, Λ). We may summarise this analysis as follows.

Theorem 5.10. *Davis-Januszkiewicz equivalence classes of omnioriented quasitoric manifolds (M, a) correspond bijectively to coincidence classes of combinatorial quasitoric pairs (P, Λ).*

We are now in a position to fulfill our promise of expressing equivariant properties of M in terms of its combinatorial data (P, Λ). Every fixed point x takes the form $M_{j_1} \cap \cdots \cap M_{j_n}$ for some sequence $j_1 < \cdots < j_n$, and it will be convenient to write Λ_x for the $n \times n$ submatrix $(\lambda_{j_1} \ldots \lambda_{j_n})$.

Proposition 5.11. *An omniorientation of M is special if and only if the column sums of Λ satisfy $\sum_{i=1}^{n} \lambda_{i,j} = 1$, for every $1 \leqslant j \leqslant m$.*

Proof. By [36, Theorem 4.8], $H^2(M; \mathbb{Z})$ is the free abelian group generated by elements $u_j := c_1(\rho_j)$ for $1 \leqslant j \leqslant m$, with n relations $\Lambda u = 0$. These give

$$u_i + \sum_{j=n+1}^{m} \lambda_{i,j} u_j \; = \; 0$$

for $1 \leqslant i \leqslant n$, whose sum is

$$(5.12) \qquad \sum_{i=1}^{n} u_i + \sum_{j=n+1}^{m} \left(\sum_{i=1}^{n} \lambda_{i,j} \right) u_j \; = \; 0 \,.$$

Furthermore, (5.5) implies that $c_1(c_\tau) = \sum_{j=1}^{m} u_j$, so the omniorientation is special if and only if $\sum_{j=1}^{m} u_j = 0$. Subtracting (5.12) gives the required formulae. $\qquad \square$

Before proceeding to the fixed point data for (M, a), we reconcile Definition 4.1 with notions of sign that appear in earlier works on toric topology. For any fixed point x the decomposition (4.2) arises by splitting $\tau_x(M)$ into subspaces normal to the M_{j_k}, leading to an isomorphism

$$(5.13) \qquad \tau_x(M) \; = \; (\nu_{j_1} \oplus \cdots \oplus \nu_{j_n})|_x \; \cong \; (\rho_{j_1} \oplus \cdots \oplus \rho_{j_n})|_x \,.$$

Lemma 5.14. *The sign of any fixed point $x \in M$ is given by: $\varsigma(x) = 1$ if the orientation of $\tau_x(M)$ induced by the orientation of M coincides with that induced by the orientations of the ρ_{j_k} in (5.13), and by $\varsigma(x) = -1$ otherwise.*

Proof. By construction, the isomorphism $\nu_j \cong \rho_j$ is of T^n-equivariant bundles for any $1 \leqslant j \leqslant m$, and the orientation of both is induced by the action of the isotropy subcircle specified by λ_j. Moreover, ρ_j is trivial over $M \setminus M_j$, so T^n acts non-trivially only on the summand $(\rho_{j_1} \oplus \cdots \oplus \rho_{j_n})|_x$. It follows that the composite map of Definition 4.1 agrees with the isomorphism

$$(5.15) \qquad \tau_x M \xrightarrow{\; \cong \;} (\rho_{j_1} \oplus \ldots \oplus \rho_{j_n})|_x$$

of (5.13), and that $\varsigma(x)$ takes values as claimed. $\qquad \square$

Lemma 5.14 confirms that Definition 4.1 is equivalent to those of [21, §5], [38], [64, §4], and [75] in the case of quasitoric manifolds.

Theorem 5.16. *For any quasitoric manifold M with combinatorial data (P, Λ) and fixed point x, let $N(P)_x$ be a matrix of column vectors normal to $F_{j_1}, \ldots, F_{j_n}$, and W_x be the matrix determined by $W_x^t \Lambda_x = I_n$; then*

    (1) *the sign $\varsigma(x)$ is given by $\mathrm{sign}\big(\det(\Lambda_x N(P)_x)\big)$*

    (2) *the weight vectors $w_1(x), \ldots w_n(x)$ are the columns of W_x.*

Proof. (1) In order to write the isomorphism (5.15) in local coordinates, recall (5.5) and (5.6). So $\tau_x(M)$ is isomorphic to $\mathbb{R}^{2n}$ by choosing coordinates for P and $\tau_1(T^n)$, and $(\rho_1 \oplus \cdots \oplus \rho_m)|_x$ is isomorphic to $\mathbb{R}^{2m}$ by choosing coordinates for $\mathbb{R}^m$ and $\tau_1(T^m)$. Then the composition

$$\tau_x(M) \longrightarrow \tau_x(M) \oplus \mathbb{R}^{2(m-n)} \xrightarrow{\; c_\tau \;} (\rho_1 \oplus \ldots \oplus \rho_m)|_x$$

is represented by the $2m \times 2n$–matrix

$$\begin{pmatrix} A_P & 0 \\ 0 & \Lambda^t \end{pmatrix} ,$$

and (5.15) is given by restriction to the relevant rows. The result follows for A_P^t, and hence for any $N(P)$.

(2) The formula follows directly from the definitions, as in [75, §1]. □

Remarks 5.17. Theorem 5.16(1) gives the following algorithm for calculating $\varsigma(x)$: write the column vectors of Λ_x in such an order that the inward pointing normal vectors to the corresponding facets give a positive basis of $\mathbb{R}^n$, and calculate the determinant of the resulting $n \times n$ matrix. This is equivalent to the definition of $\varsigma(x)$ in [20, §5.4.1] and [75, §1]. Theorem 5.16(2) states that $w_1(x), \ldots, w_n(x)$ and $\lambda_{j_1}, \ldots, \lambda_{j_n}$ are conjugate bases of $\mathbb{R}^n$.

Inserting the conclusions of Theorem 5.16 into Corollaries 4.9 and 4.20 completes our combinatorial evaluation of genera on (M, a). Of course the Conner-Floyd relations stemming from the latter must be consequences of conditions (1) and (2) of Definition 5.3 on the minors of Λ, and it is interesting to speculate on the extent to which this implication is reversible.

Example 5.18. Let P be the n–simplex Δ, geometrically represented by the standard simplex in $\mathbb{R}^n$; it is oriented and finely ordered by the standard basis. So Λ takes the form $(I_n : \epsilon)$, where ϵ denotes a column vector $(\epsilon_1, \ldots, \epsilon_n)^t$, and Condition $(*)$ confirms that $(\Delta, (I_n : \epsilon))$ is a combinatorial quasitoric pair if and only if $\epsilon_i = \pm 1$ for every $1 \leqslant i \leqslant n$. When $\epsilon_i = -1$ for all $1 \leqslant i \leqslant n$, the corresponding omnioriented quasitoric manifold $M = M(\Delta, (I_n : \epsilon))$ is the toric variety $\mathbb{C}P^n$ of Example 5.9. In general, M is obtained from $\mathbb{C}P^n$ by a change of omniorientation, in which the ith summand of the stable normal bundle is reoriented when $\epsilon_i = 1$, and unaltered otherwise; it is denoted by $\mathbb{C}P^n_\epsilon$ when convenient.

If the facets of Δ are reordered by a transposition $(j, n+1)$, the resulting pair $(\Delta', (I_n : \epsilon')$ is coincident with $(\Delta, (I_n : \epsilon))$. A simple calculation reveals that $\epsilon'_i = -\epsilon_i \epsilon_j$ for $i \neq j$, whereas $\epsilon'_j = \epsilon_j$.

The $n+1$ fixed points correspond to the vertices of Δ, and they are most conveniently labelled $x_0, \ldots, x_n$, with x_0 initial. The submatrix $(I_n : \epsilon)_{x_0}$ is I_n, and the submatrices $(I_n : \epsilon)_{x_k}$ are obtained by deleting the kth column, for $1 \leqslant k \leqslant n$. So Theorem 5.16(2) expresses the weight vectors by $w_j(x_0) = e_j$, and

$$w_j(x_k) = \begin{cases} e_j - \epsilon_j \epsilon_k e_k & \text{for } 1 \leqslant j < k \\ e_{j+1} - \epsilon_{j+1} \epsilon_k e_k & \text{for } k \leqslant j < n \\ \epsilon_k e_k & \text{for } j = n \end{cases}$$

for $1 \leqslant j, k \leqslant n$. Applying Corollary 4.20 for the augmentation genus ag then gives

$$(5.19) \qquad \frac{\varsigma(x_0)}{u_1 \ldots u_n} + \sum_{k=1}^n \left(\frac{\varsigma(x_k)}{\epsilon_k u_k} \prod_{j \neq k} \frac{1}{(u_j - \epsilon_j \epsilon_k u_k)} \right) = 0,$$

from which it follows that $\epsilon_i = -\varsigma(x_i)/\varsigma(x_0)$ for $1 \leqslant i \leqslant n$.

Formula (5.19) is closely related to (4.25) of Example 4.23. This case is of interest even for $n = 1$, and Corollary 4.9 evaluates the universal toric genus by

$$\Phi(\mathbb{C}P^1) = \frac{1}{u} + \frac{1}{\bar{u}}$$

in $\Omega_U^{-2}(BT_+)$, where $\bar{u}$ denotes the infinite series $[-1](u)$.

6. Applications and further examples

In our final section we consider applications of toric genera to questions of rigidity, and discuss additional examples that support the theory.

6.1. **Parametrised genera.** The parametrised genus $\Phi(\pi)$ is an invariant of tangentially stably complex T^k-bundles. We now return to Examples 4.17, and apply the formula (3.13) to complete the calculations. The results suggest that there exists a systematic theory of genera of bundles, with interesting applications to rigidity phenomena.

Example 6.1. The T^3-bundle $\mathbb{C}P^1 \to U(3)/T^3 \xrightarrow{\pi} \mathbb{C}P^2$ of Example 4.17(1) is tangentially complex, and $\Phi(\pi)$ lies in $\Omega_U^{-2}(EP_+)$, where EP denotes $ET^3 \times_{T^3} \mathbb{C}P^2$. Since EP is actually $\mathbb{C}P(\eta_1 \times \eta_2 \times \eta_3)$ over BT^3, there is an isomorphism

$$\Omega_U^*(EP_+) \cong \Omega_*^U[[u_1, u_2, u_3, v]] \big/ ((v + u_1)(v + u_2)(v + u_3))$$

of Ω_U^*-algebras, where $v := c_1^{MU}(\rho)$ for the canonical projective bundle ρ. So

$$(6.2) \qquad \Phi(\pi) = \sum_\omega g_{0,\omega} u^\omega + \sum_\omega g_{1,\omega} v u^\omega + \sum_\omega g_{2,\omega} v^2 u^\omega$$

holds in $\Omega_U^{-2}(EP_+)$ by (3.13), where $g_{j,\omega}$ lies in $\Omega_{2(j+|\omega|+1)}^U$, and $\omega = (\omega_1, \omega_2, \omega_3)$; in particular, $g_{0,0} + g_{1,0}v + g_{2,0}v^2$ is the cobordism class of π in $\Omega_U^{-2}(\mathbb{C}P_+^2)$.

For $0 \leqslant j \leqslant 2$, the element $a_{j,\omega}$ of Theorem 3.15 is represented by a map

$$f_{j,\omega} \colon (S^3)^\omega \times_{T^\omega} B_j \longrightarrow ET^3 \times_{T^3} \mathbb{C}P^2,$$

which combines a representative for b_ω on the first factor with a T^3-equivariant representative for $b_j \in \Omega_{2j}^U(\mathbb{C}P^2)$ on the second. It follows that $g_{j,\omega}$ is represented by the induced stably complex structure on the total space of the bundle

$$\mathbb{C}P^1 \longrightarrow (S^3)^\omega \times_{T^\omega} \mathbb{C}P(\psi_j^\perp \oplus \mathbb{C}^{2-j}) \longrightarrow (S^3)^\omega \times_{T^\omega} B_j,$$

Thus $g_{0,0} = [\mathbb{C}P^1]$ in Ω_2^U, as expected; furthermore, $g_{1,0} = [\mathbb{C}P(\zeta \oplus \mathbb{C})]$ in Ω_4^U and $g_{2,0} = [\mathbb{C}P(\psi_2^\perp)]$ in Ω_6^U, where $\zeta = \psi_1^\perp$ lies over $B_1 = S^2$. These classes may be computed in terms of standard polynomial generators for Ω_*^U as required.

The properties of $\Phi(\pi)$ may be interpreted as obstructions to the existence of equivariant bundles π. This principle is illustrated by the family of 4–dimensional omnioriented quasitoric manifolds over the combinatorial square $P = I^2$.

Example 6.3. Let the vertices of P be ordered cyclically, and write Λ as

$$\begin{pmatrix} 1 & 0 & \epsilon_1 & \delta_2 \\ 0 & 1 & \delta_1 & \epsilon_2 \end{pmatrix};$$

so (P, Λ) is a combinatorial quasitoric pair if and only if the equations

$$(6.4) \qquad \epsilon_1 = \pm 1, \quad \epsilon_2 = \pm 1, \quad \epsilon_1 \epsilon_2 - \delta_1 \delta_2 = \pm 1$$

are satisfied. The fixed points x_i of $M(P^2, \Lambda)$ correspond to the vertices of P^2, and are determined by the pairs $(i, i+1)$ of facets, where $1 \leqslant i \leqslant 4$ (and $x_5 = x_1$). We assume that $\varsigma(x_1) = 1$, by reversing the global orientation of P^2 if necessary. Then the calculations of Lemma 5.14 determine the remaining signs by

$$\varsigma(x_2) = -\epsilon_1, \quad \varsigma(x_3) = \epsilon_1 \epsilon_2 - \delta_1 \delta_2, \quad \varsigma(x_4) = -\epsilon_2,$$

and a criterion of Dobrinskaya [38, Corollary 7] confirms that $M(P^2, \Lambda)$ is the total space of a tangentially complex T^2-equivariant bundle precisely when $\delta_1 \delta_2 = 0$.

The necessity of this condition follows from Theorem 4.6, by considering the equations associated to the augmentation genus ag. Given (6.4), the contributions $k(x_i)$ of the four fixed points must always sum to zero; and if $M(P, \Lambda)$ is the total space of some T^2-bundle, then the $k(x_i)$ must split into pairs, which also sum to zero. But $k(x_1) + k(x_3) \neq 0$ by inspection, so two possibilities arise.

Firstly, $k(x_1) + k(x_4)$ and $k(x_2) + k(x_3)$ are given by

$$\frac{1}{t_1 t_2} + \frac{-\epsilon_2}{\epsilon_2 t_2 (t_1 - \epsilon_2 \delta_2 t_2)} \qquad \text{and} \qquad \frac{-\epsilon_1}{\epsilon_1 t_1 (-\epsilon_1 \delta_1 t_1 + t_2)} + \frac{\epsilon_1 \epsilon_2}{\epsilon_1 t_1 (-\epsilon_1 \epsilon_2 \delta_1 t_1 + \epsilon_2 t_2)}$$

respectively, and vanish precisely when $\delta_2 = 0$. Secondly, and by similar reasoning, $k(x_1) + k(x_2)$ and $k(x_3) + k(x_4)$ vanish precisely when $\delta_1 = 0$.

Example 6.3 extends inductively to quasitoric manifolds over higher dimensional cubes, and shows that any such manifold splits into a tower of fibre bundles if and only if the corresponding submatrix $\Lambda_\star$ may be converted to lower triangular form by simultaneous transpositions of rows and columns. Compare [65, Proposition 3.2].

6.2. **Elliptic genera.** Since the early 1980s, cobordism theory has witnessed the emergence of *elliptic genera*, taking values in rings of modular forms. The original inspiration was index-theoretical, and driven by Witten's success [87] in adopting the Atiyah-Bott fixed-point formula to compute the equivariant index of a putative Dirac operator on the free loop space of a Spin manifold. Analytic and formal group-theoretic versions of complex elliptic genera were then proposed by Ochanine [71], in the oriented category. These led directly to the homotopical constructions of Landweber, Ravenel, and Stong [61], Hopkins [52], and Ando, Hopkins, and Strickland [2], amongst others; their constructions include several variants of elliptic cohomology, such as the complex oriented theory represented by the spectrum Ell.

Mindful of Definition 3.1 and Examples 3.9, we shall focus on the homotopical elliptic genus (t^{Ell}, h^{Ell}), where Ell_* is the graded ring $\mathbb{Z}[\frac{1}{2}][\delta, \epsilon, (\delta^2 - \epsilon)^{-1}]$, and δ and ϵ have dimensions 4 and 8 respectively. Analytically, it is determined by the elliptic integral

$$(6.5) \qquad m^{H \wedge Ell}(x) \;=\; \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}}$$

over $(H \wedge Ell)_*$, so $b^{H \wedge Ell}(x)$ is a Jacobi sn function modulo constants.

We shall discuss certain generalisations of t_*^{Ell} in the toric context. Their homotopical status remains unresolved, and it is most convenient to work with the analytical and formal group theoretic methods used in their construction. Nevertheless, we proceed with the extension (3.3) closely in mind, and plan to describe integrality and homotopical properties of these genera elsewhere.

Following [16], we consider the formal group law

$$(6.6) \qquad F_b(u_1, u_2) \;=\; u_1 c(u_2) + u_2 c(u_1) - a u_1 u_2 - \frac{d(u_1) - d(u_2)}{u_1 c(u_2) - u_2 c(u_1)} u_1^2 u_2^2$$

over the graded ring $R_* = \mathbb{Z}[a, c_j, d_k : j \geqslant 2, k \geqslant 1]/J$, where $\deg a = 2$, $\deg c_j = 2j$ and $\deg d_k = 2(k+2)$; also, J is the ideal of associativity relations for (6.6), and

$$c(u) := 1 + \sum_{j \geqslant 2} c_j u^j \quad \text{and} \quad d(u) := \sum_{k \geqslant 1} d_k u^k.$$

Theorem 6.7. *The exponential series $f_b(x)$ of (6.6) may be written analytically as $\exp(ax)/\phi(x, z)$, where*

$$(6.8) \qquad \phi(x, z) \;=\; \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} \exp(\zeta(z)\,x),$$

$\sigma(z)$ is the Weierstrass sigma function, and $\zeta(z) = (\ln \sigma(z))'$.

Moreover, $R_* \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}[a, c_2, c_3, c_4]$ as graded algebras.

Proof. The formal group law may be defined by

$$(6.9) \qquad f_b\big(f_b^{-1}(u_1) + f_b^{-1}(u_2)\big) \;=\; F_b(u_1, u_2)$$

over $R_* \otimes \mathbb{Q}$, so the exponential series satisfies the functional equation

$$f(u_1 + u_2) \;=\; f(u_1)c(f(u_2)) + f(u_2)c(f(u_1)) - af(u_1)f(u_2)$$
$$- \frac{d(f(u_1)) - d(f(u_2))}{f(u_1)c(f(u_2)) - f(u_2)c(f(u_1))} f^2(u_1)f^2(u_2)$$

over $R \otimes \mathbb{C}$. Substituting $\xi_1(u) = c^2(f(u)) - af(u)c(f(u)) + d(f(u))f^2(u)$ and $\xi_2(u) = c(f(u))$ gives

$$f(u_1 + u_2) = \frac{f^2(u_1)\xi_1(u_2) - f^2(u_2)\xi_1(u_1)}{f(u_1)\xi_2(u_2) - f(u_2)\xi_2(u_1)}.$$

This equation is considered in [16], and the required form of the solution is established by [16, Theorem 1 and Corollary 8].

Because F_b is commutative, its classifying map $\Omega_*^U \to R_*$ is epimorphic, and $R_* \otimes \mathbb{Q}$ is generated by the coefficients of $f_b(x) = x + \sum_{k \geqslant 1} f_k x^{k+1}$. In order to identify polynomial generators in terms of the elements a, c_j and d_k, it therefore suffices to reduce equation (6.9) modulo decomposable coefficients, and consider

$$\sum_{k \geqslant 1} f_k \big((u_1 + u_2)^{k+1} - u_1^{k+1} - u_2^{k+1}\big)$$

$$\equiv -au_1u_2 + \sum_{k \geqslant 2} c_k\big(u_1 u_2^k + u_1^k u_2\big) - u_1^2 u_2^2 \sum_{k \geqslant 1} d_k(u_1^k - u_2^k)/(u_1 - u_2).$$

Equating coefficients in degrees 4, 6, 8, and 10 then gives

$$(6.10) \qquad 2f_1 = -a, \;\; 3f_2 \equiv c_2, \;\; 4f_3 \equiv c_3, \;\; 6f_3 \equiv -d_1, \;\; 5f_4 \equiv c_4, \;\; 10f_4 \equiv -d_2;$$

in higher degrees the only solutions are $f_k \equiv 0$, so f_k is decomposable for $k \geqslant 5$. $\square$

The function $\varphi(x, z)$ of (6.8) is known as the *Baker–Akhiezer function* associated to the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$. It satisfies the *Lamé equation*, and is important in the theory of nonlinear integrable equations. Krichever [59] studies the genus kv corresponding to the exponential series f_b, which therefore classifies the formal group law (6.6). Analytically, it depends on the four complex variables z, a, g_2 and g_3.

Corollary 6.11. *The genus* $kv \colon \Omega_*^U \to R_*$ *induces an isomorphism of graded abelian groups in dimensions* < 10.

Proof. By (6.10), $kv \otimes 1_{\mathbb{Q}}$ is monomorphic in dimensions < 10; so kv is also monomorphic, because Ω_*^U is torsion-free. Since kv is epimorphic in all dimensions, the result follows. $\square$

Many of the genera considered in previous sections are special cases of kv, and the corresponding methods of calculation may be adapted as required.

Examples 6.12.

(1) The Abel genus ab and 2-parameter Todd genus $t2$ of Examples 3.9(5) may be identified with the cases

$$d(u) = 0, \;\; a = y + z \qquad \text{and} \qquad c(u) = 1 - yzu^2, \;\; d(u) = -yz(y + z)u - y^2z^2u^2$$

respectively. The corresponding formal group laws are

$$F_{ab} = u_1 c(u_2) + u_2 c(u_1) - (y + z)u_1 u_2 \qquad \text{and} \qquad F_{t2} = \frac{u_1 + u_2 - (y + z)u_1 u_2}{1 - yzu_1 u_2}.$$

(2) The elliptic genus t_*^{Ell} of (6.5) corresponds to Euler's formal group law

$$F_{Ell}(u_1, u_2) = \frac{u_1 c(u_2) + u_2 c(u_1)}{1 - \varepsilon u_1^2 u_2^2} = u_1 c(u_2) + u_2 c(u_1) + \varepsilon \frac{u_1^2 - u_2^2}{u_1 c(u_2) - u_2 c(u_1)} u_1^2 u_2^2,$$

and may therefore be identified with the case

$$a = 0, \;\; d(u) = -\varepsilon u^2, \quad \text{and} \quad c^2(u) = R(u) := 1 - 2\delta u^2 + \varepsilon u^4.$$

Theorem 6.13. *Let* M^{2n} *be a specially omnioriented quasitoric manifold; then*
(1) *the Krichever genus* kv *vanishes on* M^{2n}
(2) M^{2n} *represents* 0 *in* Ω_{2n}^U *whenever* $n < 5$.

Proof. (1) Let $T < T^n$ be a generic subcircle, determined by a primitive integral vector $\nu = (\nu_1, \ldots, \nu_n)$. By [75, Theorem 2.1], the fixed points x of T are precisely those of T^n, and have T-weights $w_j(x) \cdot \nu$ for $1 \leqslant j \leqslant n$. By Proposition 5.11 and Theorem 5.16, the sum $s(x) := \sum_j w_j(x) \cdot \nu$ reduces to $\sum_j \nu_j$, which is independent of x and may be chosen to be nonzero. Since M is specially omnioriented, it follows from [59, Theorem 2.1] that $kv(M) = 0$.

(2) Combining (1) with Corollary 6.11 establishes (2). $\qquad\square$

Krichever [59, Lemma 2.1] proves that $s(x)$ is independent of x in more general circumstances.

Conjecture 6.14. Theorem 6.13(2) holds for all n.

6.3. **Rigidity.** We now apply kv, and $t2$ to characterise genera that are T^k-rigid on certain individual stably complex manifolds M, as specified by Definition 3.22. Up to complex cobordism, we may insist that M admits a tangentially stably complex T^k-action with isolated fixed points x, and write the fixed point data as $\{\varsigma(x), w_j(x)\}$. We continue to work analytically.

Proposition 6.15. *For any analytic exponential series f over a $\mathbb{Q}$-algebra A, the genus ℓ_f is T^k-rigid on M only if the functional equation*

$$\sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{f(w_j(x) \cdot u)} \;=\; c$$

is satisfied in $A[[u_1, \ldots, u_k]]$, for some constant $c \in A$.

Proof. Because A is a $\mathbb{Q}$-algebra, f is an isomorphism between the additive formal group law and $F_f(u_1, u_2)$. So the formula follows from Corollary 4.20. $\qquad\square$

The quasitoric examples $\mathbb{C}P^1$, $\mathbb{C}P^2$, and the T^2-manifold S^6 are all instructive.

Example 6.16. A genus ℓ_f is T-rigid on $\mathbb{C}P^1$ with the standard complex structure only if the equation

(6.17) $$\frac{1}{f(u)} + \frac{1}{f(-u)} \;=\; c,$$

holds in $A[[u]]$. The general analytic solution is of the form

$$f(u) \;=\; \frac{u}{q(u^2) + cu/2}, \quad \text{where} \quad q(0) \;=\; 1.$$

The Todd genus of Example 3.9(3) is defined over $\mathbb{Q}[z]$ by $f_{td}(u) = (e^{zu} - 1)/z$, and (6.17) is satisfied with $c = -z$. So td is T-rigid on $\mathbb{C}P^1$, and $q(u^2)$ is given by

$$(zu/2) \cdot \frac{e^{zu/2} + e^{-zu/2}}{e^{zu/2} - e^{-zu/2}}$$

in $\mathbb{Q}[z][[u]]$. In fact td is multiplicative with respect to $\mathbb{C}P^1$ by [49], so rigidity also follows from Section 3.

Example 6.18. A genus ℓ_f is T^2-rigid on the stably complex manifold $\mathbb{C}P^2_{(1,-1)}$ of Example 5.18 only if the equation

$$\frac{1}{f(u_1)f(u_2)} - \frac{1}{f(u_1)f(u_1 + u_2)} + \frac{1}{f(-u_2)f(u_1 + u_2)} \;=\; c$$

holds in $A[[u_1, u_2]]$. Repeating for $\mathbb{C}P^2_{(-1,1)}$ (or reparametrising T^2) gives

(6.19) $$\frac{1}{f(u_2)f(u_1)} - \frac{1}{f(u_2)f(u_1 + u_2)} + \frac{1}{f(-u_1)f(u_1 + u_2)} \;=\; c,$$

and subtraction yields

$$\left(\frac{1}{f(u_1)} + \frac{1}{f(-u_1)}\right)\frac{1}{f(u_1+u_2)} = \left(\frac{1}{f(u_2)} + \frac{1}{f(-u_2)}\right)\frac{1}{f(u_1+u_2)}.$$

It follows that

$$\frac{1}{f(u)} + \frac{1}{f(-u)} = c' \qquad \text{and} \qquad \frac{1}{f(-u)} = c' - \frac{1}{f(u)}$$

in $A[[u]]$, for some constant c'. Substituting in (6.19) gives

$$\left(\frac{1}{f(u_1)} + \frac{1}{f(u_2)} - c'\right)\frac{1}{f(u_1+u_2)} = \frac{1}{f(u_1)f(u_2)} - c,$$

which rearranges to

$$f(u_1+u_2) = \frac{f(u_1) + f(u_2) - c'f(u_1)f(u_2)}{1 - cf(u_1)f(u_2)}.$$

So f is the exponential series of the formal group law $F_{t2}(u_1, u_2)$, with $c' = y + z$ and $c = yz$.

Example 6.18 provides an alternative proof of a recent result of Musin [68].

Proposition 6.20. *The 2-parameter Todd genus t2 is universal for rigid genera.*

Proof. Krichever shows [58] that $t2$ is rigid on tangentially almost complex T-manifolds, by giving a formula for the genus in terms of fixed point sets of arbitrary dimension. His proof automatically extends to the stably complex situation by incorporating the signs of isolated fixed points.

The canonical stably complex structure on $\mathbb{C}P^2_{(1,-1)}$ is T^2-invariant, so universality follows from Example 6.18(2) by restricting to arbitrary subcircles $T < T^2$. □

Example 6.21. A genus ℓ_f is T^2-rigid on the almost complex manifold S^6 of Examples 4.17(2) only if the equation

$$\frac{1}{f(u_1)f(u_2)f(-u_1-u_2)} + \frac{1}{f(-u_1)f(-u_2)f(u_1+u_2)} = c$$

holds in $A[[u_1, u_2]]$, for some constant c. Rearranging as

$$\frac{f(u_1+u_2)}{f(u_1)f(u_2)} - \frac{-f(-u_1-u_2)}{(-f(-u_1))(-f(-u_2))} = cf(u_1+u_2)f(-u_1-u_2)$$

and applying the differential operator $D = \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$ yields

$$(6.22) \quad f(u_1+u_2)D\left(\frac{1}{f(u_1)f(u_2)}\right) = -f(-u_1-u_2)D\left(\frac{1}{(-f(-u_1))(-f(-u_2))}\right).$$

Substituting

$$\phi_1(u_1+u_2) = \frac{f(u_1+u_2)}{-f(-u_1-u_2)}, \quad \phi_2(u) = \frac{1}{-f(-u)}, \quad \phi_4(u) = \frac{1}{f(u)}$$

into (6.22) then gives

$$\phi_1(u_1+u_2) = \frac{\phi_2'(u_1)\phi_2(u_2) - \phi_2(u_1)\phi_2'(u_2)}{\phi_4'(u_1)\phi_4(u_2) - \phi_4(u_1)\phi_4'(u_2)},$$

which is exactly the equation [12, (1)], with $\phi_3(u) = \phi_2'(u)$ and $\phi_5(u) = \phi_4'(u)$. So by [12, Theorem 1], the general analytic solution is of the form $\exp(ax)/\phi(x,z)$, and f coincides with the exponential series f_b of Theorem 6.7.

Theorem 6.23. *The Krichever genus kv is universal for genera that are rigid on SU-manifolds.*

Proof. The rigidity of kv on almost complex manifolds with zero first Chern class is established in [59]; the proof extends immediately to general SU-manifolds, by incorporating signs as in Proposition 6.20.

On S^6, the almost complex T^2-invariant structure of Example 4.17(2) is stably SU, so universality follows from Example 6.21 by restricting to arbitrary subcircles $T < T^2$. □

An alternative characterisation of Krichever's genus is given by Totaro [85, Theorem 6.1], in the course of proving that kv extends to singular complex algebraic varieties. Building on work of Höhn [51], Totaro interprets kv as universal amongst genera that are invariant under SU-*flops*; this property is equivalent to SU-fibre multiplicativity with respect to a restricted class of stably tangentially SU bundles. A second version of the latter approach is dealt with by Braden and Feldman [13].

6.4. **Realising fixed point data.** In 1938, PA Smith considered actions of finite groups G on the sphere. In the case of two fixed points, he asked if the tangential representations necessarily coincide; this became a celebrated question that now bears his name, although the answer is known to be negative for certain G. It suggests the following realisation problem for torus actions.

Problem 6.24. For any set of signs $\varsigma(x)$ and matching weight vectors $w_j(x)$, find necessary and sufficient conditions for the existence of a tangentially stably complex T^k-manifold with the given fixed point data.

This problem was originally addressed by Novikov [69] for $\mathbb{Z}/p$ actions, and his emphasis on the necessity of the Conner-Floyd relations initiated much of the subsequent output of the Moscow school. It was also a motivating factor for tom Dieck [37], who provided a partial answer in terms of the K-theoretic Hurewicz genus and the Hattori-Stong theorem. We have introduced the relations in Proposition 4.14, and the analytic viewpoint provides an elegant way of combining their implications under the action of an appropriate genus.

Theorem 6.25. *For any exponential series f over a $\mathbb{Q}$-algebra A_*, the formal Laurent series*

$$\sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^n \frac{1}{f(w_j(x) \cdot u)}$$

lies in $A[[u_1, \ldots, u_k]]$, and has zero principal part.

As f varies over formal exponential series, Theorem (6.25) provides strong restrictions on the fixed point data.

Example 6.26. Applying Example 4.23 in case $E = M$ and $X = *$ shows that the augmentation genus ag constrains the fixed point data of M to satisfy the condition

$$\sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^n \frac{1}{w_j(x) \cdot u} = 0$$

in $\mathbb{Z}[[u_1, \ldots, u_k]]$. Homotopical versions of this description are implicit in Remarks 4.21 and Example 4.23.

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