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**Abstract**

Let \( G \) denote a split simply connected almost simple \( p \)-adic group. We study the unramified \( C^* \)-algebra of \( G \) and prove that the rank of the \( K \)-theory group \( K_0 \) is the connection index \( f(G) \).

**1 Introduction**

Let \( k \) be a global field, let \( \mathbb{A}_k \) denote ring of adeles over \( k \). Let \( \pi \) denote an automorphic representation of the adelic group \( G(\mathbb{A}_k) \). Then we have the restricted product

\[
\pi = \bigotimes_v \pi_v
\]

over all the places \( v \) of \( k \). All but finitely many of the local representations \( \pi_v \) admit \( K_v \)-fixed vectors: they are unramified. In this Note, we investigate the unramified \( C^* \)-algebra attached to a local group \( G_v \).

One point of interest is that one has to go outside the affine Weyl group \( W_a \) to the extended affine Weyl group \( W'_a \): in fact, the crucial finite group in this context is the quotient group \( W'_a/W_a \).

Quite specifically, let \( F \) be a local nonarchimedean field of characteristic 0, let \( \mathcal{G} \) be the group of \( F \)-rational points in a split, almost simple, simply connected, semisimple linear algebraic group defined over \( F \), for example \( \text{SL}(n) \). Let \( S \) denote a maximal split torus of \( \mathcal{G} \). Let \( \mathcal{G}^\vee, S^\vee \) denote the Langlands dual groups, and let \( G, T \) denote maximal compact subgroups:

\[
G \subset \mathcal{G}^\vee, \quad T \subset S^\vee.
\]

Then \( G \) is a compact Lie group with maximal torus \( T \).

We will consider the unramified \( C^* \)-algebra \( \mathfrak{A} = \mathfrak{A}(\mathcal{G}) = C^*_{nr}(\mathcal{G}) \) of \( \mathcal{G} \). Let

\[
\pi(t) := \text{Ind}^G_{S^\vee}(t \otimes 1)
\]

be the unramified unitary principal series of \( \mathcal{G} \), so that \( t \in \Psi(S) \) is an unramified unitary character of \( S \). Let \( C(t) \) be the commuting algebra of \( \pi(t) \). Then

\[
C(t) \cong \mathbb{C}[R(t)],
\]

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where $R(t)$ is the $R$-group of $t$, see [6]. The $R$-groups are tabulated by Keys [6].

Let $\Phi^\vee$ be the coroot system associated to $\Phi$. Let $Q(\Phi)$ be the root lattice, $P(\Phi)$ be the weight lattice; each weight takes integer values on the coroot lattice. Let $P(\Phi)/Q(\Phi)$ be the quotient group. The order of the group $P(\Phi)/Q(\Phi)$ is called the connection index $f = f(\mathcal{G})$ of $\mathcal{G}$.

The abelian groups $P/Q$ are tabulated in [4, Plates I–X, p.265–292]. For $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ they are, respectively

$\mathbb{Z}/(n+1)\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \ (n \text{ odd}), \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ (n \text{ even})$

$\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, 0, 0$

Here is the striking fact: the maximal $R$-groups are isomorphic to the fundamental groups $P/Q$. To see this, it is enough to compare the $R$-group computations in [6] with the above list. In this Note, we attempt to explain this fact, via the elementary geometry of the Lie algebra $t$: the vector space $t$ has the structure of a Euclidean space tessellated with alcoves. In the present context, we need a refinement of this tessellation. For example, if $\mathcal{G} = \text{SL}(3)$, then the standard tiling of the Euclidean plane by equilateral triangles is refined to produce a tiling by isosceles triangles (see the Figures in section 3). Reducibility in the unramified unitary principal series is determined by the fundamental group $P/Q$.

**Theorem 1.1.** Let $\mathcal{G}$ be simply connected, let $\mathfrak{A} = C^*_\text{nr}(\mathcal{G})$ and let $f(\mathcal{G})$ be the connection index of $\mathcal{G}$. Then we have

\[
K_0(\mathfrak{A}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (f(\mathcal{G}) \text{ copies of } \mathbb{Z})
\]

\[
K_1(\mathfrak{A}) = 0
\]

$K_0$ detects the $L$-packet in the unramified unitary principal series with the maximal number of constituents.

## 2 The alcoves in $t$

Our reference at this point is [4, Lie IX, p. 309–327]. The proof hinges on the distinction between the affine Weyl group and the extended affine Weyl group. Let $t$ denote the Lie algebra of $T$, and let $\exp : t \to T$ denote the exponential map. The kernel of $\exp$ is denoted $\Gamma(T)$. Denote by $\mathcal{N}(G,T)$ the subgroup of $\Gamma(T)$ generated by the nodal vectors. The quotient $\Gamma(T)/\mathcal{N}(G,T)$ can be identified with the fundamental group of $G$. Thanks to the identification of the nodal vector $K_\alpha$ with the coroot $\alpha^\vee$, the group
$N(G, T)$ can be identified with the group $Q(\Phi^\vee)$ generated by the set $\Phi^\vee$ of coroots.

The affine Weyl group is $W_a = N(G, T) \rtimes W$ and the extended affine Weyl group is $W'_a = \Gamma(T) \rtimes W$; the subgroup $W_a$ of $W'_a$ is normal. The group $W$ (resp. $W_a$) operates simply-transitively on the set of chambers (resp. alcoves). Let $C$ be a chamber and $A$ an alcove. Then $C$ (resp. $\overline{A}$) is a fundamental domain for the operation of $W$ on $t$ (resp. of $W_a$ on $t$). Let $H_A$ be the stabilizer of $A$ in $W'_a$. Then $W'_a$ is the semi-direct product

$$W'_a = W_a \rtimes H_A.$$ 

We have to consider the $C^*$-summand, $\mathfrak{A}(G)$ in the reduced $C^*$-algebra $C^*_r(G)$ which corresponds to the unramified unitary principal series, see [6]. The algebra $C^*_r(G)$ is defined as follows. We choose a left-invariant Haar measure on $G$, and form a Hilbert space $L^2(G)$. The left regular representation $\lambda$ of $L^1(G)$ on $L^2(G)$ is given by

$$(\lambda(f))(h) = f * h$$

where $f \in L^1(G)$, $h \in L^2(G)$ and $*$ denotes the convolution. The $C^*$-algebra generated by the image of $\lambda$ is the reduced $C^*$-algebra $C^*_r(G)$. Let $\mathcal{K}$ denote the $C^*$-algebra of compact operators on the standard Hilbert space $H$.

At this point we use Langlands duality. The Langlands dual of $G$ is the complex reductive group $G^\vee$ with maximal torus $S^\vee$. Let $G$ be a maximal compact subgroup of $G^\vee$, let $T$ be the maximal compact subgroup of $S^\vee$.

The group $\Psi(S)$ of unramified unitary characters of $S$ is isomorphic to $T$. The unramified $C^*$-algebra $C^*_{nr}(G)$ is given by the fixed point algebra

$$C^*_{nr}(G) = C(T, \mathcal{K})^W = \{ f \in C(T, \mathcal{K}) : f(wt) = \mathfrak{c}(w : t) \cdot f(t), w \in W \}$$

where $\mathfrak{a}(w : t)$ are normalized intertwining operators, and

$$\mathfrak{c}(w : t) = \text{Ad} \, \mathfrak{a}(w : t)$$

as in [7]. Then $\mathfrak{a} : W \to C(T, U(H))$ is a 1-cocycle:

$$\mathfrak{a}(w_2w_1 : t) = \mathfrak{a}(w_2 : w_1t)\mathfrak{a}(w_1 : t).$$

We have $\overline{A} \subset t$. The group $H_A$ is defined as the stabilizer in $W'_a$ of $\overline{A}$:

$$H_A := \{ w \in W'_a : w\overline{A} \subset \overline{A} \}$$

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Then $H_A$ is a finite abelian group [4, Ch.IX, p.326]. View $t$ as an additive group, and form the Euclidean group $t \rtimes O(t)$. We have $W'_a \subset t \rtimes O(t)$ and so $W'_a$ acts as affine transformations of $t$.

The **special points** of $W_a$ are the elements $x$ of $t$ such that $\exp x \in C(G)$, see [4, Ch. IX, p.326]. Let $S$ be the set of special points of $\overline{A}$. The group $H_A$ acts freely on $S$, by [4, Ch. IX, p.327]. Let $x_0$ denote the barycentre of $S$, i.e.

$$x_0 := \frac{1}{|S|} \cdot \sum_{s_j \in S} s_j.$$  

The group $H_A$ will permute the special points $s \in S$. Note that

$$w \cdot x_0 = \frac{1}{|S|} \sum_{s_j \in S} w \cdot s_j = \frac{1}{|S|} \sum_{s_j \in S} s_j = x_0$$

so that $x_0$ is $H_A$-fixed. Then $\overline{A}$ is equivariantly contractible to $x_0$:

$$r_t(x) := (1 - t)x_0 + tx$$  \hspace{1cm} (1)$$

with $0 \leq t \leq 1$. This is an affine $H_A$-equivariant contraction from $\overline{A}$ to $x_0$.

**Lemma 2.1.** Let $t_0 = \exp x_0$. There is an isomorphism of abelian groups:

$$H_A \simeq W(t_0) \simeq R(t_0)$$

where $R(t_0)$ is the $R$-group of $t_0$.

**Proof.** The exponential map $\exp : t \to T$ commutes with the action of $W$. Let $h = \gamma w = (\gamma, w) \in \Gamma(T) \rtimes W$. Then

$$hx_0 = x_0 \implies \gamma \in \Gamma(T) \implies \exp hx_0 = \exp wx_0 = w \exp x_0 = wt_0$$

This determines the map

$$H_A \longrightarrow W(t_0), \quad \gamma w \mapsto w$$

Now

$$w(t_0) = t_0 \implies (\exists \gamma \in \Gamma(T)) \gamma(wx_0) = x_0 \implies \gamma w \in H_A$$

so that $H_A \to W(t_0)$ is surjective. The map is injective because

$$(\gamma_1, w) = (\gamma_2, w) \implies \gamma_1 = \gamma_2$$

This creates the canonical isomorphism of abelian groups:

$$H_A \simeq W(t_0)$$
In fact, $W(t_0)$ is the image of $H_A$ in the canonical map $W'_a \to W$ is $W(t_0)$:

$$\Gamma(T) \times W \to W, \quad H_A \simeq W(t_0)$$

In general, we have $R(t_0) \subset W(t_0)$. The calculations of Keys [6, Theorem in §3] prove that $R(t_0) = W(t_0)$. □

**Theorem 2.2.** The group $K_0(C^*_n(G))$ is free abelian on $f(G)$ generators, and $K_1(C^*_n(G)) = 0$.

**Proof.** We have the exponential map $\exp : t \to T$. We lift $f$ from $T$ to a periodic function $F$ on $t$, and lift $a$ from a 1-cocycle $a : W \to C(T, U(H))$ to a 1-cocycle $b : W'_a \to C(t, U(H))$:

$$F(x) := f(\exp x), \quad b(w' : x) := a(w : \exp x)$$

with $w' = (\gamma, w)$. The semidirect product rule is

$$w'_1w'_2 = (\gamma_1, w_1)(\gamma_2, w_2) = (\gamma_1w_1(\gamma_2), w_1w_2).$$

Note that $b$ is still a 1-cocycle:

$$b(w'_2w'_1 : x) = a(w_2w_1 : \exp x)$$

$$= a(w_2 : \exp x)a(w_1 : \exp x)$$

$$= a(w_2 : \exp w_1x)b(w'_1 : x)$$

$$= a(w_2 : \exp \gamma w_1x)b(w'_1 : x)$$

$$= b(w'_2 : w'_1x)b(w'_1 : x)$$

Now we define

$$d(w : x) := Ad \cdot b(w : x), \quad w \in W'_a$$

The fixed algebra $C^*_n(G)$ is as follows:

$$\{ F \in C(t, \mathfrak{K}) : F(wx) = d(w : x) \cdot F(x), w \in W'_a, x \in t, F \text{ periodic} \}$$

Now $F$ is determined by its restriction to $\overline{A}$. Upon restriction, we obtain

$$C^*_n(G) \simeq \{ f \in C(\overline{A}, \mathfrak{K}) : f(wx) = d(w : x) \cdot f(x), w \in H_A, x \in \overline{A} \}$$

We want to make the $b(w, x)$ independent of $x \in \overline{A}$. At this point, we adapt a proof in Solleveld [9]. We will write $H = H_A$. Let

$$\mathfrak{C} := C(\overline{A}, \mathfrak{K}), \quad \mathfrak{B} := C(\overline{A}, \mathfrak{L}(V))$$
so that $C_{\text{nr}}^*(G) = \mathcal{C}^H$. Let $\mathcal{C} \rtimes H$ be the crossed product of $\mathcal{C}$ and $H$ with respect to the action of $H$ on $\mathcal{C}$ via the intertwiners $b(w : x)$. Let $r_t$ be defined as in Eq.(1). Define

$$p_t(x) := \frac{1}{|H|} \sum_{w \in H} b(w : r_t x) \cdot w$$  \hspace{1cm} (3)

Then $p_t \in \mathfrak{B} \rtimes H$ is an idempotent by Eq.(3). By [8], we have an isomorphism of $C^*$-algebras:

$$\phi_1 : \mathcal{C}^H \simeq p_1(\mathcal{C} \rtimes H)p_1, \quad \sigma \mapsto p_1 \sigma p_1$$

Clearly the idempotents $p_t$ are all homotopic, so they are conjugate in $\mathfrak{B} \rtimes H$, see [3, Proposition 4.3.3], e.g. $p_1 = a p_0 a^{-1}$. Now $\mathcal{C} \rtimes H$ is a $C^*$-ideal in $\mathfrak{B} \rtimes H$, and so we have

$$\mathcal{C}^H \simeq p_1(\mathcal{C} \rtimes H)p_1 = ap_0 a^{-1}(\mathcal{C} \rtimes H)ap_0 a^{-1} = ap_0(\mathcal{C} \rtimes H)p_0 a^{-1} \simeq p_0(\mathcal{C} \rtimes H)p_0$$

as conjugate $C^*$-algebras in $\mathfrak{B} \rtimes H$. Define

$$\mathcal{C}^H(0) := \{ f \in C(\overline{A}, \mathfrak{A}) : f(wx) = d(w : x_0) \cdot f(x), w \in H_A, x \in \overline{A} \}$$

We have the the $C^*$-algebra isomorphism

$$\mathcal{C}^H(0) \simeq p_0(\mathcal{C} \rtimes H)p_0, \quad \sigma \mapsto p_0 \sigma p_0$$

and so

$$\mathcal{C}^H \simeq p_0(\mathcal{C} \rtimes H)p_0 \simeq \mathcal{C}(0)^H$$

We have replaced the intertwiners $b(w : x)$ by the constant (i.e. independent of $x$) intertwiners $b(w : x_0)$.

To the algebra $\mathcal{C}(0)^H$ we will apply the homotopy $r_t$. This shows that $\mathcal{C}^H(0)$ is homotopy-equivalent to its fiber $\mathfrak{R}(V_0)^H$ over $x_0$. Each element in $\mathfrak{R}(V_0)^H$ must commute with the intertwiners $\{b(x_0 : h) : h \in H_A\}$. Now we have

$$\{b(x_0 : h) : h \in H_A\} = \{a(t_0 : w) : w \in W(t_0)\}$$

by Lemma 2.1 and Eq.(2). So each element in $\mathfrak{R}(V_0)^H$ must commute with the commuting algebra

$$\mathcal{C}(t_0) \simeq \mathbb{C}[H_A]$$

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Since $H_A$ is an abelian group of order $f$, we infer that
\[ \mathcal{R}(V_0)^H \simeq \mathcal{R} \oplus \ldots \oplus \mathcal{R} \]
with $f$ copies of $\mathcal{R}$. It is now immediate that $K_0(\mathfrak{A}) = \mathbb{Z}^f$, $K_1(\mathfrak{A}) = 0$.

The logic of the above proof is this:
\[ C^\ast_{nr}(G) = \mathcal{C}^H \simeq \mathcal{C}^H(0) \sim \mathcal{R}(V_0)^H \simeq \mathbb{R}^f \]

The unramified unitary principal series is reducible at $t_0$, and it splits the Hilbert space $V_0$ into $f$ irreducible subspaces $H_1, \ldots, H_f$. Let $E_j : V_0 \to H_j$ be a rank-one projection. Let $e_j \in \mathfrak{A}$ be such that $e_j(x_0) = E_j$. Then \{e_1, \ldots, e_f\} is a set of generators in $K_0(\mathfrak{A})$.

3 Examples

The exponential map $\exp : t \to T$ induces a homeomorphism
\[ \mathcal{A}/H_A \simeq T/W \]
by [4, Ch. IX, p.326], and $\mathcal{A}/H_A$ is equivariantly contractible. The map $r_t$ in Eq. (1) descends to the quotient $\mathcal{A}/H_A$ and shows that the quotient $\mathcal{A}/H_A$ is a contractible space. Therefore, $T/W$ is contractible.

If $G = E_8$, $F_4$ or $G_2$ then we have $P/Q = 0$ and so the connection index $f = 1$, see [4, Plates VIII, IX, X]. Therefore
\[ T/W \simeq \mathcal{A}. \]

This is a contractible space. There are no $L$-packets. The $K$-theory of the unramified unitary principal series of $G$ is that of a point.

From now on, let $G = SL(\ell)$ with $\ell$ a prime number. We have $G' = PGL(\ell, \mathbb{C})$, $G = PU(\ell, \mathbb{C})$ and $T$ is a maximal torus in $PU(\ell, \mathbb{C})$. Then the Weyl group $W$ is the symmetric group $S_\ell$. Let $\gamma$ denote the standard $\ell$-cycle $(123 \ldots \ell) \in S_\ell$. The fixed set of $\gamma$ is
\[ t_0 = \exp x_0 = (1 : \omega : \omega^2 : \cdots : \omega^{\ell-1}) \in T^\ell/\mathbb{T} \]
where $\omega = \exp(2\pi i/\ell)$. The $\mathcal{L}$-group attached to $t_0$ is
\[ \mathcal{L}(t_0) = < \omega^\text{val} > = \mathbb{Z}/\ell\mathbb{Z} \]
and the $X$-group is trivial, so that the $R$-group is given by

$$R(t_0) = \mathbb{Z}/\ell\mathbb{Z}.$$ 

The $\ell$ irreducible constituents of $\text{Ind}_R^G(t_0)$ are \textit{elliptic} representations [5].

Let $\overline{A}/H_A$ denote the extended quotient of $\overline{A}$ by $H_A$. The canonical projection

$$\pi : \overline{A}/H_A \to A/H_A$$

is a model of reducibility at the point $t_0$. The fibre $\pi^{-1}(t_0)$ contains $\ell$ points. This is very much in the spirit of the recent geometric conjecture in [2].

Note that $f = \ell$ as required. The unramified unitary principal series of $\text{SL}(\ell)$ admits a unique $L$-packet with $\ell$ elliptic constituents, and this is detected by $K_0$. For $\text{SL}(2)$ we have $K_0 = \mathbb{Z} \oplus \mathbb{Z}$, $K_1 = 0$.

Let $G = \text{SL}(3)$. Then the vector space $t$ is the Euclidean plane $\mathbb{R}^2$. The singular hyperplanes [4, Ch. IX, p.325] tessellate $\mathbb{R}^2$ into equilateral triangles, as in Figures 1 and 2. This tessellation of the plane appears in [1, p.104] as the Stiefel diagram for $\text{SU}(3,\mathbb{C})$. The interior of each equilateral triangle is an \textit{alcove}. The closure of an alcove $A$ is denoted $\overline{A}$. The affine Weyl group $W_a$ operates simply transitively on the set of alcoves, and $\overline{A}$ is a fundamental domain for the action of $W_a$ on $t$, see [4, p.326]. There is a finer tessellation into isosceles triangles, as in Figures 1 and 2. An isosceles triangle in this tessellation will be denoted $\Delta$. The extended affine Weyl group $W'_a$ acts simply transitively on the set of such isosceles triangles $\Delta$, but the closure $\overline{\Delta}$ is not a fundamental domain for the action of $W'_a$. Quotient spaces are as follows:

$$t/W'_a \simeq \overline{A}/H_A$$

The abelian group $H_A$ is the cyclic group $\mathbb{Z}/3\mathbb{Z}$ which acts on $\overline{A}$ by rotation through $2\pi/3$. If $\overline{A} = oab$ then the special points of $\overline{A}$ are the vertices $\{o, a, b\}$. The quotient $\overline{A}/H_A$ is an identification space of $\overline{\Delta}$: identify two adjacent edges of $\overline{\Delta}$. This space is contractible.

Consider the closed set $oabc$ in Figure 1. This closed set is the closure of the union of two adjacent alcoves $A_1, A_2$ and is also the closure of the union of 6 isosceles triangles $\Delta_1, \ldots, \Delta_6$. The torus $T$ is an identification space of the closed set $oabc$: identify $oa$ with $cb$ and $oc$ with $ab$. 
References


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