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# Computing combinatorial types of trajectories in Pfaffian Dynamics<sup>☆</sup>

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## ABSTRACT

Suppose that the state space of a dynamical system has a finite partition, and each element of the partition is labelled by a letter of some alphabet. Then every trajectory of the system is naturally labelled by a word in this alphabet. This word is called the *combinatorial type* of the trajectory. In applications it is important to decide whether among a certain family of trajectories there is at least one trajectory of a given type, or whether all the trajectories in this family have the same type. In this paper we construct algorithms for solving this sort of questions for a wide class of *Pfaffian dynamical systems*, which have elementary (doubly-exponential) upper complexity bounds.

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## 1. Introduction

In this paper we study continuous dynamical systems which are called Pfaffian, and first introduced in [9,10]. These systems are defined by Pfaffian functions, either implicitly (via triangular systems of ordinary differential equations) or explicitly (by means of equations and inequalities involving *Pfaffian functions*). Pfaffian functions naturally arise in applications as real analytic solutions of triangular systems of first order partial differential equations with polynomial coefficients, and include polynomials, algebraic functions, exponentials, and trigonometric functions in the appropriate domains [8]. Pfaffian functions form a large natural class of real analytic functions which have a uniform description and an explicit characterization of complexity of their representations in terms of *formats*.

One of the important problems in the theory of dynamical systems is understanding of the behavior of a dynamical system with respect to viable and invariant sets. In this paper we consider a generalization of this problem for Pfaffian dynamical systems. Viability constraints and invariants naturally arise when some trajectories of a dynamical system do not satisfy the imposed requirements. These constraints include state constraints in control theory and verification of safety-critical systems, power constraints in game theory, ecological constraint in genetics, etc. [1]. Therefore, the goal is to select trajectories which are viable in the sense that they satisfy these constraints at each point in time.

In mathematical setting we consider the following problem. Let

$$\gamma : G_1 \times (-T, T) \rightarrow G_2$$

be a continuous dynamical system, where  $G_1 \subseteq \mathbb{R}^{k_1}$  is a set of control parameters,  $(-T, T)$  is an interval of time and  $G_2 \subseteq \mathbb{R}^{k_2}$  is a state space. For a fixed parameter  $\mathbf{x} \in G_1$  the map  $\Gamma_{\mathbf{x}} : (-T, T) \rightarrow G_2, t \mapsto \gamma(\mathbf{x}, t)$  is the *trajectory* corresponding to  $\mathbf{x}$ . Let  $U \subset G_1$ . A subset  $V \subset G_2$  is *viable* under the dynamical system  $\gamma$  and the control  $U$  if there exists at least one trajectory

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$\Gamma_{\mathbf{x}}, \mathbf{x} \in U$  of the system such that  $\forall t \in T_{\gamma}(\mathbf{x}, t) \in V$ . A subset  $V \subset G_2$  is *invariant* under the dynamical system  $\gamma$  and the control  $U$  if for all trajectories  $\Gamma_{\mathbf{x}}, \mathbf{x} \in U$  of the system,  $\forall t \in T_{\gamma}(\mathbf{x}, t) \in V$ .

Generalizing these concepts, consider a partition  $\mathcal{P} := \{P_1, \dots, P_S\}$  of  $G_2$ . Then for every trajectory  $\Gamma_{\mathbf{x}}$  there exists a *word* in the alphabet of symbols  $P_1, \dots, P_S$  which corresponds to  $\Gamma_{\mathbf{x}}$ . We will say that a word  $\omega$  is *viable* (respectively, *invariant*) under the dynamical system  $\gamma$ , the control  $U$ , and the partition  $\mathcal{P}$ , if at least one trajectory  $\Gamma_{\mathbf{x}}$  (respectively, for all trajectories  $\Gamma_{\mathbf{x}}$ ) of the system, with  $\mathbf{x} \in U$ , is (are) labelled by  $\omega$ .

In this paper we assume that dynamical systems and sets, we are interested in, are semi-Pfaffian. Our goal is to construct an algorithm for checking viability and invariance with an elementary exponential upper complexity bound. To achieve our goal we use cylindrical cell decomposition for semi-Pfaffian sets [5].

The outline of the paper is as follows. Section 1 presents a brief overview of Pfaffian functions, upper bounds on topological complexities of semi- and sub-Pfaffian sets, and algorithms for computing their closures and cylindrical cell decompositions. In Section 2 we recall the notion of Pfaffian dynamical system, viable and invariant sets. We also explain how to associate a word to a trajectory. Finally, in Section 3 we propose an algorithm (with the usual for Pfaffian functions theory oracle) for checking viability and invariance. The complexity of the algorithm is doubly exponential in the format of an input system.

## 1. Basic definitions and notions

### 1.1. Pfaffian functions and related sets

In this section we overview the theory of Pfaffian functions and sets definable with Pfaffian functions. The detailed exposition can be found in the survey [5].

**Definition 1.1.** A *Pfaffian chain* of the order  $r \geq 0$  and degree  $\alpha \geq 1$  in an open domain  $G \subset \mathbb{R}^n$  is a sequence of real analytic functions  $f_1, \dots, f_r$  in  $G$  satisfying differential equations

$$\frac{\partial f_j}{\partial x_i} = g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \dots, f_j(\mathbf{x})) \tag{1.1}$$

for  $1 \leq j \leq r, 1 \leq i \leq n$ . Here  $g_{ij}(\mathbf{x}, y_1, \dots, y_j)$  are polynomials of degrees not exceeding  $\alpha$ .

A function

$$f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \dots, f_r(\mathbf{x})),$$

where  $P(\mathbf{x}, y_1, \dots, y_r)$  is a polynomial of a degree not exceeding  $\beta \geq 1$ , the sequence  $f_1, \dots, f_r$  is a Pfaffian chain of order  $r$  and degree  $\alpha$ , is called a *Pfaffian function* of order  $r$  and degree  $(\alpha, \beta)$ .

In order to illustrate the definition let us consider several examples of Pfaffian functions.

- (a) Pfaffian functions of order 0 and degree  $(1, \beta)$  are polynomials of degrees not exceeding  $\beta$ .
- (b) The exponential function  $f(x) = e^{ax}$  is a Pfaffian function of order 1 and degree  $(1, 1)$  in  $\mathbb{R}$ , due to the equation  $df(x) = af(x)dx$ . More generally, for  $i = 1, 2, \dots, r$ , let  $E_i(x) := e^{E_{i-1}(x)}, E_0(x) = ax$ . Then  $E_r(x)$  is a Pfaffian function of order  $r$  and degree  $(r, 1)$ , since  $dE_r(x) = aE_1(x) \cdot \dots \cdot E_r(x)dx$ .
- (c) The function  $f(x) = 1/x$  is a Pfaffian function of order 1 and degree  $(2, 1)$  in the domain  $\{x \in \mathbb{R} \mid x \neq 0\}$ , due to the equation  $df(x) = -f^2(x)dx$ .
- (d) The logarithmic function  $f(x) = \ln(|x|)$  is a Pfaffian function of order 2 and degree  $(2, 1)$  in the domain  $\{x \in \mathbb{R} \mid x \neq 0\}$ , due to equations  $df(x) = g(x)dx$  and  $dg(x) = -g^2(x)dx$ , where  $g(x) = 1/x$ .
- (e) The polynomial  $f(x) = x^m$  can be viewed as a Pfaffian function of order 2 and degree  $(2, 1)$  in the domain  $\{x \in \mathbb{R} \mid x \neq 0\}$  (but not in  $\mathbb{R}$ ), due to the equations  $df(x) = mf(x)g(x)dx$  and  $dg(x) = -g^2(x)dx$ , where  $g(x) = 1/x$ . In some cases a better way to deal with  $x^m$  is to change the variable  $x = e^u$  reducing this case to (b).
- (f) The function  $f(x) = \tan(x)$  is a Pfaffian function of order 1 and degree  $(2, 1)$  in the domain  $\bigcap_{k \in \mathbb{Z}} \{x \in \mathbb{R} \mid x \neq \pi/2 + k\pi\}$ , due to the equation  $df(x) = (1 + f^2(x))dx$ .
- (g) The function  $\cos(x)$  is a Pfaffian function of order 2 and degree  $(2, 1)$  in the domain  $\bigcap_{k \in \mathbb{Z}} \{x \in \mathbb{R} \mid x \neq \pi + 2k\pi\}$ , due to equations  $\cos(x) = 2f(x) - 1, df(x) = -f(x)g(x)dx$ , and  $dg(x) = \frac{1}{2}(1 + g^2(x))dx$ , where  $f(x) = \cos^2(x/2)$  and  $g(x) = \tan(x/2)$ . Also, since  $\cos(x)$  is a polynomial of degree  $m$  of  $\cos(x/m)$ , the function  $\cos(x)$  is Pfaffian of order 2 and degree  $(2, m)$  in the domain  $\bigcap_{k \in \mathbb{Z}} \{x \in \mathbb{R} \mid x \neq m\pi + 2km\pi\}$ . The same is true, of course, for any shift of this domain by a multiple of  $\pi$ . However,  $\cos(x)$  is not a Pfaffian function in the whole real line.

As we can see, apart from polynomials, the class of Pfaffian functions includes real algebraic functions, exponentials, logarithms, trigonometric functions, their compositions, and other major transcendental functions in appropriate domains (see [5,6]). Now we introduce classes of sets definable with Pfaffian functions. In the case of polynomials they reduce to *semialgebraic* sets whose quantitative and algorithmic theory is treated in [2].

**Definition 1.2.** A set  $X \subset \mathbb{R}^n$  is called *semi-Pfaffian* in an open domain  $G \subset \mathbb{R}^n$  if it consists of the points in  $G$  satisfying a Boolean combination of some atomic equations and inequalities  $f = 0, g > 0$ , where  $f, g$  are Pfaffian functions having a common Pfaffian chain defined in  $G$ . A semi-Pfaffian set  $X$  is *restricted* in  $G$  if its topological closure lies in  $G$ .

**Definition 1.3.** A set  $X \subset \mathbb{R}^n$  is called *sub-Pfaffian* in an open domain  $G \subset \mathbb{R}^n$  if it is the image of a semi-Pfaffian set under a projection into a subspace.

It is worth noting that according to the Tarski-Seidenberg Theorem, the projection of a semialgebraic set is again semialgebraic.

In the sequel we will be dealing with the following subclass of sub-Pfaffian sets.

**Definition 1.4.** Suppose  $\bar{I} \subset \mathbb{R}$  is a closed interval. Consider the closed cube  $\bar{I}^{m+n}$  in an open domain  $G \subset \mathbb{R}^{m+n}$  and the projection map  $\pi : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ . A subset  $Y \subset \bar{I}^n$  is called *restricted sub-Pfaffian* if  $Y = \pi(X)$  for a restricted semi-Pfaffian set  $X \subset \bar{I}^{m+n}$ .

Note that a restricted sub-Pfaffian set need not to be semi-Pfaffian.

**Definition 1.5.** Consider a semi-Pfaffian set

$$X := \bigcup_{1 \leq i \leq M} \{ \mathbf{x} \in \mathbb{R}^n \mid f_{i1} = 0, \dots, f_{i l_i} = 0, g_{i1} > 0, \dots, g_{i j_i} > 0 \} \subset G, \quad (1.2)$$

where  $f_{is}, g_{is}$  are Pfaffian functions with a common Pfaffian chain of order  $r$  and degree  $(\alpha, \beta)$ , defined in an open domain  $G$ . Its *format* is the tuple  $(r, N, \alpha, \beta, n)$ , where  $N \geq \sum_{1 \leq i \leq M} (l_i + j_i)$ . For  $n = m + k$  and a sub-Pfaffian set  $Y \subset \mathbb{R}^k$  such that  $Y = \pi(X)$ , its *format* is the format of  $X$ .

We will refer to the representation of a semi-Pfaffian set in the form (1.2) as to the *disjunctive normal form (DNF)*.

**Remark.** In this paper we are concerned with complexities of computations, as functions of the format. In the case of Pfaffian dynamical systems these sizes and complexities also depend on the domain  $G$ . So far our definitions imposed no restrictions on an open set  $G$ , thus allowing it to be arbitrarily complex and to induce this complexity on the corresponding semi- and sub-Pfaffian sets. To avoid this we will always assume in the context of Pfaffian dynamical systems that  $G$  is “simple”, like  $\mathbb{R}^n$ , or  $I^n$  for open  $I \subset \mathbb{R}$ .

**Remark.** In this paper we construct and examine complexities of algorithms for checking satisfiability of viability constraints. In order to estimate the “efficiency” of a computation we need to specify more precisely a *model of computation*. As such we use a *real number machine* which is an analogy of a classical Turing machine but allows the exact arithmetic and comparisons on the real numbers. Since we are interested only in upper complexity bounds for algorithms, there is no need for a formal definition of this model of computation (it can be found in [3]). In some of our computational problems we will need to modify the standard real number machine by equipping it with an *oracle* for deciding feasibility of any system of Pfaffian equations and inequalities. An oracle is a subroutine which can be used by a given algorithm any time the latter needs to check feasibility. We assume that this procedure always gives a correct answer (“true” or “false”) though we do not specify how it actually works. An *elementary step* of a real number machine is either an arithmetic operation, or a comparison (branching) operation, or an oracle call. The *complexity* of a real number machine is the number of elementary steps it makes in the worst case until termination, as a function of the format of the input.

In the special case of semialgebraic sets, the oracle can be replaced by a proper real number machine, so the algorithm for checking of satisfiability of viability constraints can be realized as a standard real number machine.

## 1.2. Cylindrical cell decompositions

Now we define cylindrical decompositions of semi- and sub-Pfaffian sets in a cube  $\bar{I}^n$ , where  $\bar{I}$  is a closed interval.

**Definition 1.6.** A *cylindrical cell* in  $\bar{I}^n$  is defined by induction as follows.

- (1) A cylindrical 0-cell in  $\bar{I}^n$  is an isolated point.
- (2) A cylindrical 1-cell in  $\bar{I}$  is an open interval  $(a, b) \subset \bar{I}$ .
- (3) For  $n \geq 2$  and  $0 \leq k < n$  a cylindrical  $(k + 1)$ -cell in  $\bar{I}^n$  is either a graph of a continuous bounded function  $f : C \rightarrow \mathbb{R}$ , where  $C$  is a cylindrical  $(k + 1)$ -cell in  $\bar{I}^{n-1}$  and  $k < n - 1$ , or else a set of the form

$$\{ (x_1, \dots, x_n) \in \bar{I}^n \mid (x_1, \dots, x_{n-1}) \in C \text{ and } f(x_1, \dots, x_{n-1}) < x_n < g(x_1, \dots, x_{n-1}) \},$$

where  $C$  is a cylindrical  $k$ -cell in  $\bar{I}^{n-1}$ , and  $f, g : C \rightarrow \bar{I}$  are continuous bounded functions such that  $f(x_1, \dots, x_{n-1}) < g(x_1, \dots, x_{n-1})$  for all points  $(x_1, \dots, x_{n-1}) \in C$ .

**Definition 1.7.** A *cylindrical cell decomposition*  $\mathcal{D}$  of a subset  $A \subset \bar{I}^n$  with respect to the variables  $x_1, \dots, x_n$  is defined by induction as follows.

- (1) If  $n = 1$ , then  $\mathcal{D}$  is a finite family of pair-wise disjoint cylindrical cells (i.e., isolated points and intervals) whose union is  $A$ .

- (2) If  $n \geq 2$ , then  $\mathcal{D}$  is a finite family of pair-wise disjoint cylindrical cells in  $\bar{I}^n$  whose union is  $A$  and there is a cylindrical cell decomposition of  $\pi(A)$  such that  $\pi(C)$  is its cell for each  $C \in \mathcal{D}$ , where  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  is the projection map onto the coordinate subspace of  $x_1, \dots, x_{n-1}$ .

**Definition 1.8.** Let  $B \subset A \subset \bar{I}^n$  and  $\mathcal{D}$  be a cylindrical cell decomposition of  $A$ . Then  $\mathcal{D}$  is *compatible* with  $B$  if for any  $C \in \mathcal{D}$  we have either  $C \subset B$  or  $C \cap B = \emptyset$  (i.e., some subset  $\mathcal{D}' \subset \mathcal{D}$  is a cylindrical cell decomposition of  $B$ ).

**Definition 1.9.** For a given finite family  $f_1, \dots, f_N$  of Pfaffian functions in an open domain  $G$  we define its *consistent sign assignment* as a non-empty semi-Pfaffian set in  $G$  of the kind

$$\{\mathbf{x} \in G \mid f_{i_1} = 0, \dots, f_{i_{N_1}} = 0, f_{i_{N_1+1}} > 0, \dots, f_{i_{N_2}} > 0, f_{i_{N_2+1}} < 0, \dots, f_{i_N} < 0\},$$

where  $i_1, \dots, i_{N_1}, \dots, i_{N_2}, \dots, i_N$  is a permutation of  $1, \dots, N$ .

**Theorem 1.10** [6,12]. Let  $f_1, \dots, f_N$  be a family of Pfaffian functions in an open domain  $G \subset \mathbb{R}^n$ ,  $G \supset \bar{I}^n$  having a common Pfaffian chain of order  $r$ , and degrees  $(\alpha, \beta)$ . Then there is an algorithm (with the oracle) producing a cylindrical cell decomposition of  $\bar{I}^n$  which is compatible with each consistent sign assignment of  $f_1, \dots, f_N$ . Each cell is a sub-Pfaffian set represented as a projection of a semi-Pfaffian set in DNF. The number of cells, the components of their formats and the complexity of the algorithm are less than

$$N^{(r+n)^{O(n)}} (\alpha + \beta)^{(r+n)^{O(n^3)}}.$$

We summarize main properties of Pfaffian functions in the following propositions.

- Pfaffian functions can be considered as generalisation of algebraic functions.
- Pfaffian functions have the uniform description and the explicit characterization of complexity of their representations.
- The class of Pfaffian functions includes *exp*, trigonometrical functions defined in appropriate domains, and more generally solutions of a large class of differential equations.
- The structure  $\mathbb{R} = \langle \mathbb{R}, +, *, 0, 1, <, \{f_1, \dots, f_N\} \rangle$  is o-minimal, i.e. definable sets have only a finite number of connected components, in the other words, it has finiteness property.

## 2. Pfaffian dynamical systems

### 2.1. Pfaffian dynamics and related sets

We now recall definitions concerning Pfaffian dynamical systems.

**Definition 2.1.** Let  $G_1 \subset \mathbb{R}^{k_1}$  and  $G_2 \subset \mathbb{R}^{k_2}$  be open domains. A *Pfaffian dynamical system* is a map

$$\gamma : G_1 \times (-T, T) \rightarrow G_2$$

with a semi-Pfaffian graph, where  $G_1$  is a set of control parameters,  $(-T, T)$  is an interval of time, and  $G_2$  is a state space.

For a given  $\mathbf{x} \in G_1$  the set

$$\Gamma_{\mathbf{x}} = \{\mathbf{y} \mid \exists t \in (-T, T) (\gamma(\mathbf{x}, t) = \mathbf{y})\} \subset G_2$$

is called the *trajectory* (or evolution) determined by  $\mathbf{x}$ , and the graph

$$\hat{\Gamma}_{\mathbf{x}} = \{(t, \mathbf{y}) \mid \gamma(\mathbf{x}, t) = \mathbf{y}\} \subset (-T, T) \times G_2$$

is called the *integral curve* determined by  $\mathbf{x}$ .

**Definition 2.2.** Let  $U \subseteq G_1$ . A set  $V \subseteq G_2$  is called *viable* under the dynamical system  $\gamma$  and the control  $U$  if there exists  $\mathbf{x} \in U$  such that for all  $t \in T$ ,  $\gamma_{\mathbf{x}}(t) \in V$ . We say a *subset*  $U \subseteq G_1$  *satisfies the constraint*  $V$  if  $V$  is viable under  $U$  and the dynamical system  $\gamma$ .

**Definition 2.3.** Let  $U \subseteq G_1$ . A set  $Inv \subseteq G_2$  is called *invariant* under the dynamical system  $\gamma$  and the control  $U$  if for all  $\mathbf{x} \in U$  and for all  $t \in T$ ,  $\gamma_{\mathbf{x}}(t) \in Inv$ .

In the next sections we investigate the behavior of a Pfaffian dynamical system with respect to a given semi-Pfaffian viability constraint.

### 2.2. Encoding trajectories by words

We now introduce, following [4,9], a technique of encoding trajectories of dynamical systems by words. Consider a Pfaffian dynamical system  $\gamma : G_1 \times (-T, T) \rightarrow G_2$ , where  $G_1 \subset \mathbb{R}^{k_1}$  and  $G_2 \subset \mathbb{R}^{k_2}$  are open domains, and a partition  $\mathcal{P} := \{P_1, \dots, P_S\}$

of  $G_2$  into  $s$  semi-Pfaffian sets  $P_j$ . Let the graph of  $\gamma$  and each set  $P_j$  have a format  $(r, N, \alpha, \beta, n)$ , where  $n \geq k_1 + k_2 + 1$ , and all Pfaffian functions involved have a common Pfaffian chain. Fix  $\mathbf{x} \in G_1$ . Define the set of points and open intervals in  $\mathbb{R}$ :

$$\mathcal{F}_{\mathbf{x}} := \{J \mid J \text{ is a point or an interval in } (-T, T) \text{ maximal w.r.t. inclusion for the property } \exists i \in \{1, \dots, s\} \forall t \in J (\gamma(\mathbf{x}, t) \in P_i)\}.$$

Let the cardinality  $|\mathcal{F}_{\mathbf{x}}| = r$  and  $y_1 < \dots < y_r$  be the set of representatives of  $\mathcal{F}_{\mathbf{x}}$  such that  $\gamma(\mathbf{x}, y_j) \in P_{i_j}$ . Then define the word  $\omega := P_{i_1} \dots P_{i_r}$  in the alphabet  $\mathcal{P}$ . Informally,  $\omega$  is the list of names of elements of the partition in the order they are visited by the trajectory  $\Gamma_{\mathbf{x}}$ . In our setting  $\omega$  is called the *type of trajectory*  $\Gamma_{\mathbf{x}}$ . Introduce the set of words  $\Omega := \{\omega \mid \mathbf{x} \in G_1\}$ .

**Theorem 2.4** [4,9]. *The set  $\Omega$  is finite and the number of different trajectory types of  $\gamma$  with respect to the partition  $\mathcal{P}$  is less than*

$$(sN)^{(r+n)^{0(m)}} (\alpha + \beta)^{(r+n)^{0(m^3)}}. \quad (2.1)$$

**Theorem 2.5.** *There is a cell decomposition of the control parameter space  $G_1$  such that if  $\mathbf{x}_1$  and  $\mathbf{x}_2$  belong to the same cell then  $\Gamma_{\mathbf{x}_1}$  and  $\Gamma_{\mathbf{x}_2}$  are labelled by the same word.*

**Proof.** Consider the family  $\mathcal{F} = \{f_1, \dots, f_k\}$  of Pfaffian functions in the domain  $G_1 \times (-T, T) \times G_2$  consisting of all functions in variables  $\mathbf{x}, t, \mathbf{y}$  involved in the defining formulas for the graph of the map  $\gamma : (\mathbf{x}, t) \mapsto \mathbf{y}$ , and for all sets  $P_j$ . According to Theorem 1.10, there is a cylindrical decomposition  $\mathcal{D}$  of  $G_1 \times (-T, T) \times G_2$  with respect to the variables  $\mathbf{x}, t, \mathbf{y}$  having the following properties.

- (1)  $\mathcal{D}$  is compatible with each consistent sign assignment of  $f_1, \dots, f_k$ .
- (2) There are at most (2.1) cylindrical cells.
- (3) Each of these cells is sub-Pfaffian.
- (4)  $\mathcal{D}$  induces a cylindrical decomposition on  $G_1$  which we denote by  $\mathcal{E}$ .

We claim that for any cell  $C \in \mathcal{E}$  and any two points  $\mathbf{x}_1, \mathbf{x}_2 \in C$  the trajectories  $\Gamma_{\mathbf{x}_1}, \Gamma_{\mathbf{x}_2} \in G_2$  are intersecting sets  $P_1, \dots, P_s$  in the same order (i.e., are encoded by the same word from  $\Omega$ ). Indeed, let  $\pi : G_1 \times (-T, T) \times G_2 \rightarrow G_1$  be the projection on  $G_1$ . The decomposition  $\mathcal{D}$  induces cylindrical decompositions  $\mathcal{D}_1$  and  $\mathcal{D}_2$  on  $\pi^{-1}(\mathbf{x}_1)$  and  $\pi^{-1}(\mathbf{x}_2)$  respectively. In particular, each of the integral curves  $\widehat{\Gamma}_{\mathbf{x}_1}$  and  $\widehat{\Gamma}_{\mathbf{x}_2}$  is decomposed into a sequence of alternating points and open intervals. Due to basic properties of cylindrical decomposition, there is a natural bijection  $\psi : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  such that

- (i) the restriction of  $\psi$  to the set of all cells in  $\widehat{\Gamma}_{\mathbf{x}_1}$  is a bijection onto the set of all cells in  $\widehat{\Gamma}_{\mathbf{x}_2}$ ;
- (ii) for each  $1 \leq j \leq s$  the restriction of  $\psi$  to the set of all cells in  $(-T, T) \times P_j \cap \pi^{-1}(\mathbf{x}_1)$  is a bijection onto the set of all cells in  $(-T, T) \times P_j \cap \pi^{-1}(\mathbf{x}_2)$ .
- (iii) the bijection  $\psi$  preserves the order in which cells appear in the trajectories.

It follows that if a cell  $B \in \mathcal{D}_1$  is a subset of  $\widehat{\Gamma}_{\mathbf{x}_1} \cap ((-T, T) \times P_j)$  for some  $1 \leq j \leq s$ , then  $\psi(B) \subset \widehat{\Gamma}_{\mathbf{x}_2} \cap ((-T, T) \times P_j)$ . Moreover, if for cells  $B_1, B_2 \in \mathcal{D}_1$  there exist  $t_1, t_2 \in (-T, T)$  such that  $t_1 < t_2$  and  $\gamma(\mathbf{x}_1, t_1) \in B_1 \wedge \gamma(\mathbf{x}_1, t_2) \in B_2$  then there exist  $t'_1, t'_2 \in (-T, T)$  such that  $t'_1 < t'_2$  and  $\gamma(\mathbf{x}_2, t'_1) \in \psi(B_1) \wedge \gamma(\mathbf{x}_2, t'_2) \in \psi(B_2)$ . The claim is proved.

It follows that the cardinality of  $\Omega$  does not exceed the cardinality of  $\mathcal{E}$  which does not exceed the cardinality of  $\mathcal{D}$  which in turn is at most (2.1).  $\square$

### 3. An algorithm for checking viability and invariance

Consider a Pfaffian dynamical system  $\gamma : G_1 \times (-T, T) \rightarrow G_2$ , a semi-Pfaffian subset of control parameters  $U \subseteq G_1$ , a partition  $\mathcal{P} = \{P_1, \dots, P_s\}$  of  $G_2$ , and a word  $\omega$  in the alphabet of symbols  $P_1, \dots, P_s$ . Let the graph of  $\gamma$  and the sets  $U, P_1, \dots, P_s$  have a format  $(r, N, \alpha, \beta, n)$ , and all Pfaffian functions involved have a common Pfaffian chain.

**Theorem 3.1.** *There is an algorithm which checks whether the control  $U$  satisfies the viability or the invariance constraint  $\omega$  under the partition  $\mathcal{P}$ . The complexity of this algorithm does not exceed*

$$(sN)^{(r+n)^{0(m)}} (\alpha + \beta)^{(r+n)^{0(m^3)}}. \quad (3.1)$$

**Proof.** We are going to show the main steps of our algorithm. First the algorithm produces the set of words  $\Omega$  corresponding to the Pfaffian dynamical system  $\gamma : G_1 \times (-T, T) \rightarrow G_2$  and the partition  $\mathcal{P}$ . Consider the family of Pfaffian functions in the domain  $G_1 \times (-T, T) \times G_2$  consisting of all functions in variables  $\mathbf{x}, t, \mathbf{y}$  involved in the defining formulas for the graph of the map  $\gamma : (\mathbf{x}, t) \mapsto \mathbf{y}$ , for the set  $V$ , and for the partition  $\mathcal{P}$ . According to Theorem 1.10, there is a cylindrical decomposition  $\mathcal{D}$  with respect to  $(\mathbf{x}, t, \mathbf{y})$  which is compatible with this family and consists of at most (3.1) cylindrical cells.

This cell decomposition  $\mathcal{D}$  induces the cell decomposition  $\mathcal{E}$  (see the proof of Theorem 2.5). Using the oracle, which decides feasibility of any system of Pfaffian equations and inequalities, the algorithm selects the cells from  $\mathcal{D}$  which are subsets of  $\{(\mathbf{x}, t, \mathbf{y}) \mid \mathbf{y} = \gamma(\mathbf{x}, t)\}$ . Denote the set of the selected cells by  $\mathcal{B}$ . Observe that for any fixed  $\mathbf{x}' \in G_1$  the set  $\bigcup_{B \in \mathcal{B}} B \cap \{(\mathbf{x}, t, \mathbf{y}) \mid \mathbf{x} = \mathbf{x}'\}$

coincides with the integral curve  $\widehat{\Gamma}_{\mathbf{x}'}$ . Then the algorithm determines the order in which the cells  $B \in \mathcal{B}$  intersected with  $\{(\mathbf{x}, t, \mathbf{y}) \mid \mathbf{x} = \mathbf{x}'\}$  appear in the trajectory  $\Gamma_{\mathbf{x}'}$ .

More precisely, for each pair of distinct cells  $B_1, B_2 \in \mathcal{B}$  the algorithm decides, using the oracle, whether

$$\exists \mathbf{x} \exists t_1 \exists t_2 \exists \mathbf{y}_1 \exists \mathbf{y}_2 ((\mathbf{x}, t_1, \mathbf{y}_1) \in B_1 \wedge (\mathbf{x}, t_2, \mathbf{y}_2) \in B_2 \wedge (t_1 < t_2)).$$

For a given  $C \in \mathcal{E}$ , after all pairs of cells are processed we get the ordered set of cells  $B_1, \dots, B_k$  in  $\mathcal{D}$  such that for any  $1 \leq i \leq k$  and any  $\mathbf{x}' \in C$  the sequence of points and intervals

$$B_1 \cap \{(\mathbf{x}, t, \mathbf{y}) \mid \mathbf{x} = \mathbf{x}'\}, \dots, B_k \cap \{(\mathbf{x}, t, \mathbf{y}) \mid \mathbf{x} = \mathbf{x}'\}$$

forms the integral curve  $\widehat{\Gamma}_{\mathbf{x}'}$ . By the definition of cylindrical decomposition, for any pair  $B_i, P_j$  either  $B_i \subset (C \times (-T, T) \times P_j)$  or  $B_i \cap (C \times (-T, T) \times P_j) = \emptyset$ . The algorithm uses the oracle to decide for every pair which of these two cases takes place. As the result, the sequence  $B_1, \dots, B_k$  becomes partitioned into subsequences of the kind

$$(B_1, \dots, B_{k_1}), (B_{k_1+1}, \dots, B_{k_2}), \dots, (B_{k_{\ell-1}+1}, \dots, B_k),$$

where for any  $i$ ,  $0 \leq i \leq \ell - 1$ , the cells  $B_{k_i+1}, \dots, B_{k_{i+1}}$  lie in  $C \times (-T, T) \times P_{j_i}$  for some  $j_i$ , while  $B_{k_i} \cap C \times (-T, T) \times P_{j_i} = \emptyset$  and  $B_{k_{i+1}+1} \cap C \times (-T, T) \times P_{j_i} = \emptyset$ . Then the word  $\omega := P_{j_0} \dots P_{j_{\ell-1}}$  corresponds to the cell  $C$ . Considering all cells in  $\mathcal{E}$  the algorithm finds  $\Omega$ .

Then the algorithm collects all cells from  $\mathcal{E}$  such that their union is  $U$ . If at least one of these cells corresponds to the word  $\omega$ , then  $\omega$  is viable. If all of these cells corresponds to  $\omega$ , then  $\omega$  is invariant. This completes the description of the algorithm.

A straightforward analysis shows that the complexity of the algorithm does not exceed (3.1), taking into account the bounds from Theorem 1.10.  $\square$

**Corollary 3.2.** *There is an algorithm checking viability and invariance of a set of state space  $V$  under the dynamics  $\gamma$  and the control  $U$ . The complexity of this algorithm does not exceed  $N^{(r+n)^{O(m)}} (\alpha + \beta)^{(r+n)^{O(m^3)}}$ .*

#### 4. Conclusion and future research

We have proposed an algorithm for checking viability and invariance in a Pfaffian dynamical system. This research has been motivated by verification problems of safety-critical large scale continuous and hybrid systems. First step in the suggested procedure is to construct a cylindrical cell decomposition which is compatible with each sign assignment of the Pfaffian functions involved in the definitions of a continuous dynamic and a viability constraint. In the second step we encode trajectories of the Pfaffian dynamical system by finite words. By the construction of cylindrical cell decomposition, the space of parameters is decomposed to cells in such a way that each cell corresponds to one word. In other words, if points  $\mathbf{x}_1$  and  $\mathbf{x}_2$  belong to the same cell the trajectories  $\Gamma_{\mathbf{x}_1}$  and  $\Gamma_{\mathbf{x}_2} \in G_2$  are encoded by the same word. This induces a natural marking the cells of parameters by the words. In the final step we check intersections of a given set of control parameters and the cells of parameters which marked by the special word. If at least one of them is nonempty, then the given set of control parameters satisfies the viability constraint. If all of these intersections are nonempty, then this set of control parameters satisfies the invariance constraint. This algorithm is based on the cylindrical cell decomposition technique and, accordingly, has a double exponential upper complexity bound. It seems feasible to construct an algorithm with *single exponential* complexity using the approach employed in the paper [10].

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