K-theory and the connection index

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Abstract

Let $G$ denote a split simply connected almost simple $p$-adic group. We study the unramified $C^*$-algebra of $G$ and prove that the rank of the $K$-theory group $K_0$ is the connection index $f(G)$.

1 Introduction

Let $k$ be a global field, let $\mathbb{A}_k$ denote ring of adeles over $k$. Let $\pi$ denote an automorphic representation of the adelic group $G(\mathbb{A}_k)$. Then we have the restricted product

$$\pi = \bigotimes_v \pi_v$$

over all the places $v$ of $k$. All but finitely many of the local representations $\pi_v$ admit $K_v$-fixed vectors: they are unramified. In this Note, we investigate the unramified $C^*$-algebra attached to a local group $G_v$.

One point of interest is that one has to go outside the affine Weyl group $W_a$ to the extended affine Weyl group $W'_a$: in fact, the crucial finite group in this context is the quotient group $W'_a/W_a$.

Quite specifically, let $F$ be a local nonarchimedean field of characteristic 0, let $G$ be the group of $F$-rational points in a split, almost simple, simply connected, semisimple linear algebraic group defined over $F$, for example $\text{SL}(n)$. Let $S$ denote a maximal split torus of $G$. Let $G^\vee, S^\vee$ denote the Langlands dual groups, and let $G, T$ denote maximal compact subgroups:

$$G \subset G^\vee, \quad T \subset S^\vee.$$ 

Then $G$ is a compact Lie group with maximal torus $T$.

We will consider the unramified $C^*$-algebra $\mathfrak{A} = \mathfrak{A}(G) = C^*_u(G)$ of $G$. Let

$$\pi(t) := \text{Ind}_{S^\vee}^G(t \otimes 1)$$

be the unramified unitary principal series of $G$, so that $t \in \Psi(S)$ is an unramified unitary character of $S$. Let $\mathcal{C}(t)$ be the commuting algebra of $\pi(t)$. Then

$$\mathcal{C}(t) \cong \mathbb{C}[R(t)],$$
where \( R(t) \) is the \( R \)-group of \( t \), see [6]. The \( R \)-groups are tabulated by Keys [6].

Let \( \Phi^\vee \) be the coroot system associated to \( \Phi \). Let \( Q(\Phi) \) be the root lattice, \( P(\Phi) \) be the weight lattice; each weight takes integer values on the coroot lattice. Let \( P(\Phi)/Q(\Phi) \) be the quotient group. The order of the group \( P(\Phi)/Q(\Phi) \) is called the connection index \( f = f(\mathcal{G}) \) of \( \mathcal{G} \).

The abelian groups \( P/Q \) are tabulated in [4, Plates I–X, p.265–292]. For \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \) they are, respectively

\[
\mathbb{Z}/(n+1)\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \quad (n \text{ odd}), \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad (n \text{ even})
\]

\[
\mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad 0, \quad 0, \quad 0
\]

Here is the striking fact: the maximal \( R \)-groups are isomorphic to the fundamental groups \( P/Q \). To see this, it is enough to compare the \( R \)-group computations in [6] with the above list. In this Note, we attempt to explain this fact, via the elementary geometry of the Lie algebra \( \mathfrak{t} \): the vector space \( \mathfrak{t} \) has the structure of a Euclidean space tessellated with alcoves. In the present context, we need a refinement of this tessellation. For example, if \( \mathcal{G} = \text{SL}(3) \), then the standard tiling of the Euclidean plane by equilateral triangles is refined to produce a tiling by isosceles triangles (see the Figures in section 3). Reducibility in the unramified unitary principal series is determined by the fundamental group \( P/Q \).

**Theorem 1.1.** Let \( \mathcal{G} \) be simply connected, let \( \mathfrak{a} = C^*_{nr}(\mathcal{G}) \) and let \( f(\mathcal{G}) \) be the connection index of \( \mathcal{G} \). Then we have

\[
K_0(\mathfrak{a}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (f(\mathcal{G}) \text{ copies of } \mathbb{Z})
\]

\[
K_1(\mathfrak{a}) = 0
\]

\( K_0 \) detects the \( L \)-packet in the unramified unitary principal series with the maximal number of constituents.

## 2 The alcoves in \( \mathfrak{t} \)

Our reference at this point is [4, Lie IX, p. 309–327]. The proof hinges on the distinction between the affine Weyl group and the extended affine Weyl group. Let \( \mathfrak{t} \) denote the Lie algebra of \( T \), and let \( \exp : \mathfrak{t} \to T \) denote the exponential map. The kernel of \( \exp \) is denoted \( \Gamma(T) \). Denote by \( N(G, T) \) the subgroup of \( \Gamma(T) \) generated by the nodal vectors. The quotient \( \Gamma(T)/N(G, T) \) can be identified with the fundamental group of \( G \). Thanks to the identification of the nodal vector \( K_\alpha \) with the coroot \( \alpha^\vee \), the group
$N(G,T)$ can be identified with the group $Q(\Phi^\vee)$ generated by the set $\Phi^\vee$ of coroots.

The affine Weyl group is $W_a = N(G,T) \rtimes W$ and the extended affine Weyl group is $W'_a = \Gamma(T) \rtimes W$; the subgroup $W_a$ of $W'_a$ is normal. The group $W$ (resp. $W_a$) operates simply-transitively on the set of chambers (resp. alcoves). Let $C$ be a chamber and $A$ an alcove. Then $\overline{C}$ (resp. $\overline{A}$) is a fundamental domain for the operation of $W$ on $t$ (resp. of $W_a$ on $t$). Let $H_A$ be the stabilizer of $A$ in $W'_a$. Then $W'_a$ is the semi-direct product $W'_a = W_a \rtimes H_A$.

We have to consider the $C^*$-summand, $\mathfrak{A}(G)$ in the reduced $C^*$-algebra $C^*_r(G)$ which corresponds to the unramified unitary principal series, see [6]. The algebra $C^*_r(G)$ is defined as follows. We choose a left-invariant Haar measure on $G$, and form a Hilbert space $L^2(G)$. The left regular representation $\lambda$ of $L^1(G)$ on $L^2(G)$ is given by

$$(\lambda(f))(h) = f * h$$

where $f \in L^1(G)$, $h \in L^2(G)$ and $*$ denotes the convolution. The $C^*$-algebra generated by the image of $\lambda$ is the reduced $C^*$-algebra $C^*_r(G)$. Let $\mathcal{K}$ denote the $C^*$-algebra of compact operators on the standard Hilbert space $H$.

At this point we use Langlands duality. The Langlands dual of $G$ is the complex reductive group $G^\vee$ with maximal torus $S^\vee$. Let $G$ be a maximal compact subgroup of $G^\vee$, let $T$ be the maximal compact subgroup of $S^\vee$.

The group $\Psi(S)$ of unramified unitary characters of $S$ is isomorphic to $T$. The unramified $C^*$-algebra $C^*_{nr}(G)$ is given by the fixed point algebra

$$C^*_{nr}(G) = C(T,\mathcal{K})^W = \{ f \in C(T,\mathcal{K}) : f(wt) = c(w:t) \cdot f(t), w \in W \}$$

where $a(w:t)$ are normalized intertwining operators, and

$$c(w:t) = \text{Ad } a(w:t)$$

as in [7]. Then $a : W \to C(T,U(H))$ is a 1-cocycle:

$$a(w_2w_1:t) = a(w_2 : w_1t)a(w_1 : t).$$

We have $\overline{A} \subset t$. The group $H_A$ is defined as the stabilizer in $W'_a$ of $\overline{A}$:

$$H_A := \{ w \in W'_a : w\overline{A} \subset \overline{A} \}$$
Then $H_A$ is a finite abelian group [4, Ch.IX, p.326]. View $t$ as an additive group, and form the Euclidean group $t \rtimes O(t)$. We have $W'_a \subset t \rtimes O(t)$ and so $W'_a$ acts as affine transformations of $t$.

The special points of $W_a$ are the elements $x$ of $t$ such that $\exp x \in C(G)$, see [4, Ch. IX, p.326]. Let $S$ be the set of special points of $\overline{A}$. The group $H_A$ acts freely on $S$, by [4, Ch. IX, p.327]. Let $x_0$ denote the barycentre of $S$, i.e.

$$ x_0 := \frac{1}{|S|} \sum_{s_j \in S} s_j. $$

The group $H_A$ will permute the special points $s \in S$. Note that

$$ w \cdot x_0 = \frac{1}{|S|} \sum_{s_j \in S} w \cdot s_j = \frac{1}{|S|} \sum_{s_j \in S} s_j = x_0 $$

so that $x_0$ is $H_A$-fixed. Then $\overline{A}$ is equivariantly contractible to $x_0$:

$$ r_t(x) := (1 - t)x_0 + tx $$

with $0 \leq t \leq 1$. This is an affine $H_A$-equivariant contraction from $\overline{A}$ to $x_0$.

**Lemma 2.1.** Let $t_0 = \exp x_0$. There is an isomorphism of abelian groups:

$$ H_A \simeq W(t_0) \simeq R(t_0) $$

where $R(t_0)$ is the $R$-group of $t_0$.

**Proof.** The exponential map $\exp : t \to T$ commutes with the action of $W$. Let $h = \gamma w = (\gamma, w) \in \Gamma(T) \rtimes W$. Then

$$ h x_0 = x_0 \implies \gamma \in \Gamma(T) \implies \exp h x_0 = \exp w x_0 = w \exp x_0 = w t_0 $$

This determines the map

$$ H_A \longrightarrow W(t_0), \quad \gamma w \mapsto w $$

Now

$$ w(t_0) = t_0 \implies (\exists \gamma \in \Gamma(T)) \gamma(w x_0) = x_0 \implies \gamma w \in H_A $$

so that $H_A \to W(t_0)$ is surjective. The map is injective because

$$ (\gamma_1, w) = (\gamma_2, w) \implies \gamma_1 = \gamma_2 $$

This creates the canonical isomorphism of abelian groups:

$$ H_A \simeq W(t_0) $$
In fact, $W(t_0)$ is the image of $H_A$ in the canonical map $W'_a \to W$ is $W(t_0)$:

$$\Gamma(T) \times W \to W, \quad H_A \simeq W(t_0)$$

In general, we have $R(t_0) \subset W(t_0)$. The calculations of Keys [6, Theorem in §3] prove that $R(t_0) = W(t_0)$.

**Theorem 2.2.** The group $K_0(C^*_\text{nr}(G))$ is free abelian on $f(G)$ generators, and $K_1(C^*_\text{nr}(G)) = 0$.

**Proof.** We have the exponential map $\exp : t \to T$. We lift $f$ from $T$ to a periodic function $F$ on $t$, and lift $a$ from a 1-cocycle $a : W \to C(T, U(H))$ to a 1-cocycle $b : W'_a \to C(t, U(H))$:

$$F(x) := f(\exp x), \quad b(w' : x) := a(w : \exp x)$$

with $w' = (\gamma, w)$. The semidirect product rule is

$$w'_1w'_2 = (\gamma_1w_1)(\gamma_2w_2) = (\gamma_1w_1(\gamma_2), w_1w_2).$$

Note that $b$ is still a 1-cocycle:

$$b(w'_2w'_1 : x) = a(w_2w_1 : \exp x)$$

$$= a(w_2 : w_1(\exp x))a(w_1 : \exp x)$$

$$= a(w_2 : \exp w_1x)b(w'_1 : x)$$

$$= a(w_2 : \exp \gamma w_1x)b(w'_1 : x)$$

$$= b(w'_2 : w'_1x)b(w'_1 : x)$$

Now we define

$$d(w : x) := Adb(w : x), \quad w \in W'_a$$

(2)

The fixed algebra $C^*_\text{nr}(G)$ is as follows:

$$\{F \in C(t, \mathfrak{R}) : F(wx) = \varnothing(w : x) \cdot F(x), w \in W'_a, x \in t, F \text{ periodic}\}$$

Now $F$ is determined by its restriction to $\overline{A}$. Upon restriction, we obtain

$$C^*_\text{nr}(G) \simeq \{f \in C(\overline{A}, \mathfrak{R}) : f(wx) = \varnothing(w : x) \cdot f(x), w \in H_A, x \in \overline{A}\}$$

We want to make the $b(w, x)$ independent of $x \in \overline{A}$. At this point, we adapt a proof in Solleveld [9]. We will write $H = H_A$. Let

$$\mathfrak{C} := C(\overline{A}, \mathfrak{R}), \quad \mathfrak{B} := C(\overline{A}, \mathcal{L}(V))$$

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so that $C^*_{nr}(G) = \mathcal{C}^H$. Let $\mathcal{C} \rtimes H$ be the crossed product of $\mathcal{C}$ and $H$ with respect to the action of $H$ on $\mathcal{C}$ via the intertwiners $b(w : x)$. Let $r_t$ be defined as in Eq.(1). Define

$$p_t(x) := \frac{1}{|H|} \sum_{w \in H} b(w : r_t x) \cdot w$$

(3)

Then $p_t \in \mathfrak{B} \rtimes H$ is an idempotent by Eq.(3). By [8], we have an isomorphism of $C^*$-algebras:

$$\phi_1 : \mathcal{C}^H \simeq p_1(\mathcal{C} \rtimes H)p_1, \quad \sigma \mapsto p_1 \sigma p_1$$

Clearly the idempotents $p_t$ are all homotopic, so they are conjugate in $\mathfrak{B} \rtimes H$, see [3, Proposition 4.3.3], e.g. $p_1 = a p_0 a^{-1}$. Now $\mathcal{C} \rtimes H$ is a $C^*$-ideal in $\mathfrak{B} \rtimes H$, and so we have

$$\mathcal{C}^H \simeq p_1(\mathcal{C} \rtimes H)p_1$$

$$= a p_0 a^{-1}(\mathcal{C} \rtimes H) a p_0 a^{-1}$$

$$= a p_0(\mathcal{C} \rtimes H) p_0 a^{-1}$$

$$\simeq p_0(\mathcal{C} \rtimes H)p_0$$

as conjugate $C^*$-algebras in $\mathfrak{B} \rtimes H$. Define

$$\mathcal{C}^H(0) := \{ f \in C(\mathcal{A}, \mathfrak{K}) : f(wx) = v(w : x_0) \cdot f(x), w \in H_A, x \in \overline{A} \}$$

We have the the $C^*$-algebra isomorphism

$$\mathcal{C}^H(0) \simeq p_0(\mathcal{C} \rtimes H)p_0, \quad \sigma \mapsto p_0 \sigma p_0$$

and so

$$\mathcal{C}^H \simeq p_0(\mathcal{C} \rtimes H)p_0 \simeq \mathcal{C}(0)^H$$

We have replaced the intertwiners $b(w : x)$ by the constant (i.e. independent of $x$) intertwiners $b(w : x_0)$.

To the algebra $\mathcal{C}(0)^H$ we will apply the homotopy $r_t$. This shows that $\mathcal{C}^H(0)$ is homotopy-equivalent to its fiber $\mathfrak{K}(V_0)^H$ over $x_0$. Each element in $\mathfrak{K}(V_0)^H$ must commute with the intertwiners $\{ b(x_0 : h) : h \in H_A \}$. Now we have

$$\{ b(x_0 : h) : h \in H_A \} = \{ a(t_0 : w) : w \in W(t_0) \}$$

by Lemma 2.1 and Eq.(2). So each element in $\mathfrak{K}(V_0)^H$ must commute with the commuting algebra

$$\mathcal{C}(t_0) \simeq \mathbb{C}[H_A]$$
Since $H_A$ is an abelian group of order $f$, we infer that
\[ \mathcal{R}(V_0)^H \cong \mathcal{R} \oplus \ldots \oplus \mathcal{R} \]
with $f$ copies of $\mathcal{R}$. It is now immediate that $K_0(\mathfrak{A}) = \mathbb{Z}^f$, $K_1(\mathfrak{A}) = 0$.

The logic of the above proof is this:
\[ C^*_n(G) \cong \mathcal{C}^H \cong \mathcal{C}^H(0) \sim \mathcal{R}(V_0)^H \cong \mathcal{R}^f \]

The unramified unitary principal series is reducible at $t_0$, and it splits the Hilbert space $V_0$ into $f$ irreducible subspaces $H_1, \ldots, H_f$. Let $E_j : V_0 \rightarrow H_j$ be a rank-one projection. Let $e_j \in \mathfrak{A}$ be such that $e_j(x_0) = E_j$. Then \( \{e_1, \ldots, e_f\} \) is a set of generators in $K_0(\mathfrak{A})$.

3 Examples

The exponential map $\exp : t \rightarrow T$ induces a homeomorphism
\[ \bar{\mathfrak{A}}/H_A \cong T/W \]
by [4, Ch. IX, p.326], and $\bar{\mathfrak{A}}/H_A$ is equivariantly contractible. The map $r_t$ in Eq. (1) descends to the quotient $\bar{\mathfrak{A}}/H_A$ and shows that the quotient $\bar{\mathfrak{A}}/H_A$ is a contractible space. Therefore, $T/W$ is contractible.

If $G = E_8$, $F_4$ or $G_2$ then we have $P/Q = 0$ and so the connection index $f = 1$, see [4, Plates VIII, IX, X]. Therefore
\[ T/W \cong \bar{\mathfrak{A}}. \]

This is a contractible space. There are no $L$-packets. The $K$-theory of the unramified unitary principal series of $G$ is that of a point.

From now on, let $G = \text{SL}(\ell)$ with $\ell$ a prime number. We have $G' = \text{PGL}(\ell, \mathbb{C})$, $G = \text{PU}(\ell, \mathbb{C})$ and $T$ is a maximal torus in $\text{PU}(\ell, \mathbb{C})$. Then the Weyl group $W$ is the symmetric group $S_\ell$. Let $\gamma$ denote the standard $\ell$-cycle $(123\ldots \ell) \in S_\ell$. The fixed set of $\gamma$ is
\[ t_0 = \exp x_0 = (1 : \omega : \omega^2 : \cdots : \omega^{\ell-1}) \in T^\ell/\mathbb{T} \]
where $\omega = \exp(2\pi i/\ell)$. The $L$-group attached to $t_0$ is
\[ \bar{L}(t_0) = < \omega^{\text{val}F} > \cong \mathbb{Z}/\ell\mathbb{Z} \]
and the $X$-group is trivial, so that the $R$-group is given by

$$R(t_0) = \mathbb{Z}/\ell\mathbb{Z}.$$  

The $\ell$ irreducible constituents of $\text{Ind}_B^G(t_0)$ are elliptic representations [5].

Let $\overline{A}/H_A$ denote the extended quotient of $\overline{A}$ by $H_A$. The canonical projection

$$\pi : \overline{A}/H_A \to A/H_A$$

is a model of reducibility at the point $t_0$. The fibre $\pi^{-1}(t_0)$ contains $\ell$ points. This is very much in the spirit of the recent geometric conjecture in [2].

Note that $f = \ell$ as required. The unramified unitary principal series of $\text{SL}(\ell)$ admits a unique $L$-packet with $\ell$ elliptic constituents, and this is detected by $K_0$. For $\text{SL}(2)$ we have $K_0 = \mathbb{Z} \oplus \mathbb{Z}$, $K_1 = 0$.

Let $G = \text{SL}(3)$. Then the vector space $t$ is the Euclidean plane $\mathbb{R}^2$. The singular hyperplanes [4, Ch. IX, p.325] tessellate $\mathbb{R}^2$ into equilateral triangles, as in Figures 1 and 2. This tessellation of the plane appears in [1, p.104] as the Stiefel diagram for $\text{SU}(3, \mathbb{C})$. The interior of each equilateral triangle is an alcove. The closure of an alcove $A$ is denoted $\overline{A}$. The affine Weyl group $W_a$ operates simply transitively on the set of alcoves, and $\overline{A}$ is a fundamental domain for the action of $W_a$ on $t$, see [4, p.326]. There is a finer tessellation into isosceles triangles, as in Figures 1 and 2. An isosceles triangle in this tessellation will be denoted $\Delta$. The extended affine Weyl group $W'_a$ acts simply transitively on the set of such isosceles triangles $\overline{\Delta}$, but the closure $\overline{\Delta}$ is not a fundamental domain for the action of $W'_a$. Quotient spaces are as follows:

$$t/W'_a \simeq \overline{A}/H_A$$

The abelian group $H_A$ is the cyclic group $\mathbb{Z}/3\mathbb{Z}$ which acts on $\overline{A}$ by rotation through $2\pi/3$. If $\overline{A} = oab$ then the special points of $\overline{A}$ are the vertices $\{o, a, b\}$. The quotient $\overline{A}/H_A$ is an identification space of $\overline{\Delta}$: identify two adjacent edges of $\overline{\Delta}$. This space is contractible.

Consider the closed set $oabc$ in Figure 1. This closed set is the closure of the union of two adjacent alcoves $A_1, A_2$ and is also the closure of the union of 6 isosceles triangles $\Delta_1, \ldots, \Delta_6$. The torus $T$ is an identification space of the closed set $oabc$: identify $oa$ with $cb$ and $oc$ with $ab$. 

8
References


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