Hermitian Quadratic Matrix Polynomials: Solvents and Inverse Problems

Lancaster, Peter and Tisseur, Françoise

2010

MIMS EPrint: **2010.10**

ISSN 1749-9097

Hermitian Quadratic Matrix Polynomials: Solvents and Inverse Problems[☆]

Peter Lancaster [a,1], Françoise Tisseur[*,b]

[a] *Department of Mathematics and Statistics, University of Calgary, AB T2N 1N4, Canada*
[b] *School of Mathematics, The University of Manchester, Manchester, M13 9PL, UK*

Abstract

A monic quadratic Hermitian matrix polynomial $L(\lambda)$ can be factorized into a product of two linear matrix polynomials, say $L(\lambda) = (I\lambda - S)(I\lambda - A)$. For the inverse problem of finding a quadratic matrix polynomial with prescribed spectral data (eigenvalues and eigenvectors) it is natural to prescribe a right solvent A and then determine compatible left solvents S. This problem is explored in the present paper. The splitting of the spectrum between real eigenvalues and nonreal conjugate pairs plays an important role. Special attention is paid to the case of real-symmetric quadratic polynomials.

Key words: quadratic eigenvalue problem, Hermitian quadratic matrix polynomial, inverse problem, vibrating systems, solvent, igenvalue type, sign characteristic
2000 MSC: 74A15, 15A29

1. Introduction

In mechanics, Hermitian quadratic matrix polynomials of the form

$$L(\lambda) := M\lambda^2 + D\lambda + K, \qquad M > 0,\ D^* = D,\ K^* = K, \tag{1}$$

frequently arise and we refer to (1) as a *system*. Here $M > 0$ means Hermitian positive definite. Because the coefficient matrices are Hermitian, the spectrum of $L(\lambda)$ is symmetric about the real axis in the complex plane. For simplicity, it will be assumed here that the system is reduced to the monic case, $M = I_n$, which is always possible since $M > 0$. We recall that if $\det L(\lambda_0) = 0$ and $L(\lambda_0)x = 0$ for some nonzero x then λ_0 and x are known as an eigenvalue and (right) eigenvector of $L(\lambda)$, respectively.

The general factorization theorem [3, Thm. 11.2] says that there is a factorization

$$L(\lambda) = \lambda^2 I_n + \lambda D + K = (\lambda I_n - S)(\lambda I_n - A). \tag{2}$$

[☆] Version of January 10, 2010.
[*] Corresponding author
Email address: ftisseur@ma.man.ac.uk (Françoise Tisseur)
[1] The work of this author was supported by the Natural Sciences and Engineering Research Council of Canada.

The monic pencils $(\lambda I_n - S)$ and $(\lambda I_n - A)$ are called left and right linear divisors of L, respectively. Straightforward calculations show that S and A satisfy

$$S^2 + SD + K = 0, \qquad A^2 + DA + K = 0.$$

We refer to S and A as left and right solvents of $L(\lambda)$, respectively.

In applications it is often the case that only a few of the $2n$ eigenvalues of $L(\lambda)$ can be predicted with any confidence. Also, they are sometimes the eigenvalues closest to the origin (associated with the fundamental modes). In the present context these may be assigned as eigenvalues of A and the eigenvalues of S adjusted to be more distant from the origin (possibly associated with "high frequency noise").

It is our objective to take advantage of the factorization (2) in the design of techniques for solving the inverse problem: find a system with prescribed eigenvalues and eigenvectors. The strategy is first to assign the matrix A of (2) (and hence n eigenvalues and (possibly generalised) right eigenvectors) and then to determine a class of matrices S for which (2) holds for Hermitian D and K. Solutions S will then determine the remaining n eigenvalues directly—and the complementary right eigenvectors (indirectly).

The nature of these problems depends on the distribution of the eigenvalues to be admitted. In particular, a problem may be "mixed" in the sense that there are both real and nonreal eigenvalues, or there may be no real eigenvalues (as in the case of elliptic problems considered in [7]), or all the eigenvalues may be real (as in the case of quasihyperbolic and hyperbolic systems considered in [1], [4] and [5], for example). We will use the theory of Hermitian matrix polynomials as developed in [3]. In particular, the notion of the *sign characteristics* of simple real eigenvalues will play an important role. We refer the reader to [8] for a careful discussion of canonical structures and the theory behind them.

2. Left solvents from right

It follows from the factorization (2) that, given a right solvent $A \in \mathbb{F}^{n\times n}$, where $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$, we are to find matrices $S \in \mathbb{F}^{n\times n}$ such that $A + S = -D$ and $SA = K$ are Hermitian (symmetric when $\mathbb{F} = \mathbb{R}$). Thus

$$A + S = A^* + S^*, \qquad SA = A^*S^*. \tag{3}$$

This problem has the obvious solution $S = A^*$, which ensures that the real eigenvalues of $L(\lambda)$ are just those of A (if any). We draw attention to the fact that, although λ may be, by hypothesis, a *simple real* eigenvalue of A, it is necessarily a *defective real* eigenvalue of multiplicity two of $L(\lambda) = (I\lambda - A^*)(I\lambda - A)$ [3, Thm. 12.8]. For this reason, we focus on solutions (admitting real eigenvalues) other than $S = A^*$.

In contrast, Lancaster and Maroulas [6] have considered (3) under the assumption that A is nonsingular with all eigenvalues in the upper half plane. In this case the last equation in (3) is equivalent to $S = A^*H$ for some Hermitian H and the first equation in (3) becomes $A^*H - HA = A^* - A$. The strategy there is to solve this equation for H in terms of A and then obtain S from $S = A^*H$.

The line of attack here is different and requires no assumptions on A. We make the decomposition

$$S = S_1 + S_2, \qquad S_1 = S_1^*, \quad S_2 = -S_2^*,$$

that is, $S_1 = \frac{1}{2}(S + S^*)$ and $S_2 = \frac{1}{2}(S - S^*)$. Then the first of equations (3) simply says that S_2 is determined by A. Indeed,

$$S_2 = -\frac{1}{2}(A - A^*). \tag{4}$$

Now the second equation, $SA = A^*S^*$, is equivalent to $(S_1 + S_2)A = A^*(S_1 + S_2)^* = A^*(S_1 - S_2)$ and hence, $S_1A - A^*S_1 = -(S_2A + A^*S_2)$ which, on using (4), becomes

$$S_1A - A^*S_1 = \frac{1}{2}(A^2 - (A^*)^2). \tag{5}$$

Theorem 1 *Given a matrix $A \in \mathbb{F}^{n\times n}$, a matrix $S \in \mathbb{F}^{n\times n}$ is such that both $S + A$ and SA are Hermitian if and only if*

$$S = S_1 - \frac{1}{2}(A - A^*),$$

where $S_1 \in \mathbb{F}^{n\times n}$ is a Hermitian solution of (5).

Proof. The statement is already proved in one direction. For the converse, we have

$$S + A = S_1 - \frac{1}{2}(A - A^*) + A = S_1 + \frac{1}{2}(A + A^*),$$

and since $S_1^* = S_1$, we see that $S + A$ is Hermitian. Then

$$SA = \Big(S_1 - \frac{1}{2}(A - A^*)\Big)A = S_1A - \frac{1}{2}A^2 + \frac{1}{2}A^*A.$$

But (5) gives $S_1A = A^*S_1 + \frac{1}{2}(A^2 - (A^*)^2)$ so that

$$\begin{aligned} SA &= A^*S_1 - \frac{1}{2}(A^*)^2 + \frac{1}{2}A^*A, \\ &= A^*\Big(S_1 + \frac{1}{2}(A - A^*)\Big), \\ &= A^*\Big(S_1 - \frac{1}{2}(A - A^*)\Big)^*, \ \text{ since } S_1^* = S_1, \\ &= A^*S^*, \quad \text{by definition of } S. \qquad \square \end{aligned}$$

The *existence* of a Hermitian solution of (5) is not in question because, clearly, there is always a solution $S_1 = \frac{1}{2}(A + A^*)$. Also, it is well-known that equation (5) has a *unique* solution S_1 if and only if A and A^*, have no eigenvalues in common. In particular, this happens when the right solvent A has all of its eigenvalues in the open upper half of the complex plane. The resulting matrix polynomial $L(\lambda) = I\lambda^2 - \lambda(A + A^*) + A^*A$ has no real eigenvalues: it is *elliptic* since $L(\lambda) > 0$ for all real λ.

Thus, in order to generate a polynomial with mixed real and nonreal spectrum it is necessary to consider the solutions of a *singular* Lyapunov equation. The following simple example indicates some of the issues to be resolved in our analysis.

Example 1 Consider a right solvent determined by the real matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. In this way, we specify a pair of nonreal eigenvalues $\pm i$, where $i = \sqrt{-1}$. Easy calculations show that the general solution of (5) has the form

$$S_1 = \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{bmatrix}, \qquad \xi_1,\, \xi_2 \in \mathbb{R}.$$

We obtain a family of left solvents

$$S = S_1 + S_2 = \begin{bmatrix} \xi_1 & -1+\xi_2 \\ 1+\xi_2 & -\xi_1 \end{bmatrix}, \qquad \xi_1,\, \xi_2 \in \mathbb{R}$$

whose eigenvalues satisfy $\lambda^2 = \xi_1^2 + \xi_2^2 - 1$, and are real or complex depending on the choice of the two real parameters ξ_1, ξ_2. Furthermore, since A and S are both real, so are the coefficients of $L(\lambda) = (\lambda I - S)(\lambda I - A) = \lambda^2 I + \lambda D + K$,

$$D = \begin{bmatrix} -\xi_1 & -\xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix}, \qquad K = \begin{bmatrix} 1-\xi_2 & \xi_1 \\ \xi_1 & 1+\xi_2 \end{bmatrix} \qquad \xi_1,\, \xi_2 \in \mathbb{R}.$$

3. Hermitian systems

Let $A = XJX^{-1}$ be the Jordan canonical decomposition of $A \in \mathbb{C}^{n\times n}$ and consider the set

$$\mathcal{C}(J) := \{B \in \mathbb{C}^{n\times n} : BJ = (BJ)^*,\ B^* = B\}. \tag{6}$$

Lemma 2 *If $A \in \mathbb{C}^{n\times n}$ has Jordan canonical decomposition $A = XJX^{-1}$ then Z is a Hermitian solution of the homogeneous equation $ZA - A^*Z = 0$ if and only if $Z = X^{-*}BX^{-1}$ for some $B \in \mathcal{C}(J)$.*

Proof. If $B \in \mathcal{C}(J)$ and $Z = X^{-*}BX^{-1}$ then

$$ZA = (X^{-*}BX^{-1})(XJX^{-1}) = X^{-*}(BJ)X^{-1} \tag{7}$$
$$= X^{-*}(J^*B)X^{-1} = (X^{-*}J^*X^*)(X^{-*}BX^{-1}) = A^*Z. \tag{8}$$

Conversely, if $ZA = A^*Z$ for some Hermitian Z then we have $Z(XJX^{-1}) = (X^{-*}J^*X^*)Z$ whence $(X^*ZX)J = J^*(X^*ZX)$ and $X^*ZX \in \mathcal{C}(J)$. □

Theorem 3 *Given a matrix $A \in \mathbb{C}^{n\times n}$ with Jordan canonical decomposition $A = XJX^{-1}$, all matrices S for which $L(\lambda) = (\lambda I - S)(\lambda I - A)$ is Hermitian have the form*

$$S = A^* + X^{-*}BX^{-1}, \tag{9}$$

for some $B \in \mathcal{C}(J)$.

Proof. There is a particular solution $S_1 = \frac{1}{2}(A + A^*)$ of (5) so, using Lemma 2, the general solution to (5) has the form

$$\frac{1}{2}(A + A^*) + X^{-*}BX^{-1}, \qquad B \in \mathcal{C}(J).$$

Then, by Theorem 1, the class of all matrices for which $S + A$ and SA are Hermitian has the form

$$\frac{1}{2}(A + A^*) + X^{-*}BX^{-1} - \frac{1}{2}(A - A^*) = A^* + X^{-*}BX^{-1},$$

as required. $\square$

If one has the freedom to assign eigenvalues to a system it is most likely that they will be chosen to be distinct. So we now assume that $A = XJX^{-1} \in \mathbb{C}^{n\times n}$ has distinct eigenvalues and write

$$J = \mathrm{diag}\big(\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_{2s}}_{\text{conj.pairs}}, \underbrace{\lambda_{2s+1}, \ldots, \lambda_{2s+r}}_{\text{nonreal}}, \underbrace{\lambda_{2s+r+1}, \ldots, \lambda_{2s+r+t}}_{\text{real}}\big), \tag{10}$$

where the $2s + r$ nonreal eigenvalues consists of s complex conjugate pairs $(\lambda_{2j-1}, \lambda_{2j})$ with $\lambda_{2j} = \overline{\lambda_{2j-1}}$, $j = 1{:}\, s$ and r nonreal eigenvalues satisfying $\lambda_{2s+j} \neq \overline{\lambda_{2s+k}}$, $j, k = 1{:}\, r$. Under these hypotheses it is easily seen that $\mathcal{C}(J)$ in (6) is a $(2s+t)$-dimensional manifold:

$$\mathcal{C}(J) = \Bigg\{\mathrm{diag}\Bigg(\begin{bmatrix} 0 & \gamma_1 \\ \overline{\gamma_1} & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \gamma_s \\ \overline{\gamma_s} & 0 \end{bmatrix}, 0, \ldots, 0, \delta_1, \ldots \delta_t\Bigg) \in \mathbb{C}^{n\times n} :$$
$$\gamma_j = \alpha_j + i\beta_j,\ \alpha_j, \beta_j, \delta_k \in \mathbb{R},\ (\beta_j \neq 0),\ j = 1{:}\, s,\ k = 1{:}\, t\Bigg\}. \tag{11}$$

In particular, we see from Theorem 3 that, when A has no real eigenvalues ($t = 0$) and no complex conjugate pairs ($s = 0$), then the trivial solution $S = A^*$ is unique.

If J has the form (10), then B is tridiagonal (as in (11)) and we interpret Theorem 3 in the following way: Since $X^{-1}A = JX^{-1}$, the determination of coefficients D and K is reduced essentially to the calculation of the left eigenvectors (rows of X^{-1}) associated with the eigenvalues of A. Indeed, in the Jordan canonical decomposition of A, the rows of

$$X^{-1} = \begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix} \tag{12}$$

define a complete set of left eigenvectors of A and we can rewrite S in (9) as

$$S = A^* + \sum_{j=1}^{s}\Big((\alpha_j + i\beta_j)Y_j + (\alpha_j - i\beta_j)Y_j^*\Big) + \sum_{j=1}^{t}\delta_j Y_{2s+r+j}, \tag{13}$$

with rank-one matrices

$$Y_j = \begin{cases} y_{2j-1}y_{2j}^*, & j = 1{:}\, s, \\ y_j y_j^*, & j = 2s + r + 1{:}\, n. \end{cases} \tag{14}$$

With these constructions, Theorem 3 leads to the next result.

Corollary 4 *Let $A = XJX^{-1} \in \mathbb{C}^{n\times n}$ with distinct eigenvalues and J be as in* (10)*. If Y is defined from X^{-1} as in* (12) *and* (14)*, then the coefficients of a Hermitian system*

$L(\lambda) = \lambda^2 I + \lambda D + K$ *having* A *as right solvent can be written in the form*

$$-D = A + A^* + \sum_{j=1}^{s} \Big((\alpha_j + i\beta_j)Y_j + (\alpha_j - i\beta_j)Y_j^* \Big) + \sum_{j=1}^{t} \delta_j Y_{2s+r+j},$$

$$K = A^*A + \sum_{j=1}^{s} \Big(\lambda_{2j}(\alpha_j + i\beta_j)Y_j + \overline{\lambda_{2j}}(\alpha_j - i\beta_j)Y_j^* \Big) + \sum_{j=1}^{t} \lambda_{2s+r+j}\delta_j Y_{2s+r+j},$$

where the $2s + t$ *scalars* α_j, β_j, $j = 1{:}\,s$, *and* δ_k, $k = 1{:}\,t$ *are arbitrary real parameters.*

The next corollary indicates how the spectrum of the left divisor is determined by the right divisor and the choice of $B \in \mathcal{C}(J)$.

Corollary 5 *With the hypotheses of Theorem* 3*, the spectrum of the left solvent,* S*, is that of*

$$J^* + B(X^*X)^{-1}, \qquad B \in \mathcal{C}(J).$$

Proof. Since $A^* = X^{-*}J^*X^*$, it follows from (9) and the fact that $B^* = B$ that

$$S = X^{-*}(J^* + BX^{-1}X^{-*})X^* = X^{-*}(J^* + B(X^*X)^{-1})X^*,$$

which is a similarity—so the statement follows. □

Example 2 Consider the right solvent with "mixed" spectrum:

$$A = \begin{bmatrix} 2 & 2+i & -1-2i \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}.$$

It is easily seen that we may take

$$y_1^* = [\,0 \quad 0 \quad 1\,], \quad y_2^* = [\,1 \quad 2+i \quad -i\,], \quad y_3^* = [\,0 \quad -1 \quad 0\,]$$

as left eigenvectors corresponding to the eigenvalues $i, 2, 1$, respectively. With our conventions, left solvents will have the form $S = A^* + \sum_{j=2}^{3} \delta_j y_j y_j^*$. Taking $\delta_2 = \delta_3 = 1$ leads to

$$S = \begin{bmatrix} 3 & 2+i & -i \\ 4-2i & 7 & -1-2i \\ -1+3i & -1+2i & 1-i \end{bmatrix}$$

and then we obtain the Hermitian matrices

$$D = -(S+A) = \begin{bmatrix} -5 & -4-2i & 1+3i \\ & -8 & 1+2i \\ & & -1 \end{bmatrix}, \quad K = SA = \begin{bmatrix} 6 & 8+4i & -2-6i \\ & 17 & -6-7i \\ & & 8 \end{bmatrix}.$$

Finally, $\sigma(L) = \sigma(A) \cup \sigma(S) \approx \{i, 2, 1\} \cup \{-i, 1.3647, 9.6533\}$.

4. Real symmetric systems

The given matrix A is now to be a *real* right divisor and we seek real left divisors S (so that coefficients D and K will be real). We can follow the earlier line of argument and obtain an analogue of Theorem 3.

Theorem 6 *Given a matrix $A \in \mathbb{R}^{n\times n}$ with real Jordan canonical decomposition $A = X_R J_R X_R^{-1}$, all real matrices S for which $A+S$ and AS are real and symmetric have the form*

$$S = A^T + X_R^{-T} B_R X_R^{-1}, \tag{15}$$

for some $B_R \in \mathcal{C}_R(J_R)$, where

$$\mathcal{C}_R(J_R) := \{B_R \in \mathbb{R}^{n\times n} : B_R J_R = (B_R J_R)^T,\ B_R^T = B_R\}.$$

Explicit formulae for the real symmetric system coefficients generated in this way are

$$\begin{aligned} D &= -(S+A) = -(A+A^T) - X_R^{-T} B_R X_R^{-1}, \\ K &= SA = A^T A + X_R^{-T}(B_R J_R) X_R^{-1}. \end{aligned}$$

Now assume that all eigenvalues of A are simple. Thus, $A = X_R J_R X_R^{-1}$ with

$$J_R = \mathrm{diag}\Big(\begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix}, \dots \begin{bmatrix} \sigma_s & \omega_s \\ -\omega_s & \sigma_s \end{bmatrix}, \lambda_{2s+1}, \dots, \lambda_{2s+t}\Big) \in \mathbb{R}^{n\times n}, \tag{16}$$

where $(\lambda_{2k-1}, \lambda_{2k})$ with $\lambda_{2k-1} = \sigma_k + i\omega_k$ and $\lambda_{2k} = \overline{\lambda_{2k-1}}$, $k = 1{:}\,s$ are the complex conjugate pairs of eigenvalues. Then it is easy to show that

$$\begin{aligned} \mathcal{C}_R(J_R) = \Big\{ &\mathrm{diag}\Big(\begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & -\alpha_1 \end{bmatrix}, \dots \begin{bmatrix} \alpha_s & \beta_s \\ \beta_s & -\alpha_s \end{bmatrix}, \delta_1, \dots, \delta_t\Big) \in \mathbb{R}^{n\times n} : \\ &\alpha_j, \beta_j, \delta_k \in \mathbb{R},\ j = 1{:}\,s,\ k = 1{:}\,t \Big\}. \end{aligned} \tag{17}$$

Example 3 Assigning

$$J_R = \mathrm{diag}\Big(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, -2, -3\Big), \quad X_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

determines the right divisor

$$A = X_R J_R X_R^{-1} = \begin{bmatrix} -2 & 2 & 1 & -4 \\ -1 & 0 & 1 & -3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

with eigenvalues $-1 \pm i$, -2 -3. If we assign

$$B_R = \mathrm{diag}\Big(\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}, -4, -4\Big),$$

we obtain a matching left divisor

$$S = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -4 & 0 & 0 & -2 \\ 1 & 1 & -9 & 13 \\ -4 & -5 & 10 & -22 \end{bmatrix}$$

with eigenvalues -28.90, 1.54, -4.84, -2.80 to two decimal places. The real symmetric system coefficients obtained are

$$D = \begin{bmatrix} 6 & -3 & -1 & 4 \\ & 0 & -1 & 5 \\ & & 10 & -11 \\ & & & 26 \end{bmatrix}, \quad K = \begin{bmatrix} 7 & -8 & -3 & 13 \\ & 8 & 2 & -8 \\ & & 24 & -41 \\ & & & 99 \end{bmatrix}.$$

Notice that, although a stable right solvent, A, is assigned (i.e. with eigenvalues in the open left-half plane), the choice of B_R produces an unstable eigenvalue in S (and hence in $L(\lambda)$). As Corollary 5 shows, one way to ensure that *all* the eigenvalues of L are in the left half plane (i.e. the system is stable) would be to first choose A to be stable, and then confine $B_R \in \mathcal{C}_R(J_R)$ to matrices for which $X^{-T}B_RX^{-1}$ is sufficiently small.

To illustrate, replacing B_R in this example by the smaller matrix $B_R/4$, results in a left divisor with the stable eigenvalues, -9.34, -0.43, -1.83, -2.39 to two decimal places.

5. Sign characteristics

The *sign characteristic* is an intrinsic property for Hermitian matrix polynomials with real eigenvalues, and plays an essential role in the development of canonical forms and in perturbation theory applied to real eigenvalues. It consists of a vector with components $+1$ or -1, one component for each elementary divisor corresponding to a real eigenvalue. For a simple real eigenvalue μ of $L(\lambda) = \lambda^2 I + \lambda D + K$ with eigenvector x the corresponding sign in the sign characteristics is just

$$\operatorname{sign}\big(x^* L^{(1)}(\mu)x\big) = \operatorname{sign}\big(2\mu(x^*x) + x^*Dx\big). \tag{18}$$

The simple eigenvalue μ is said to be of *positive type* (or *negative type*) if the sign in (18) is positive (or negative) and this is well defined in the sense that $x^*L^{(1)}(\mu)x \neq 0$ for a simple eigenvalue [2, Thm. 3.2].

Note the fact that, if (λ_k, x_k) is an eigenpair of A then we have an eigenvalue and right eigenvector of *three* different matrix functions of interest to us, namely,

$$\lambda I - A, \quad L(\lambda) = (\lambda I - S)(\lambda I - A), \quad L_0(\lambda) := (\lambda I - A^*)(\lambda I - A). \tag{19}$$

We note again that, although λ_k is, by hypothesis, a *simple* real eigenvalue of A, it is a *defective* real eigenvalue of multiplicity two of $L_0(\lambda)$ [3, Thm. 12.8]. In this case we have

$$x_k^* L_0^{(1)}(\lambda_k)x_k = 2\lambda_k(x_k^*x_k) - x_k^*(A + A^*)x_k = 0. \tag{20}$$

Now suppose that A has distinct eigenvalues with Jordan canonical decomposition $A = XJX^{-1}$ and J as in (10). Let a left solvent S be constructed as in Theorem 3. Then, for the resulting $L(\lambda)$ we have

$$L^{(1)}(\lambda) = 2I\lambda - (A + S) = 2I\lambda - (A + A^*) - X^{-*}BX^{-1}, \tag{21}$$

for some $B \in \mathcal{C}(J)$. Using this in (20) we find that for a simple real eigenvalue λ_k with eigenvector x_k,

$$x_k^* L^{(1)}(\lambda_k)x_k = -x_k^*(X^{-*}BX^{-1})x_k = -e_k^* B e_k = -\delta_k, \tag{22}$$

(e_k is a unit coordinate vector). Comparing with (18), we see that the free parameters $\delta_1, \ldots, \delta_t$ in the definition of $B \in \mathcal{C}(J)$ (see (11)) determine the types of the real eigenvalues of L associated with A.

Theorem 7 *Let $A \in \mathbb{C}^{n\times n}$ have distinct eigenvalues, Jordan canonical decomposition $A = XJX^{-1}$ and J as in* (10). *Write $S = A^* + X^{-*}BX^{-1}$ for some $B \in \mathcal{C}(J)$ as in* (11). *Then the jth real eigenvalue of A (namely λ_{2s+r+j}, $1 \le j \le t$) is an eigenvalue of $L(\lambda) = (\lambda I - S)(\lambda I - A)$ of positive type if $\delta_j < 0$ and negative type if $\delta_j > 0$.*

An obvious analogue of Theorem 7 holds when $A \in \mathbb{R}^{n\times n}$.

It follows from Theorem 7 that if $\delta_j < 0$ (or $\delta_j > 0$) for all $j = 1{:}t$ then all the real eigenvalues of the right divisor $\lambda I - A$ are of positive type (or negative type). Then there must be exactly t real eigenvalues of the left divisor $\lambda I - S$, and they must be a complementary set in the sense that they are all of negative type (or positive type, respectively). In particular, if $s = 0$ in (11), (i.e. A has no complex conjugate eigenvalue pairs) and all the δ_j are negative then S has exactly t real eigenvalues of negative type. The following example is instructive.

Example 4 Let a right divisor $I\lambda - A$ be defined by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

The eigenvalues of A are $1, 2, 3$ (so that $s = 0$, $t = 3$ in (11) and (13)) with left eigenvectors

$$y_1^* = [0 \quad 1/2 \quad -1], \quad y_2^* = [1 \quad 0 \quad 1], \quad y_3^* = [1 \quad 1/2 \quad 1],$$

respectively. Hence, in equation (14) we have

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/4 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 1/4 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix}.$$

(a) If we choose $\delta_1 = \delta_2 = \delta_3 = 1$ in (13), then

$$S = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix} + \sum_{j=1}^{3} \delta_j Y_j = \begin{bmatrix} 3 & 5/2 & 3 \\ -1/2 & 7/2 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

In this way we generate a system $L(\lambda)$ for which the eigenvalues of A have negative type and the eigenvalues of S have positive type. Note that the eigenvalues of S, which are $1.78,\ 2.69,\ 7.02$ to two decimal places, interlace those of A. The quadratic L is quasihyperbolic [1, Sec. 4.2].

(b) If we let $\delta_1 = \delta_2 = \delta_3 = 10$, then

$$S = \begin{bmatrix} 21 & 7 & 11 \\ 4 & 8 & 1 \\ 20 & 2 & 32 \end{bmatrix}$$

with truncated eigenvalues 42.83, 12.91, 5.25, all having positive type. This choice of parameters determines a *hyperbolic* system since all eigenvalues are real and, with the eigenvalues in increasing order, $L(\lambda)$ has 3 consecutive eigenvalues of one type followed by 3 consecutive eigenvalues of the other type with a gap between the 3rd and 4th eigenvalues [1].

(c) If we let $\delta_1 = \delta_2 = 1, \delta_3 = -1$, then

$$S = \begin{bmatrix} 1 & 3/2 & 1 \\ -3/2 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

with a real eigenvalue and a conjugate pair—the spectrum is "mixed".

6. Conclusions

The well-known result asserting that a selfadjoint quadratic matrix polynomial can be factorized as a product of linear polynomials has been used to investigate the inverse (quadratic) eigenvalue problem. We have shown that, when a linear right divisor is specified, there is a class of compatible left divisors. Special attention is given to the ways in which conjugate pairs of eigenvalues and the two distinctive types of real eigenvalues are distributed between the left and right divisors, and also to the determination of *stable* systems in the sense that all eigenvalues have negative real parts.

References

[1] M. Al-Ammari and F. Tisseur. Hermitian matrix polynomials with real eigenvalues of definite type. Part I: Classification. MIMS EPrint 2010.9, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, Jan. 2010. 24 pp.

[2] A. L. Andrew, K. E. Chu, and P. Lancaster. Derivatives of eigenvalues and eigenvectors of matrix functions. *SIAM J. Matrix Anal. Appl.*, 14(4):903–926, 1993.

[3] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2009. ISBN 0-898716-81-8. xxiv+409 pp. Unabridged republication of book first published by Academic Press in 1982.

[4] C.-H. Guo, N. J. Higham, and F. Tisseur. Detecting and solving hyperbolic quadratic eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 30(4):1593–1613, 2009.

[5] C.-H. Guo and P. Lancaster. Algorithms for hyperbolic quadratic eigenvalue problems. *Math. Comp.*, 74(252):1777–1791, 2005.

[6] P. Lancaster and J. Maroulas. Inverse eigenvalue problems for damped vibrating systems. *J. Math. Anal. and Appl.*, 123:238–261, 1987.

[7] P. Lancaster and U. Prells. Inverse problems for damped vibrating systems. *Journal of Sound and Vibration*, 283:891–914, 2005.

[8] P. Lancaster and I. Zaballa. Canonical forms for selfadjoint matrix polynomials, 2010. Submitted.