# A NEW BOUND FOR THE SMALLEST $X$ WITH $\pi(X) > LI(X)$

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In this dissertation, we study the problem of changes of sign of $\pi(x) - li(x)$. We focus on two very recent articles: Chao-Plymen, Int. J. Number Theory July 2010 and Saouter and Demichel, Math. Comp. July 2010. We give a detailed exposition of these papers, finding and correcting several minor errors. We improved [10, Theorem 6.4] resulting in a theorem, which is new and is the best known result.

# Declaration

No portion of the work referred to in this dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

# Copyright Statement

# Acknowledgements

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# Chapter 1

# Introduction

## 1.1 A NEW BOUND FOR THE SMALLEST $x$ WITH $\pi(x) > li(x)$

Denote

$\pi(x) =$ the number of primes less than or equal to x.

$li(x) =$ the logarithmic integral.

Define $f(x) = \Omega_{\pm} g(x)$ means

$$\lim_{x \longrightarrow \infty} \sup \frac{f(x)}{g(x)} > 0$$

$$\lim_{x \longrightarrow \infty} \inf \frac{f(x)}{g(x)} < 0$$

In 1914, numerical evidence proved that $\pi(x) < li(x)$ for all $x$. Littlewood [7] announced that

$$\pi(x) - li(x) = \Omega_{\pm}(x^{\frac{1}{2}} (\log x)^{-1} \log\log\log x)$$

which means:

$$\lim_{x \longrightarrow \infty} \sup \frac{(\pi(x) - li(x)) \log x}{x^{\frac{1}{2}} \log\log\log x} > 0$$

$$\lim_{x \longrightarrow \infty} \inf \frac{(\pi(x) - li(x)) \log x}{x^{\frac{1}{2}} \log\log\log x} < 0$$

which implies that $\pi(x) - li(x)$ changes sign infinitely many times. The smallest value of x with $\pi(x) \geq li(x)$ is denoted by $\Xi$ . Kotnik [5] proves $10^{14} < \Xi =$ smallest $x$ with $\pi(x) \geq li(x)$.

Lehman's theorem is an 'integrated version of the Riemann explicit formula'. His method was to integrate the function $u \longrightarrow \pi(e^u) - li(e^u)$ against a Gaussian kernel over a carefully chosen interval $[\omega - \eta, \omega + \eta]$. Here the Gaussian kernel is defined by the following:

$$\frac{e^{-kx^2}}{\int e^{-kx^2}}$$

The definite integral obtained is denoted $I(\omega, \eta)$. We let $\rho = \frac{1}{2} + i\gamma$ denote a Riemann zero with $\gamma > 0$ and let

$$H(T, \omega) = -2R \sum_{0<\gamma\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\frac{-\gamma^2}{2\alpha}}$$

where $T$ = height, $R$ = real part and $\alpha$ = related to the kernel chosen. Lehman [6] proved the following equality:

$$I(\omega, \eta) = -1 + H(T, \omega) + R$$

together with an explicit estimate $|R| < \epsilon$. This creates the inequality

$$I(\omega, \eta) \geq H(T, \omega) - (1 + \epsilon)$$

The problem now is to prove

$$H(T, \omega) > 1 + \epsilon$$

Hence, if the above holds then $I(\omega, \eta) > 0$. So there exists $x \in [e^{\omega-\eta}, e^{\omega+\eta}]$ for which $\pi(x) > li(x)$. In order to prove $H(T, \omega) > 1 + \epsilon$, numerical values of the Riemann zeros with $|\gamma| < T$ are required. Each term in $H(T, \omega)$ is a complex number determined by a Riemann zero. It is essential that the real parts of these complex numbers (which are spiralling $\longrightarrow 0$) reinforce each other sufficiently for $H(T, \omega) > 1 + \epsilon$ to hold. This is done by numerical computation. When $T$ is large, then a computer is required!

Bays and Hudson [1] made the following selection:

$$\omega = 727.95209, \eta = 0.002$$

The interval is therefore

$$[\omega - \eta, \omega + \eta] = [727.95009, 727.95409]$$

$$[e^{\omega-\eta}, e^{\omega+\eta}] = [e^{727.95009}, e^{727.95409}]$$

Plymen and Chao [2] reduce the leading term in Lehman's Theorem. This enabled them to make the following selection:

$$\omega = 727.952018, \eta = 0.00016$$

Hence the interval is given by

$$[\omega - \eta, \omega + \eta] = [727.951858, 727.952178]$$

so we have

$$[e^{\omega-\eta}, e^{\omega+\eta}] = [e^{727.951858}, e^{727.952178}]$$

An upper bound for the first crossover is:

$$\Xi < e^{727.952178} < 1.398344 \times 10^{316}$$

We see that their interval is strictly a sub-interval of the Bays-Hudson interval. It is narrower by a factor of 12 and creates the smallest known upper bound.

However, Yannick Saouter and Patrick Demichel [10] state their results in the following theorem.

**Theorem 1.1.1.** *There exists at least one value $x$ in the interval [exp(727.9513130), exp(727.9513586)] for which $\pi(x) > li(x)$ holds. Moreover, there are more than $6.09 \times 10^{150}$ successive integers in the vicinity of exp(727.951335792) where the inequality holds.*

# Chapter 2

# The Leading Term

**Theorem 2.0.2.** *Lehman's Theorem*

*Let $A$ be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $0 < \gamma \le A$. Let $\alpha, \eta$ and $\omega$ be positive numbers such that $\omega - \eta > 1$ and*

$$\frac{2}{A} \le \frac{2A}{\alpha} < \eta \le \frac{\omega}{2} \tag{2.1}$$

*Let*

$$K(y) := \sqrt{\frac{\alpha}{2\pi}} e^{\frac{-\alpha y^2}{2}}$$

$$I(\omega, \eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{\frac{-u}{2}} (\pi(e^u) - li(e^u)) du$$

*Then for $2\pi e < T \le A$*

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\le T} \frac{e^{i\gamma\omega}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} + R = -1 - H(T, \alpha, \omega) + R$$

*where $|R| \le s_1 + s_2 + s_3 + s_4 + s_5 + s_6$ with*

$$s_1 = \frac{3.05}{\omega - \eta}$$

$$s_2 = 4(\omega + \eta) e^{\frac{-(\omega-\eta)}{6}}$$

$$s_3 = \frac{2e^{\frac{-\alpha\eta^2}{2}}}{\sqrt{2\pi\alpha}\eta}$$

$$s_4 = 0.08\sqrt{\alpha} e^{\frac{-\alpha\eta^2}{2}}$$

$$s_5 = e^{\frac{-T^2}{2\alpha}} \left\{ \frac{\alpha}{\pi T^2} \log \frac{T}{2\pi} + \frac{8 \log T}{T} + \frac{4\alpha}{T^3} \right\}$$

$$s_6 = A \log A e^{(\frac{-A^2}{2\alpha} + \frac{(\omega+\eta)}{2})} \left\{ 4\alpha^{\frac{-1}{2}} + 15\eta \right\}$$

*If the Riemann hypothesis holds, then* (2.1) *and the term* $s_6$ *in the estimate for* $R$ *may be omitted.*

Plymen and Chao refined a part of Lehman's proof, which allowed them to reduce the term $s_1$ in Lehman's Theorem. We see their approach below.

The logarithmic integral is defined as:

$$li(e^z) := \int_{-\infty+iy}^{x+iy} \frac{e^t}{t} dt$$

where $z = x + iy, y \neq 0$. For $x > 1, li(x)$ is defined as

$$li(x) := \frac{1}{2} [li(x + i0) + li(x - i0)]$$

This leads us to recover the classical definition of $li(x)$ as an integral principal value

$$li(x) = \lim_{\epsilon \longrightarrow 0} \left( \int_0^{1-\epsilon} \frac{e^t}{t} dt + \int_{1+\epsilon}^{x} \frac{e^t}{t} dt \right)$$

We start off by defining the following

$$J(x) := \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + ...$$

and recall the Riemann-von Mangoldt explicit formula

$$J(x) = li(x) - \sum_\rho li(x^\rho) + \int_x^\infty \frac{du}{(u^2-1)u \log u} - \log 2$$

valid for $x > 1$. We also define the following by J. B. Rosser and L. Schoenfeld [9]

$$\pi(x) = \frac{x}{\log x} + \frac{3\theta_1(x)\frac{x}{2}}{(\log x)^2} \tag{2.2}$$

$$\pi(x) = \frac{\theta_2(x) 2x}{\log x} \tag{2.3}$$

with $|\theta_1(x)| < 1$, $\theta_2(x) < 0.62753$. We know there are at most $\frac{\log x}{\log 2}$ terms in $J(x)$. This is because we know

$$x^{\frac{1}{k}} < 2$$

$$\log(x^{\frac{1}{k}}) < \log 2$$

$$\frac{1}{k} \log x < \log 2$$

$$\log x < k \log 2$$

$$\frac{\log x}{\log 2} < k$$

This helps us to define $\theta_3(x)$ by the following

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\theta_3(x)\pi(x^{\frac{1}{3}})\left(\frac{\log x}{\log 2}\right)$$

with $\theta_3(x) < 1$. Now we look at the following

$$J(x) = J(x)$$

$$li(x) - \sum_\rho li(x^\rho) + \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2 = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\theta_3(x)\pi(x^{\frac{1}{3}})\left(\frac{\log x}{\log 2}\right)$$

$$\pi(x) - li(x) = -\sum_\rho li(x^\rho) + \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2 - \frac{1}{2}\pi(x^{\frac{1}{2}}) - \frac{1}{3}\theta_3(x)\pi(x^{\frac{1}{3}})\left(\frac{\log x}{\log 2}\right)$$

Using (2.2):

$$\begin{aligned}
\pi(x^{\frac{1}{2}}) &= \frac{x^{\frac{1}{2}}}{\log x^{\frac{1}{2}}} + \frac{3\theta_1(x^{\frac{1}{2}})\frac{x^{\frac{1}{2}}}{2}}{(\log x^{\frac{1}{2}})^2} \\
&= \frac{x^{\frac{1}{2}}}{\frac{1}{2}\log x} + \frac{\frac{3}{2}\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\frac{1}{2}\log x)^2} \\
&= 2\frac{x^{\frac{1}{2}}}{\log x} + \frac{\frac{3}{2}\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{\frac{1}{4}(\log x)^2} \\
&= 2\frac{x^{\frac{1}{2}}}{\log x} + 6\frac{\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\log x)^2}
\end{aligned}$$

Using (2.3):

$$\begin{aligned}
\pi(x^{\frac{1}{3}}) &= \frac{\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log x^{\frac{1}{3}}} \\
&= \frac{\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\frac{1}{3}\log x} \\
&= \frac{3\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log x}
\end{aligned}$$

$$\begin{aligned}
\pi(x) - li(x) &= -\sum_\rho li(x^\rho) + \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2 - \frac{1}{2}\left(2\frac{x^{\frac{1}{2}}}{\log x} + 6\frac{\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\log x)^2}\right) \\
&\quad -\frac{1}{3}\theta_3(x)\left(3\frac{\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log x}\right)\left(\frac{\log x}{\log 2}\right) \\
&= -\sum_\rho li(x^\rho) + \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2 - \left(\frac{x^{\frac{1}{2}}}{\log x} + 3\frac{\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\log x)^2}\right) \\
&\quad -\theta_3(x)\left(\frac{\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log 2}\right)
\end{aligned}$$

Now let,

$$\theta_4(x) := \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2$$

For $x > 2$, we have the following bounds:

$$-\log 2 < \theta_4(x) < \frac{1}{2} - \log 2$$

To see this we look at,

$$\pi(x) - li(x) = -\sum_\rho li(x^\rho) + \theta_4(x) - \left(\frac{x^{\frac{1}{2}}}{\log 2} + 3\frac{\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\log x)^2}\right) - \frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log 2}$$

At this point, Plymen and Chao define $\theta(x)$ as follows;

$$4\theta(x)x^{\frac{1}{3}} := \theta_4(x) - \frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log 2}$$

We then see,

$$\begin{aligned} |\theta(x)| &\leq \frac{1}{4x^{\frac{1}{3}}}\theta_4(x) - \frac{1}{4x^{\frac{1}{3}}}\frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})2x^{\frac{1}{3}}}{\log 2} \\ &\leq \frac{1}{4x^{\frac{1}{3}}}\theta_4(x) - \frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})}{2\log 2} \\ &\leq \frac{1}{4x^{\frac{1}{3}}}\left(\frac{1}{2} - \log 2\right) - \frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})}{2\log 2} \end{aligned}$$

The reason for the last inequality is because we have $\theta_4(x) < \frac{1}{2} - \log 2$ and

$$\frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})}{2\log 2} < \frac{1}{2}$$

The latter holds since $\theta_3(x) < 1$ and $\theta_2(x) < 0.62753$ so we have $\theta_2(x^{\frac{1}{3}}) < 0.62753$ (for all $x$). Hence the following:

$$\frac{1 \times 0.62753}{2\log 2} = 0.452667209504526$$

which shows

$$\frac{\theta_3(x)\theta_2(x^{\frac{1}{3}})}{2\log 2} < \frac{1}{2}$$

$$\implies |\theta(x)| < \frac{1}{4x^{\frac{1}{3}}}\left(\frac{1}{2} - \log 2\right) + \frac{1}{2} < \left(\frac{1}{2} - \log 2\right) + \frac{1}{2} < 1 - \log 2 < 1$$

$$\implies |\theta(x)| < 1 \text{ for all } x > 2$$

Therefore, we have the following

$$\pi(x) - li(x) = -\sum_{\rho} li(x^{\rho}) - \frac{x^{\frac{1}{2}}}{\log x} - 3\frac{\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\log x)^2} + \theta(x)4x^{\frac{1}{3}}$$

Plymen and Chao improve the bound for $\theta_1(x)$. They quote a result of Panaitopol [8]:

$$\pi(x) < \frac{x}{\log x - 1 - (\log x)^{\frac{-1}{2}}}$$

for all $x \geq 6$.

We can use the above and $\pi(x) = \frac{x}{\log x} + \frac{\frac{3}{2}x\theta_1(x)}{(\log x)^2}$ to see the following

$$\frac{x}{\log x} + \frac{\frac{3}{2}x\theta_1(x)}{\log^2 x} < \frac{x}{\log x - 1 - (\log x)^{\frac{-1}{2}}}$$

We now denote

$$y(x) := (\log x)^{\frac{1}{2}}$$

$\Longrightarrow \frac{1}{y} = \frac{1}{(\log x)^{\frac{1}{2}}} = (\log x)^{\frac{-1}{2}}$ and $y^2 = \log x$. Therefore,

$$\begin{aligned}
\frac{x}{y^2} + \frac{\frac{3}{2}x\theta_1(x)}{y^4} &< \frac{x}{y^2 - 1 - \frac{1}{y}} \\
x\left(\frac{1}{y^2} + \frac{\frac{3}{2}\theta_1(x)}{y^4}\right) &< \frac{x}{\frac{y^3-y-1}{y}} \\
\frac{1}{y^2} + \frac{\frac{3}{2}\theta_1(x)}{y^4} &< \frac{y}{y^3 - y - 1} \\
\frac{y^2 + \frac{3}{2}\theta_1(x)}{y^4} &< \frac{y}{y^3 - y - 1} \\
y^2 + \frac{3}{2}\theta_1(x) &< \frac{y^5}{y^3 - y - 1} \\
\frac{3}{2}\theta_1(x) &< \frac{y^5}{y^3 - y - 1} - y^2 = \frac{y^5 - (y^3 - y - 1)y^2}{y^3 - y - 1} \\
&< \frac{y^5 - y^5 + y^3 + y^2}{y^3 - y - 1} \\
&< \frac{y^3 + y^2}{y^3 - y - 1}
\end{aligned}$$

Hence,

$$0 < \theta_1(x) < \frac{2}{3}\frac{y^3 + y^2}{y^3 - y - 1}$$

for all $x \geq 6$.

Now, let

$$F(y) = \frac{y^3 + y^2}{y^3 - y - 1}$$

We can see that $F(y) > 1, F(y) \longrightarrow 1$ as $y \longrightarrow \infty$.

$$\begin{aligned} F'(y) &= \frac{(3y^2+2y)(y^3-y-1)-(y^3+y^2)(3y^2-1)}{(y^3-y-1)^2} \\ &= \frac{3y^5-3y^3-3y^2+2y^4-2y^2-2y-3y^5-y^3-3y^4-y^2}{(y^3-y-1)^2} \\ \implies F'(y) &= \frac{-y^4-4y^3-6y^2-2y}{(y^3-y-1)^2} < 0 \end{aligned}$$

Hence, $F$ is a monotone decreasing function.

We know $\theta_1(x) < \frac{2}{3}F(y)$ where $y = (\log x)^{\frac{1}{2}}$. Let $x = e^v$ so $y = (\log e^v)^{\frac{1}{2}} = v^{\frac{1}{2}} = \sqrt{v}$. $\implies \theta_1(e^v) < \frac{2}{3}F(\sqrt{v})$. Then we can clearly see

$$\begin{aligned} \theta_1(e^{\frac{u}{2}}) &< \frac{2}{3}F\left(\sqrt{\frac{u}{2}}\right) \\ 3\theta_1(e^{\frac{u}{2}}) &< 2F\left(\sqrt{\frac{u}{2}}\right) \end{aligned}$$

if $u \geq 727$ then

$$\begin{aligned} 3\theta_1(e^{\frac{u}{2}}) < 2F\left(\sqrt{\frac{727}{2}}\right) &< 2\left(\frac{\left(\sqrt{\frac{727}{2}}\right)^3 + \left(\sqrt{\frac{727}{2}}\right)^2}{\left(\sqrt{\frac{727}{2}}\right)^3 - \sqrt{\frac{727}{2}} - 1}\right) \\ &< 2\left(\frac{6930.373213 + 363.5}{6930.373213 - 19.06567596 - 1}\right) \\ &< 2\left(\frac{7293.873213}{6910.307537}\right) \\ &< 2(1.055506311) = 2.111012621043308 \end{aligned}$$

$$\implies 3\theta_1(e^{\frac{u}{2}}) < 2.1111$$

Looking back we have,

$$\pi(x) - li(x) = -\sum_{\rho} li(x^{\rho}) - \frac{x^{\frac{1}{2}}}{\log x} - 3\frac{\theta_1(x^{\frac{1}{2}})x^{\frac{1}{2}}}{(\log x)^2} + \theta(x)4x^{\frac{1}{3}}$$

Now, we let $x = e^u$

$$\pi(e^u) - li(e^u) = -\sum_{\rho} li(e^{\rho u}) - \frac{e^{\frac{u}{2}}}{\log e^u} - 3\frac{\theta_1(e^{\frac{u}{2}})e^{\frac{u}{2}}}{(\log e^u)^2} + \theta(e^u)4e^{\frac{u}{3}}$$

$$[\pi(e^u) - li(e^u) = -\sum_\rho li(e^{\rho u}) - \frac{e^{\frac{u}{2}}}{u} - 3\frac{\theta_1(e^{\frac{u}{2}})e^{\frac{u}{2}}}{u^2} + \theta(e^u)4e^{\frac{u}{3}}] \times ue^{\frac{-u}{2}}$$

$$ue^{\frac{-u}{2}}\{\pi(e^u) - li(e^u)\} = -\sum_\rho ue^{\frac{-u}{2}} li(e^{\rho u}) - ue^{\frac{-u}{2}}\frac{e^{\frac{u}{2}}}{u} - 3\frac{\theta_1(e^{\frac{u}{2}})ue^{\frac{-u}{2}}e^{\frac{u}{2}}}{u^2} + ue^{\frac{-u}{2}}\theta(e^u)4e^{\frac{u}{3}}$$

$$= -\sum_\rho ue^{\frac{-u}{2}} li(e^{\rho u}) - 1 - 3\frac{\theta_1(e^{\frac{u}{2}})}{u} + 4\theta(e^u)ue^{\frac{-u}{6}}$$

$$= -1 - \sum_\rho ue^{\frac{-u}{2}} li(e^{\rho u}) - 3\frac{\theta_1(e^{\frac{u}{2}})}{u} + 4\theta(e^u)ue^{\frac{-u}{6}}$$

We already know $K$ is a standard Gaussian distribution so we have the following

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)du = \int_{-\eta}^{\eta} K(v)dv < 1$$

and also because $K$ satisfies $\frac{e^{-kx^2}}{\int e^{-kx^2}} = 1$. As you can see we take the following $v = u - \omega$, $\omega = 0$ and $u = \omega - \eta$.

By Lehman's Theorem, if $\omega - \eta > 727$ then we have the estimate

$$I(\omega,\eta) = \left| \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\{\pi(e^u) - li(e^u)\}du \right|$$

$$\leq \left| \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(-3\frac{\theta_1(e^{\frac{u}{2}})}{u} + 4\theta(e^u)ue^{\frac{-u}{6}}\right)du \right|$$

We note here that, $\theta(e^u) < 1$ since $|\theta(x)| < 1$ for all $x > 2$.

$$\implies I(\omega,\eta) \leq \frac{2.1111}{\omega-\eta} + 4(\omega+\eta)e^{\frac{-(\omega-\eta)}{6}}$$

$$\leq s_1{}' + s_2$$

Hence, Plymen and Chao replace $s_1$ by $s_1{}'$

$$s_1{}' = \frac{2.1111}{\omega-\eta}$$

Hence, following the steps in Lehman's proof, we are led to a new estimate for $|R'|$:

$$|R'| \leq s_1{}' + s_2 + s_3 + s_4 + s_5 + s_6$$

**Theorem 2.0.3.** *Let $A$ be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $0 < \gamma \leq A$. Let $\alpha, \eta$ and $\gamma$ be positive numbers such that $\omega - \eta > 727$ and*

$$\frac{2}{A} \leq \frac{2A}{\alpha} \leq \eta \leq \frac{\omega}{2} \tag{2.4}$$

*Let $K(y)$ and $I(\omega, \eta)$ be defined as in Lehman's Theorem. Then for $2\pi e < T \leq A$ we have*

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|<T} \frac{e^{i\gamma\omega}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} + R'(= -1 - H(T, \alpha, \omega) + R')$$

*where $|R'| = {s_1}' + s_2 + s_3 + s_4 + s_5 + s_6$. If the Riemann hypothesis holds, then* (2.4) *and the term $s_6$ in the estimate for $R'$ may be omitted.*

We note here that without the Riemann hypothesis, Lehman proves by means of several intricate estimates that the inequality (2.4) are a sufficient condition for the following crucial estimate:

$$\left| \sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{\frac{-u}{2}} li(e^{\rho u}) du \right| \leq s_6$$

# Chapter 3

# Numerical Results (i)

Plymen and Chao exploit the reduced term ${s_1}'$ to obtain improved numerical results. They start by setting

$$H(T,\alpha,\omega) = -\sum_{0<|\gamma|\leq T} e^{\frac{-\gamma^2}{2\alpha}} \frac{e^{i\omega\gamma}}{\rho} = -\sum_{0<\gamma\leq T} t(\gamma,\alpha,\omega)$$

where

$$t(\gamma,\alpha,\omega) = e^{\frac{-\gamma^2}{2\alpha}} \frac{e^{i\omega\gamma}}{\rho}$$

Now we know that

$$\begin{aligned}
\frac{e^{i\omega\gamma}}{\rho} &= \frac{e^{i\omega\gamma}}{\beta+i\gamma} + \frac{e^{-i\omega\gamma}}{\beta-i\gamma} \\
&= \frac{\cos(\omega\gamma)+i\sin(\omega\gamma)}{\beta+i\gamma} + \frac{\cos(\omega\gamma)-i\sin(\omega\gamma)}{\beta-i\gamma} \\
&= \frac{(\beta-i\gamma)(\cos(\omega\gamma)+i\sin(\omega\gamma))+(\beta+i\gamma)(\cos(\omega\gamma)-i\sin(\omega\gamma))}{\beta^2-i^2\gamma^2} \\
&= \frac{\beta\cos(\omega\gamma)-i\gamma\cos(\omega\gamma)+i\beta\sin(\omega\gamma)-i^2\gamma\sin(\omega\gamma)+\beta\cos(\omega\gamma)}{\beta^2+\gamma^2} \\
&\quad + \frac{i\gamma\cos(\omega\gamma)-i\beta\sin(\omega\gamma)-i^2\gamma\sin(\omega\gamma)}{\beta^2+\gamma^2} \\
&= \frac{2\beta\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)}{\beta^2+\gamma^2}
\end{aligned}$$

We let $\beta = \frac{1}{2}$

$$\begin{aligned}
\Longrightarrow t(\gamma,\alpha,\omega) &= e^{\frac{-\gamma^2}{2\alpha}} \frac{e^{i\omega\gamma}}{\rho} \\
&= e^{\frac{-\gamma^2}{2\alpha}} \frac{\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2}
\end{aligned}$$

Plymen and Chao decided to fix $N = 2,000,000$. We then denote ${\gamma_i}^*$ to be the approximations to the true values of $\gamma_i$, correct to nine decimal places (computed by Odlyzko) for $1 \leq i \leq N$ and set $T = 1131944.47182487 > \gamma_N$.

Let $H^*(T, \alpha, \omega)$ be the value obtained by taking the sum up to $T$ using $t(\gamma^*, \alpha, \omega)$ and let $H_M^*(T, \alpha, \omega)$ be the result of computing the same sum by machine. In the case of $H$ the quantity we wish to control is

$$|H - {H_M}^*| \leq |H - H^*| + |H^* - {H_M}^*|$$

From H. J. J. Riele's paper [12] 'On the sign of the difference $\pi(x) - li(x)$' we have $\gamma < \alpha$

$$|H(T, \alpha, \omega) - H^*(T, \alpha, \omega)| < \sum_{0<\gamma\leq T} |\gamma - \gamma^*| M(\gamma, \alpha, \omega)$$

with

$$M(\gamma, \alpha, \omega) = e^{\frac{-\gamma^2}{2\alpha}} \left( \frac{2\omega}{\gamma} + \frac{\omega}{\gamma^2} + \frac{2}{\alpha} + \frac{2}{\gamma^3} + \frac{4}{\gamma^2} \right) < \frac{3\omega + 8}{\gamma}$$

However, I should mention here that there is a mistake in the above expression (as you will see later). The above should actually read:

$$M(\gamma, \alpha, \omega) = e^{\frac{-\gamma^2}{2\alpha}} \left( \frac{2\omega}{\gamma} + \frac{1}{\alpha\gamma} + \frac{\omega}{\gamma^2} + \frac{2}{\alpha} + \frac{2}{\gamma^3} + \frac{6}{\gamma^2} \right) < \frac{3\omega + 11}{\gamma}$$

Anyhow, as the results above do not differ much, this does not make a difference to the rest of the paper. Carrying on, we know from before

$$H(T, \alpha, \omega) = - \sum_{0<|\gamma|\leq T} e^{\frac{-\gamma^2}{2\alpha}} \frac{e^{i\omega\gamma}}{\rho} = - \sum_{0<\gamma\leq T} t(\gamma, \alpha, \omega)$$

where

$$t(\gamma, \alpha, \omega) = e^{\frac{-\gamma^2}{2\alpha}} \frac{\cos(\omega\gamma) + 2\gamma \sin(\omega\gamma)}{\frac{1}{4} + \gamma^2}$$

We can now define the following

$$H^*(T, \alpha, \omega) = - \sum_{0<|\gamma^*|\leq T} e^{\frac{-\gamma^{*2}}{2\alpha}} \frac{e^{i\omega\gamma^*}}{\rho} = - \sum_{0<\gamma^*\leq T} t(\gamma^*, \alpha, \omega)$$

where

$$t(\gamma^*, \alpha, \omega) = e^{\frac{-\gamma^{*2}}{2\alpha}} \frac{\cos(\omega\gamma^*) + 2\gamma^* \sin(\omega\gamma^*)}{\frac{1}{4} + \gamma^{*2}}$$

In the mean time, lets change the notation above slightly

$$H(T,\alpha,\omega) = -\sum_{0<\gamma\leq T} t(\gamma)$$

where

$$t(\gamma) = e^{\frac{-\gamma^2}{2\alpha}} \frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2}$$

By the Mean-value theorem,

$$|t(\gamma^*) - t(\gamma)| = |\gamma^* - \gamma||t'(\bar{\gamma})|$$

with $|\bar{\gamma} - \gamma| < |\gamma^* - \gamma|$. Now we look at $t'(\gamma)$:

$$t(\gamma) = e^{\frac{-\gamma^2}{2\alpha}} \frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2}$$

$$\begin{aligned}
t'(\gamma) &= \frac{e^{\frac{-\gamma^2}{2\alpha}}\{[-\omega\sin(\omega\gamma) + 2\omega\gamma\cos(\omega\gamma) + 2\sin(\omega\gamma)]\left[\frac{1}{4}+\gamma^2\right]}{(\frac{1}{4}+\gamma^2)^2} \\
&\quad -\frac{[\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)][2\gamma]\}}{(\frac{1}{4}+\gamma^2)^2} \\
&\quad +(\frac{-2\gamma}{2\alpha}e^{\frac{-\gamma^2}{2\alpha}})\left[\frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2}\right] \\
&= e^{\frac{-\gamma^2}{2\alpha}}\frac{[2\omega\gamma\cos(\omega\gamma) + (2-\omega)\sin(\omega\gamma)]\left[\frac{1}{4}+\gamma^2\right]}{(\frac{1}{4}+\gamma^2)^2} \\
&\quad -e^{\frac{-\gamma^2}{2\alpha}}\frac{[2\gamma\cos(\omega\gamma) + 4\gamma^2\sin(\omega\gamma)]}{(\frac{1}{4}+\gamma^2)^2} \\
&\quad -e^{\frac{-\gamma^2}{2\alpha}}\frac{\left[\frac{\gamma}{\alpha}\cos(\omega\gamma) + \frac{2\gamma^2}{\alpha}\sin(\omega\gamma)\right]}{\frac{1}{4}+\gamma^2} \\
\Longrightarrow t'(\gamma) &= e^{\frac{-\gamma^2}{2\alpha}}\frac{\left[\cos(\omega\gamma)(2\omega\gamma - \frac{\gamma}{\alpha}) - \sin(\omega\gamma)(\omega - 2 + \frac{2\gamma^2}{\alpha})\right]}{\frac{1}{4}+\gamma^2} \\
&\quad -e^{\frac{-\gamma^2}{2\alpha}}\frac{[2\gamma\cos(\omega\gamma) + 4\gamma^2\sin(\omega\gamma)]}{(\frac{1}{4}+\gamma^2)^2}
\end{aligned}$$

Then we see

$$\begin{aligned}
|t'(\gamma)| &< e^{\frac{-\gamma^2}{2\alpha}} \left[ \frac{2\omega\gamma + \frac{\gamma}{\alpha} + \omega + 2 + \frac{2\gamma^2}{\alpha}}{\frac{1}{4} + \gamma^2} + \frac{2\gamma + 4\gamma^2}{(\frac{1}{4} + \gamma^2)^2} \right] \\
&< e^{\frac{-\gamma^2}{2\alpha}} \left[ \frac{2\omega\gamma + \frac{\gamma}{\alpha} + \omega + 2 + \frac{2\gamma^2}{\alpha}}{\gamma^2} + \frac{2\gamma + 4\gamma^2}{\gamma^4} \right] \\
&< e^{\frac{-\gamma^2}{2\alpha}} \left[ \frac{2\omega}{\gamma} + \frac{1}{\alpha\gamma} + \frac{\omega}{\gamma^2} + \frac{2}{\gamma^2} + \frac{2}{\alpha} + \frac{2}{\gamma^3} + \frac{4}{\gamma^2} \right] \\
&< e^{\frac{-\gamma^2}{2\alpha}} \left[ \frac{2\omega}{\gamma} + \frac{1}{\alpha\gamma} + \frac{\omega}{\gamma^2} + \frac{2}{\alpha} + \frac{2}{\gamma^3} + \frac{6}{\gamma^2} \right]
\end{aligned}$$

So we have

$$\begin{aligned}
M(\gamma,\alpha,\omega) &= e^{\frac{-\gamma^2}{2\alpha}} \left( \frac{2\omega}{\gamma} + \frac{1}{\alpha\gamma} + \frac{\omega}{\gamma^2} + \frac{2}{\alpha} + \frac{2}{\gamma^3} + \frac{6}{\gamma^2} \right) \\
&< \frac{3\omega + 11}{\gamma} = \frac{3(728) + 11}{\gamma} = \frac{2195}{\gamma}
\end{aligned}$$

This is because we know $\gamma < \alpha$ (for example; $\frac{1}{\alpha\gamma} < \frac{1}{\gamma^2} < \frac{1}{\gamma}$ and hence we do the same for the rest).

Now we can see the following

$$\begin{aligned}
|H(T,\alpha,\omega) - H^*(T,\alpha,\omega)| &< \sum_{0<\gamma\leq T} |\gamma - \gamma^*|.M(\gamma,\alpha,\omega) \\
&< 10^{-9} \times 2195 \sum_{i=1}^{N} \frac{1}{\gamma_i}
\end{aligned}$$

I should also mention here that, Chao-Plymen stated a slightly different result in their paper. Instead of 2195 they state 2192, and the reason for this is because they assumed te Riele's result: $M < \frac{3\omega+8}{\gamma}$. However, this being said the slight differences of our results does not make any difference in the results that Chao-Plymen achieved.

Looking at the inequality above, we see that te Riele yields an approximation $\gamma_i^*$ to $\gamma_i$ for which $|{\gamma_i}^* - \gamma_i| < 10^{-9}$. Plymen and Chao found numerically that $\sum_{i=1}^{N} \frac{1}{\gamma_i} < 12$ so that $|H - H^*|$ is bounded above by $3 \times 10^{-5}$.

$$|H(T,\alpha,\omega) - H^*(T,\alpha,\omega)| < 10^{-9} \times 2195 \times 12 = 2.6340 \times 10^{-5} < 3 \times 10^{-5}$$

Plymen and Chao used machine computation for $H^*$ and for the quantities ${s_1}', s_2, s_3, s_4, s_5, s_6$. They denote $R'_M = {s_1}' + s_2 + s_3 + s_4 + s_5 + s_6$. They chose specific values for $(\omega, \alpha, \eta, A)$ so that the following inequality is satisfied

$$H_M^* - (1 + R'_M) > 1 \times 10^{-4}$$

They state confidently that the cumulative effects of adverse numerical phenomena lie well below the threshold of $1 \times 10^{-6}$. A simple strategy for selecting suitable $(\omega, \alpha, \eta, A)$ is to make order of magnitude estimates of the exponential factors in the terms $s_2, s_3, s_4, s_5, s_6$.

**Remark.** Should be noted here that with $N = 2,000,000$ and $\omega$ near 728, the exponential in $s_5 = e^{\frac{-T^2}{2\alpha}}\{\frac{\alpha}{\pi T^2}\log\frac{T}{2\pi} + \frac{8\log T}{T} + \frac{4\alpha}{T^3}\}$ will be of the same order of magnitude as the leading term $s_1{}' = \frac{2.1111}{\omega-\eta}$ when $\alpha$ is near $10^{11}$.

A larger $\alpha$ means that $\eta$ may be taken smaller, resulting in a narrower interval $(\omega-\eta, \omega+\eta)$. After a lot of work, Plymen and Chao chose the following values to keep $H_M^* - (1 + R'_M)$ safely above $1 \times 10^{-4}$.

$$\omega = 727.952018$$
$$\eta = 0.00016$$
$$\alpha = 1.34 \times 10^{11}$$
$$A = 1.022 \times 10^{7}$$

They also found that $H_M^* > 1.006569$. This value was obtained by computing the sum up to $T$ using $t(\gamma^*, \alpha, \omega)$ by machine. Hence, Plymen and Chao made the following estimates:

$$s_1{}' < 0.002901$$
$$s_2 < 10^{-49}$$
$$s_3 < 10^{-746}$$
$$s_4 < 10^{-740}$$
$$s_5 < 0.003380$$
$$s_6 < 10^{-5}$$

Using the Symbolic Math Toolbox, version 5.4 in MATLAB R2010a, I double-checked the above estimates and found the following (which confirmed the above),

$$s_1 = \frac{2.1111}{(\omega - \eta)}$$
$$= 0.002900054415412 < 0.002901$$
$$s_2 = 4(\omega + \eta)e^{\frac{-(\omega-\eta)}{6}}$$
$$= 5.932671993417954 \times 10^{-50} < 10^{-49}$$
$$s_3 = \frac{2e^{\frac{-\alpha\eta^2}{2}}}{\sqrt{2\pi\alpha}\eta} = \frac{670^{\frac{1}{2}}}{1072\pi^{\frac{1}{2}}e^{\frac{8576}{5}}}$$

The above expression was obtained through the following commands on MATLAB

```
omega = 727.952018;
eta = 0.00016;
alpha = 1.34e11;
sym s3;
a = sym(alpha);
e = sym(eta);
digits(747);
zero=0;i=1;
while (zero==0)
    i=i+1;
    vpa(sym((2*exp((-a*e^2)/2))/(sqrt(2*pi*a)*e))
    *sym(1000000000000)^sym(i))
    zero=double(vpa(sym((2*exp((-a*e^2)/2))/(sqrt(2*pi*a)*e))
    *sym(1000000000000)^sym(i)));
    %vpa()
end;
zero
i
vpa(sym((2*exp((-a*e^2)/2))/(sqrt(2*pi*a)*e)))
```

We then have the value of $s_3$;

$$\begin{aligned} s_3 &= \frac{1.707545634263533 \times 10^{-315}}{(10^{12})^{36}} \\ &= \frac{1.707545634263533 \times 10^{-315}}{10^{432}} \\ &= 1.707545634263533 \times 10^{-747} < 10^{-746} \\ s_4 &= 0.08\sqrt{\alpha}e^{\frac{-\alpha\eta^2}{2}} = \frac{1600(335)^{\frac{1}{2}}}{e^{\frac{8576}{5}}} \end{aligned}$$

The above expression was obtained through the following commands on MATLAB

```
omega = 727.952018;
eta = 0.00016;
alpha = 1.34e11;
sym s4;
a = sym(alpha);
e = sym(eta);
digits(741);
zero=0;i=1;
while (zero==0)
    i=i+1;
    vpa(sym((0.08)*sqrt(a)*exp((-a*e^2)/2))*sym(1000000000000)^sym(i))
    zero=double(vpa(sym((0.08)*sqrt(a)*exp((-a*e^2)/2))
    *sym(1000000000000)^sym(i)));
    %vpa()
end;
zero
i
vpa(sym((0.08)*sqrt(a)*exp((-a*e^2)/2)))
```

We then have the value of $s_4$;

$$\begin{aligned}
s_4 &= \frac{3.670907748600462 \times 10^{-321}}{(10^{12})^{35}} \\
&= \frac{3.670907748600462 \times 10^{-321}}{10^{420}} \\
&= 3.670907748600462 \times 10^{-741} < 10^{-740} \\
s_5 &= e^{\frac{-T^2}{2\alpha}} \left\{ \frac{\alpha}{\pi T^2} \log \frac{T}{2\pi} + \frac{8 \log T}{T} + \frac{4\alpha}{T^3} \right\} \\
&= 0.003379923143389 < 0.003380 \\
s_6 &= A \log A e^{(\frac{-A^2}{2\alpha} + \frac{\omega+\eta}{2})} \left\{ 4\alpha^{\frac{-1}{2}} + 15\eta \right\} \\
&= 2.591323639104662 \times 10^{-6} < 10^{-5}
\end{aligned}$$

Using Lehman's theorem Plymen and Chao see that

$$\begin{aligned}
I(\omega, \eta) &\geq H - (1 + |R'|) \\
&\geq H_M^* - (1 + R'_M) - |H - H^*| - 1 \times 10^{-6} \\
&\geq (1 \times 10^{-4}) - (3 \times 10^{-5}) - (1 \times 10^{-6}) \\
&\geq 6.900000000000001 \times 10^{-5} \\
&\geq 6 \times 10^{-5} > 0
\end{aligned}$$

which shows there's a value of $u$ in the interval

$$(\omega - \eta, \omega + \eta) = (727.951858, 727.952178)$$

for which $\pi(e^u) > li(e^u)$.

However, we see here that Chao-Plymen obtained a slightly different result compared to what I have just obtained above. Namely, $I(\omega, \eta) \geq 2 \times 10^{-4} > 0$. But, we also know that since both our results are positive, then this does not make much difference to the rest of their paper.

Now we define:

$$\begin{aligned}
F(u) :&= u e^{\frac{-u}{2}} \{\pi(e^u) - li(e^u)\} \\
&= u e^{\frac{-u}{2}} (\pi - li)(e^u)
\end{aligned}$$

By Lehman's theorem,

$$\begin{aligned} I(\omega,\eta) &= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\{\pi(e^u)-li(e^u)\}du \\ &= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)F(u)du \geq \delta \end{aligned}$$

where $\delta = 6{\times}10^{-5}$. So we have the following, (since $K(u)du$ is a probability measure):

$$0 < \delta \leq \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sup F(u)du < \sup F(u)$$

The reason for the above is, we know that $\int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \leq 1$ since $\frac{e^{-kx^2}}{\int e^{-kx^2}} = 1$.

Then we also have $F(u) > \delta$,

$$\begin{aligned} F(u) &= ue^{\frac{-u}{2}}\{\pi(e^u)-li(e^u)\} \\ \delta &= ue^{\frac{-u}{2}}\{\pi(e^u)-li(e^u)\} \\ \frac{\delta}{ue^{\frac{-u}{2}}} &= \pi(e^u)-li(e^u) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \pi(e^u)-li(e^u) &> \frac{\delta}{u}e^{\frac{u}{2}} > \frac{6\times10^{-5}}{727}e^{\frac{727}{2}} \\ \pi(e^u)-li(e^u) &> \frac{6\times10^{-5}}{727}e^{\frac{727}{2}} = 6.062629262085750\times10^{150} \\ \pi(e^u)-li(e^u) &> 6\times10^{150} \end{aligned}$$

Now we see

$$li(N+r)-li(N)-\frac{r}{\log N} = \int_N^{N+r}\frac{du}{\log u} - \frac{r}{\log N} \leq 0 \leq \pi(N+r)-\pi(N)$$

So we have,

$$\pi(N+r)-li(N+r) \geq \pi(N)-li(N)-\frac{r}{\log N}$$

By definition we have,

$$\begin{aligned} li(N+r) &= \int_0^{N+r}\frac{du}{\log u} \\ li(N) &= \int_0^{N}\frac{du}{\log u} \end{aligned}$$

$$\Longrightarrow li(N+r)-li(N) = \int_0^{N+r}\frac{du}{\log u} - \int_0^{N}\frac{du}{\log u} = \int_N^{N+r}\frac{du}{\log u}$$

We also see

$$\int_N^{N+r}\frac{du}{\log u} \leq \frac{r}{\log N}$$

$$\Longrightarrow li(N+r) - li(N) - \frac{r}{\log N} = \int_N^{N+r} \frac{du}{\log u} - \frac{r}{\log N} \leq 0 \leq \pi(N+r) - \pi(N)$$

Therefore, the above shows us that if $\pi(e^u) - li(e^u) > 6 \times 10^{150} > 0$ then $\pi(e^u) - li(e^u)$ will remain positive for another $[6 \times 10^{150} \times u]$ consecutive integers.

Here

$$6 \times 10^{150} \times 727 = 4.362000000000000 \times 10^{153} > 10^{153}$$

Hence we have the following theorem

**Theorem 3.0.4.** *There is a value of $u$ in the interval*

$$(727.951858, 727.952178)$$

*for which $\pi(e^u) - li(e^u) > 6 \times 10^{150}$. There are at least $10^{153}$ successive integers $x$ between $exp(727.951858)$ and $exp(727.952178)$ for which $\pi(x) > li(x)$.*

It should be noted here that, this theorem above is phrased different to Chao-Plymen's theorem since our values for $\pi(e^u) - li(e^u)$ differ.

I will now state Lehman's Theorem again, since in Yannick Saouter and Patrick Demichel's paper the terms are defined slightly different.

**Theorem 3.0.5.** *Lehman's Theorem*

*Let $A$ be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\rho = \beta + i\gamma$ of Riemann zeta function $\zeta(s)$ for which $0 < \gamma \leq A$. Let $\alpha, \eta$ and $\omega$ be positive values such that $\omega - \eta > 1$ and the following conditions hold:*

$$\frac{4A}{\omega} \leq \alpha \leq A^2 \tag{3.1}$$

$$\frac{2A}{\alpha} \leq \eta \leq \frac{\omega}{2} \tag{3.2}$$

*Let*

$$K(y) = \sqrt{\frac{\alpha}{2\pi}} e^{\frac{-\alpha y^2}{2}}$$

$$I(\omega, \eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{\frac{-u}{2}} \{\pi(e^u) - li(e^u)\}.du \tag{3.3}$$

*Then for $2\pi e < T \leq A$, we have*

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} + R \tag{3.4}$$

*where* $|R| \leq S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ *with*

$$S_1 = \frac{3}{\omega - \eta} + 4(\omega + \eta)e^{\frac{-(\omega-\eta)}{6}} = s_1 - \frac{0.05}{\omega - \eta} + s_2$$

$$S_2 = \frac{2e^{\frac{-\alpha\eta^2}{2}}}{\sqrt{2\pi\alpha}\eta} = s_3$$

$$S_3 = 0.08\sqrt{\alpha}e^{\frac{-\alpha\eta^2}{2}} = s_4$$

$$S_4 = e^{\frac{-T^2}{2\alpha}}\left\{\frac{\alpha}{\pi T^2}\log\frac{T}{2\pi} + \frac{8\log T}{T} + \frac{4\alpha}{T^3}\right\} = s_5$$

$$S_5 = \frac{0.05}{\omega - \eta} = s_1 - \frac{3}{\omega - \eta}$$

$$S_6 = A\log Ae^{(\frac{-A^2}{2\alpha} + \frac{(\omega+\eta)}{2})}\left\{4\alpha^{\frac{-1}{2}} + 15\eta\right\} = s_6$$

*If the Riemann hypothesis holds, then conditions* (3.1) *and* (3.2) *may be omitted and the term* $S_6$ *may be omitted in the upper bound for* $R$*.*

**Remark.** Note here that $s_1, s_2, s_3, s_4, s_5, s_6$ should be referenced to Theorem 2.0.2.

The complete proof of Lehman's paper can be found in his paper on the difference $\pi(x) - li(x)$. Saouter-Demichel see that the application of Lehman's theorem makes two essential assumptions. First of all, the Riemann hypothesis should be checked up to height $A$. Second, explicit values for the zeros of $\zeta$ have to be known up to height $T$. They state if these both conditions are met then one can estimate the integral (3.3) and (3.4).

We see here that Lehman's method amounts to finding appropriate values for $\alpha$ and $\omega$ such that the first two terms in the right-hand side of (3.4) sum to a positive value larger than the associated error term $|R|$. The integral (3.3) is established to be positive and thus by the fact that $K$ is positive, the term $\{\pi(e^u) - li(e^u)\}$ must admit some positive values for $u$ in the interval $[\omega - \eta, \omega + \eta]$.

# Chapter 4

# Improvements

Saouter-Demichel suggest that improvements on the error term $R$ are possible. In fact, the dominating term in $R$ is generally $S_1 = \frac{3}{\omega-\eta} + 4(\omega+\eta)e^{\frac{-(\omega-\eta)}{6}}$. We note here that Lehman (in his seminal work) derived $S_1$ from an upper bound for $\pi(x)$ that was obtained by Rosser and Schoenfeld. Chao-Plymen derived a tighter bound by using recent results of Panaitopol. By doing this, they could lower the constant 3 in the first term of $S_1$ to 2.1457. Now at this point Saouter-Demichel use a result by Dusart [3] to prove that this constant can be replaced by 2 (with some other terms in $S_1$).

Moreover, they state that this value cannot be improved. The reason behind this is, because if we look at Dusart's theorem below for $x \geq 355991$, when we look at $\pi(x^{\frac{1}{2}})$ then we see that the 1 in $\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right)$ is replaced by 2. i.e. $\pi(x^{\frac{1}{2}}) = \frac{x^{\frac{1}{2}}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right)$, therefore the value cannot be improved.

**Theorem 4.0.6.** *Dusart's Theorem*

*If* $x \geq 32299$ *we have*

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x)$$

*If* $x \geq 355991$, *we have*

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right)$$

With the help of this theorem, Saouter-Demichel prove the following theorem.

**Theorem 4.0.7.** *Under the hypothesis of Lehman's theorem and if* $\omega - \eta > 25.57$ *equation* (3.4) *still holds if* $S_1$ *is replaced by*

$$S_1' = \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \log 2(\omega+\eta)e^{\frac{-(\omega-\eta)}{2}} + \frac{2}{\log 2}(\omega+\eta)e^{\frac{-(\omega-\eta)}{6}}$$

*Proof.* Proceeding from Lehman's approach. Let

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + ...$$

and let

$$J_0(x) = lim_{\epsilon \longrightarrow 0}\frac{1}{2}\left\{J(x+\epsilon) + J(x-\epsilon)\right\}$$

The Riemann-von Mangoldt formula states that for $x > 1$,

$$J_0(x) = li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{+\infty} \frac{du}{(u^2-1)u\log u} - \log 2$$

where $\rho$ runs over the zeros of the function $\zeta$ in the critical strip. So we have the following

$$\frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + ... \leq \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}})\left[\frac{\log x}{\log 2}\right]$$

because, we know that $J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + ...$ has at most $\frac{\log x}{\log 2}$ terms.

Then we use Dusart's theorem and the following classic bound $\pi(x) \leq \frac{2x}{\log x}$.

Thus, if $x \geq 355991$ we see

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right)$$

If $x \geq 355991^2$ we have

$$\begin{aligned}
\pi(x^{\frac{1}{2}}) &\leq \frac{x^{\frac{1}{2}}}{\log x^{\frac{1}{2}}}\left\{1 + \frac{1}{\log x^{\frac{1}{2}}} + \frac{2.51}{(\log x^{\frac{1}{2}})^2}\right\} \\
&\leq \frac{x^{\frac{1}{2}}}{\frac{1}{2}\log x}\left\{1 + \frac{1}{\frac{1}{2}\log x} + \frac{2.51}{(\frac{1}{2}\log x)^2}\right\} \\
&\leq \frac{2x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{2.51}{\frac{1}{4}\log^2 x}\right\} \\
&\leq \frac{2x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\}
\end{aligned}$$

Now we use the classic bound to find the term $\pi(x^{\frac{1}{3}})$.

$$\begin{aligned}\pi(x) &\leq \frac{2x}{\log x}\\ \Longrightarrow \pi(x^{\frac{1}{3}}) &\leq \frac{2x^{\frac{1}{3}}}{\log x^{\frac{1}{3}}}\\ &\leq \frac{2x^{\frac{1}{3}}}{\frac{1}{3}\log x}\\ &\leq \frac{6x^{\frac{1}{3}}}{\log x}\end{aligned}$$

Therefore, looking back we have

$$\begin{aligned}\frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots &\leq \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}})\left[\frac{\log x}{\log 2}\right]\\ \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots &\leq \frac{1}{2}\frac{2x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\} + \frac{1}{3}\frac{6x^{\frac{1}{3}}}{\log x}\left[\frac{\log x}{\log 2}\right]\\ \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots &\leq \frac{x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\} + \frac{2x^{\frac{1}{3}}}{\log x}\left[\frac{\log x}{\log 2}\right]\\ \Longrightarrow \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots &\leq \frac{x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\} + \frac{2x^{\frac{1}{3}}}{\log 2}\end{aligned}$$

Now we see the following:

$$\begin{aligned}\pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots &= li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{\infty} \frac{du}{(u^2-1)u\log u} - \log 2\\ \pi(x) &\geq li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{\infty} \frac{du}{(u^2-1)u\log u} - \log 2\\ &\quad -\frac{1}{2}\pi(x^{\frac{1}{2}}) - \frac{1}{3}\pi(x^{\frac{1}{3}}) - \ldots\\ \pi(x) &\geq li(x) - \sum_{\rho} li(x^{\rho}) - \frac{x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\}\\ &\quad -\frac{2x^{\frac{1}{3}}}{\log 2} + \int_x^{\infty} \frac{du}{(u^2-1)u\log u} - \log 2\end{aligned}$$

$$\Longrightarrow \pi(x) \geq li(x) - \sum_{\rho} li(x^{\rho}) - \frac{x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\} - \frac{2x^{\frac{1}{3}}}{\log 2} - \log 2 \qquad (4.1)$$

Since $x > 1$ in the Riemann-von Mangoldt formula then the term $\int_x^{\infty} \frac{du}{(u^2-1)u\log u}$ is positive. Hence we suppress this term in the above inequality as we replace the " $=$ " sign by " $\geq$ " sign.

We now put $x = e^u$ in (4.1) and then if $u \geq 25.57$ we have the following

$$\begin{aligned}
\pi(x) &\geq li(x) - \sum_\rho li(x^\rho) - \frac{x^{\frac{1}{2}}}{\log x}\left\{1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right\} \\
&\quad -\frac{2x^{\frac{1}{3}}}{\log 2} - \log 2 \\
\pi(e^u) &\geq li(e^u) - \sum_\rho li(e^{\rho u}) - \frac{e^{\frac{u}{2}}}{\log e^u}\left\{1 + \frac{2}{\log e^u} + \frac{10.04}{(\log e^u)^2}\right\} \\
&\quad -\frac{2e^{\frac{u}{3}}}{\log 2} - \log 2 \\
&\geq li(e^u) - \sum_\rho li(e^{\rho u}) - \frac{e^{\frac{u}{2}}}{u}\left\{1 + \frac{2}{u} + \frac{10.04}{u}\right\} - \frac{2e^{\frac{u}{3}}}{\log 2} \\
&\quad - \log 2 \\
\pi(e^u) - li(e^u) &\geq -\sum_\rho li(e^{\rho u}) - \frac{e^{\frac{u}{2}}}{u}\left\{1 + \frac{2}{u} + \frac{10.04}{u^2}\right\} - \frac{2e^{\frac{u}{3}}}{\log 2} - \log 2 \\
ue^{\frac{-u}{2}}\{\pi(e^u) - li(e^u)\} &\geq -\sum_\rho ue^{\frac{-u}{2}} li(e^{\rho u}) - ue^{\frac{-u}{2}}\frac{e^{\frac{u}{2}}}{u} - ue^{\frac{-u}{2}}\frac{2e^{\frac{u}{2}}}{u^2} \\
&\quad -ue^{\frac{-u}{2}}\frac{10.04e^{\frac{u}{2}}}{u^3} - ue^{\frac{-u}{2}}\frac{2e^{\frac{u}{3}}}{\log 2} - ue^{\frac{-u}{2}}\log 2 \\
&\geq -\sum_\rho ue^{\frac{-u}{2}} li(e^{\rho u}) - 1 - \frac{2}{u} - \frac{10.04}{u^2} - \frac{2ue^{\frac{-u}{6}}}{\log 2} \\
&\quad - \log 2ue^{\frac{-u}{2}} \\
\Longrightarrow ue^{\frac{-u}{2}}\{\pi(e^u) - li(e^u)\} &\geq -1 - \sum_\rho ue^{\frac{-u}{2}} li(e^{\rho u}) - \frac{2}{u} - \frac{10.04}{u^2} - \frac{2u}{\log 2}e^{\frac{-u}{6}} \\
&\quad - \log 2ue^{\frac{-u}{2}}
\end{aligned}$$

Saouter-Demichel follow Lehman's proof and they derive the equation

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} + R$$

with the same bounding terms $S_2, S_3, S_4, S_5$ and $S_6$. They suggest that term $S_2$ comes from bounding the two tail integrals $\int_{-\infty}^{\omega-\eta} K(u-\omega).du$ and $\int_{\omega+\eta}^{+\infty} K(u-\omega).du$. The reason behind this is because we see from Lehman's theorem that

$$\begin{aligned}
I(\omega,\eta) &= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\{\pi(e^u)-li(e^u)\}.du \\
&\geq \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\Big[-1-\sum_{\rho} ue^{\frac{-u}{2}}li(e^{\rho u})-\Big(\frac{2}{u}+\frac{10.04}{u^2}+\frac{2u}{\log 2}e^{\frac{-u}{6}} \\
&\quad +\log 2ue^{\frac{-u}{2}}\Big)\Big].du \\
&\geq -\int_{\omega-\eta}^{\omega+\eta} K(u-\omega).du-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho} ue^{\frac{-u}{2}}li(e^{\rho u}).du \\
&\quad -\int_{\omega_\eta}^{\omega+\eta} K(u-\omega)\left(\frac{2}{u}+\frac{10.04}{u^2}+\frac{2u}{\log 2}e^{\frac{-u}{6}}+\log 2ue^{\frac{-u}{2}}\right).du \\
&\geq -\int_{\omega-\eta}^{\omega+\eta} K(u-\omega).du-\sum_{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}li(e^{\rho u}).du \\
&\quad -\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{2}{u}+\frac{10.04}{u^2}+\frac{2u}{\log 2}e^{\frac{-u}{6}}+\log 2ue^{\frac{-u}{2}}\right).du
\end{aligned}$$

So its easier to see that the term $S_2$ comes from bounding the two tail integrals $\int_{-\infty}^{\omega-\eta} K(u-\omega).du$ and $\int_{\omega+\eta}^{+\infty} K(u-\omega).du$. Likewise, they also stated that the terms $S_3, S_4, S_5$ and $S_6$ come from the estimate of $\sum_\rho \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}li(e^{\rho u}).du$. Back to concentrating on the first term namely $S_1$, this term comes from bounding the following expression:

$$(I(\omega,\eta)=)J=\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{2}{u}+\frac{10.04}{u^2}+\frac{2u}{\log 2}e^{\frac{-u}{6}}+\log 2ue^{\frac{-u}{2}}\right).du$$

Both terms in the integral are positive and $\int_{-\infty}^{+\infty} K(y)dy=1$ where $K(y)=\sqrt{\frac{\alpha}{2\pi}}e^{\frac{-\alpha y^2}{2}}$.

$$\begin{aligned}
\int_{-\infty}^{+\infty}\sqrt{\frac{\alpha}{2\pi}}e^{\frac{-\alpha}{2}y^2}.dy &= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{-\alpha}{2}y^2}.dy \\
&= \sqrt{\frac{\alpha}{2\pi}}\left(\sqrt{\frac{2\pi}{\alpha}}\right)=1
\end{aligned}$$

The reason for the above is because of the following well-known result:

Let $I=\int_{-\infty}^{+\infty} e^{-x^2}.dx$ so we have

$$\begin{aligned}
I^2 &= \int_{-\infty}^{+\infty} e^{-x^2}.dx\int_{-\infty}^{+\infty} e^{-y^2}.dy \\
&= \iint_{R^2} e^{-(x^2+y^2)}.dxdy \\
&= \iint_{R^2} e^{-r^2}r.drd\theta
\end{aligned}$$

Taking polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$ then $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2$. Hence, we have the following

$$\begin{aligned}\implies I^2 &= 2\pi\int_0^\infty e^{-r^2} r.dr \\ &= \pi\int_0^\infty e^{-t}.dt = \pi\end{aligned}$$

Therefore, $I = \sqrt{\pi}$. Using this we know that

$$\int_{-\infty}^{+\infty}\sqrt{\frac{\alpha}{2\pi}}e^{\frac{-\alpha y^2}{2}}.dy = 1$$

Hence, now we see that

$$\begin{aligned} J &\leq \frac{2}{u} + \frac{10.04}{u^2} + \frac{2u}{\log 2}e^{\frac{-u}{6}} + \log 2ue^{\frac{-u}{2}} \\ \implies J &\leq \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \frac{2(\omega+\eta)}{\log 2}e^{\frac{-(\omega-\eta)}{6}} + \log 2(\omega+\eta)e^{\frac{-(\omega-\eta)}{2}}\end{aligned}$$

In the first term we have that $u = \omega-\eta$, as it is decreasing. Similarly, for the second term we also have $u = \omega-\eta$. In the third term, since it is increasing we have $u = \omega+\eta$ and in the exponential terms we have $u = \omega-\eta$, since they are decreasing terms. □

# Chapter 5

# Numerical Results (ii)

Saouter-Demichel mention that the use of the previous theorems presupposes numerical verifications of the Riemann hypothesis up to height $A$. We know that Lehman made a verification on his own on the first 250,000 zeros, giving $A = 170571.35$. Since then, a lot of work has been done to check numerically the Riemann hypothesis up to larger and larger heights. We find that in 2001, van de Lune established that the conjecture is confirmed for the first 10,000,000,000 zeros up to height $A = 3293531632.415$. It was stated that this value can in fact set an upper bound for the value of $A$ that can be used in Lehman's theorem.

Now the following two important recent verifications should be mentioned. The first was performed by Gourdon and Demichel [4] using a fast multiple evaluation algorithm for $\zeta$ invented by Odlyzko with their implementation, the conjecture has been verified up to the $10^{13}$-th zero. The second is the distributed ZetaGrid project managed by Wedeniwski [13] which was active between 2002 and 2005.

However, we now note that the official status of these verifications are not clear. Gourdon and Demichel's work has never been independently verified and in the case of the ZetaGrid project, it was not established that all zeros were checked.

For $0 < T \leq A$, we know that the real part of zeros $\rho = \beta + i\gamma$ of $\zeta$ such that $|\gamma| < T$ is equal to $\frac{1}{2}$ ($\beta = \frac{1}{2}$). Moreover, zero of $\zeta$ in the critical strip occur as conjugate pairs, so the sum to evaluate is:

$$\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} = \sum_{0<\gamma\leq T} \frac{\cos(\gamma\omega) + 2\gamma\sin(\gamma\omega)}{\frac{1}{4}+\gamma^2} e^{\frac{-\gamma^2}{2\alpha}}$$

This is because we look at the following:

$$\frac{e^{i\omega\gamma}}{\rho} = \frac{e^{i\omega\gamma}}{\beta + i\gamma} + \frac{e^{-i\omega\gamma}}{\beta - i\gamma}$$

and also using the identity

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

then we have

$$\begin{aligned}
\frac{e^{i\omega\gamma}}{\rho} &= \frac{e^{i\omega\gamma}}{\beta + i\gamma} + \frac{e^{-i\omega\gamma}}{\beta - i\gamma} \\
&= \frac{\cos(\omega\gamma) + i\sin(\omega\gamma)}{\beta + i\gamma} + \frac{\cos(\omega\gamma) - i\sin(\omega\gamma)}{\beta - i\gamma} \\
&= \frac{(\beta - i\gamma)(\cos(\omega\gamma) + i\sin(\omega\gamma)) + (\beta + i\gamma)(\cos(\omega\gamma) - i\sin(\omega\gamma))}{(\beta + i\gamma)(\beta - i\gamma)} \\
&= \frac{\beta\cos(\omega\gamma) - i\gamma\cos(\omega\gamma) + i\beta\sin(\omega\gamma) - i^2\gamma\sin(\omega\gamma) + \beta\cos(\omega\gamma)}{\beta^2 - i^2\gamma^2} \\
&\quad + \frac{i\gamma\cos(\omega\gamma) - i\beta\sin(\omega\gamma) - i^2\gamma\sin(\omega\gamma)}{\beta^2 - i^2\gamma^2} \\
&= \frac{2\beta\cos(\omega\gamma) + 2\gamma\sin\omega\gamma}{\beta^2 + \gamma^2} \\
&= \frac{2(\frac{1}{2})\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{(\frac{1}{2})^2 + \gamma^2} \\
\implies \frac{e^{i\omega\gamma}}{\rho} &= \frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4} + \gamma^2}
\end{aligned}$$

Therefore, we have

$$\sum_{0<|\gamma|\leq T} \frac{e^{i\omega\gamma}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} = \sum_{0<|\gamma|\leq T} \frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2} e^{\frac{-\gamma^2}{2\alpha}}$$

For their numerical computations, Saouter-Demichel computed the first 22,000,000 zeros of $\zeta$. This was completed in two phases. First, an approximation was computed by the Riemann-Siegel formula and then second of all precision was improved up to 9 decimal digits using correction terms in this formula. The Riemann-Siegel formula here states, if $M$ and $N$ are non-negative integers, then the zeta function is equal to

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \gamma(1-s)\sum_{n=1}^{M} \frac{1}{n^{1-s}} + R(s)$$

where

$$\gamma(s) = \pi^{\frac{1}{2}-s} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}$$

is the factor appearing in the functional equation $\zeta(s) = \gamma(s)\zeta(1-s)$ and

$$R(s) = \frac{-\Gamma(1-s)}{2\pi i} \int \frac{(-x)^{s-1}e^{-Nx}}{e^x - 1}.dx$$

is a contour integral whose contour starts and ends at $+\infty$ and circles the singularities of absolute value at most $2\pi M$.

The last zero of their database gave them the following value for $T$,

$T = 10379599.727431060$. The relative precision that can be expected when computing

$$\sum_{0<|\gamma|\leq T} \frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2} e^{\frac{-\gamma^2}{2\alpha}}$$

is then bounded by

$$\Delta I = 10^{-9} \sum_{0<\gamma\leq T} \frac{\partial}{\partial\gamma}\left\{\frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2} e^{\frac{-\gamma^2}{2\alpha}}\right\}$$

The reason for the above is because we see from the mean-value theorem that

$|t(\gamma^*) - t(\gamma)| = |\gamma^* - \gamma||t'(\bar{\gamma})|$ with $|\bar{\gamma} - \gamma| < |\gamma^* - \gamma|$ where $\gamma < \bar{\gamma}$ and we let

$$t(\gamma) = \frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2} e^{\frac{-\gamma^2}{2\alpha}}$$

Then we find in te-Riele's paper that he yields an approximation for $|\gamma^* - \gamma| < 10^{-9}$. So it follows that

$$\begin{aligned}
\Delta I &= |\gamma^* - \gamma||t'(\gamma)| \\
&= 10^{-9} \sum_{0<\gamma\leq T} \frac{\partial}{\partial\gamma} t(\gamma) \\
&= 10^{-9} \sum_{0<\gamma\leq T} \frac{\partial}{\partial\gamma}\left\{\frac{\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2} e^{\frac{-\gamma^2}{2\alpha}}\right\} \\
&= 10^{-9} \sum_{0<\gamma\leq T} \left\{ e^{\frac{-\gamma^2}{2\alpha}} \left[\frac{\cos(\omega\gamma)(2\omega\gamma - \frac{\gamma}{\alpha}) + \sin(\omega\gamma)(2 - \omega - \frac{2\gamma^2}{\alpha})}{\frac{1}{4}+\gamma^2}\right]\right\} \\
&\quad -10^{-9} \sum_{0<\gamma\leq T} \left\{ e^{\frac{-\gamma^2}{2\alpha}} \left[\frac{2\gamma\cos(\omega\gamma) + 4\gamma^2\sin(\omega\gamma)}{(\frac{1}{4}+\gamma^2)^2}\right]\right\}
\end{aligned}$$

In Saouter-Demichel's computations they set $\alpha = 6 \times 10^{12}$ and $\omega \approx 727.95$. They computed the associated precision given in the equation

$$\begin{aligned} \Delta I &= 10^{-9} \sum_{0<\gamma\leq T} \Big\{ e^{\frac{-\gamma^2}{2\alpha}} \Big[ \frac{(2\omega\gamma - \frac{\gamma}{\alpha})\cos(\omega\gamma) + (2 - \omega - \frac{2\gamma^2}{\alpha})\sin(\omega\gamma)}{\frac{1}{4}+\gamma^2} \Big] \\ &\quad - e^{\frac{-\gamma^2}{2\alpha}} \Big[ \frac{2\gamma\cos(\omega\gamma) + 4\gamma^2\sin(\omega\gamma)}{(\frac{1}{4}+\gamma^2)^2} \Big] \Big\} \end{aligned}$$

Using the following MATLAB commands, we get the following:

```
T = 10379599.727431060;
syms gamma omega alpha s c pf;
omega = sym(727.95);
alpha = sym(6*10^(12));
c = cos(omega*gamma);
s = sin(omega*gamma);
pf= exp((-gamma^2)/(2*alpha));
I1 = pf*((2*omega*gamma-gamma/alpha)*c
+(2-omega-(2*gamma^2)/(alpha))*s)/(1/4+gamma^2);
I2 = pf*(2*gamma*c+4*gamma^2*s)/(1/4+gamma^2)^2;
%I=10^(-9)*(I1 - I2)
S = sym(10^(-9))*(pf*((2*omega*gamma-gamma/alpha)*c
+(2-omega-(2*gamma^2)/(alpha))*s)/(1/4+gamma^2)
-pf*(2*gamma*c+4*gamma^2*s)/(1/4+gamma^2)^2);
int(S, gamma, 0, T)

ans = 0.0000000039999999999903698814458835062958
```

$$\Longrightarrow \Delta I < 7 \times 10^{-7}$$

Saouter-Demichel chose to make the computation explicitly, instead of using bounds as in previous work, in order to obtain the best possible precision. On the last page there are figures 2-4 which show the different regions where $\pi(x) - li(x)$ is positive by Chao-Plymen.

Numerically, the least value for $\omega$ giving a positive value for $I(\omega, \gamma)$ is $\omega = 727.951335792$ (this being observed from the figures 2-4). By studying the remainder terms and $S_6$, Saouter-Demichel found that $A = 6.85 \times 10^7$ is the value minimising the interval length through long detailed computation. We then see

$$\begin{aligned} \eta = \frac{2A}{\alpha} &= \frac{2(6.85 \times 10^7)}{6 \times 10^{12}} \\ &= 2.283333333333333 \times 10^{-5} \end{aligned}$$

and so the following conditions are satisfied:

(1)

$$\begin{aligned} \frac{4A}{\omega} &\leq \alpha \leq A^2 \\ \frac{4(6.85 \times 10^7)}{727.951335792} &\leq 6 \times 10^{12} \leq (6.85 \times 10^7)^2 \\ 3.763987872924117 \times 10^5 &\leq 6 \times 10^{12} \leq 4.692250000000000 \times 10^{15} \end{aligned}$$

(2)

$$\begin{aligned} \frac{2A}{\alpha} &\leq \eta \leq \frac{\omega}{2} \\ \frac{2(6.85 \times 10^7)}{6 \times 10^{12}} &\leq \eta \leq \frac{727.951335792}{2} \\ 2.283333333333333 \times 10^{-5} &\leq \eta \leq 3.639756678960000 \times 10^2 \end{aligned}$$

We also see the condition of Theorem 3.2 is met too, i.e. $\omega - \eta = 727.951335792 - 0.00002283333334 = 727.9513129586667 > 25.57$.

By computation, they see

$$\sum_{0<|\gamma|\leq T} \frac{e^{i\omega\gamma}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} = -1.002906086981405$$

Here we see immediately that

$$\begin{aligned} I^*(\omega, \eta) &= -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\omega\gamma}}{\rho} e^{\frac{-\gamma^2}{2\alpha}} \\ &= -1 - (-1.002906086981405) \\ &= 0.002906086981405 \end{aligned}$$

then this gives an estimate for $I(\omega, \eta)$. Hence we also see that,

$$I(\omega, \eta) \geq I^*(\omega, \eta) - \Delta I - S_1' - S_2 - S_3 - S_4 - S_5 - S_6 \geq 0$$

where $I^*(\omega, \eta)$ is the approximate values up to 22 million zeros and $-S_1' - S_2 - S_3 - S_4 - S_5 - S_6$ is the possible ways to round up/down the 22 million zeros. Therefore, $I(\omega, \eta)$ is positive.

Numerically, Saouter-Demichel compute the following:

$$\Delta I = 7.1645945511 \times 10^{-7}$$

$$S_1' = 0.002766382992$$

$$S_2 = 7.612616047 \times 10^{-682}$$

$$S_3 = 1.045693526 \times 10^{-674}$$

$$S_4 = 0.00003202055301$$

$$S_5 = 0.00006868591225$$

$$S_6 = 7.640973098 \times 10^{-7}$$

Using the following MATLAB commands, I double-checked the above computations.

```
omega = 727.951335792;
eta = 0.00002283333334;
alpha = 6e12;
T = 10379599.727431060;
A = 6.85e7;
S'1 = 2/(omega - eta) + 10.04/(omega - eta)^2 + log(2)*(omega + eta)
*exp(-(omega - eta)/2) + (2/log(2))*(omega + eta)*exp(-(omega - eta)/6)
S4 = ((alpha/(pi*T^2))*log(T/(2*pi))+8*(log(T)/T)+(4*alpha)/(T^3))
*exp((-T^2)/(2*alpha))
S5 = 0.05/(omega - eta)
S6 = A*log(A)*(4*alpha^(-1/2)+15*eta)*exp(-A^2/(2*alpha)+(omega + eta)/2)
```

Hence we get:

$$S_1' = 0.002766382992008$$

$$S_4 = 3.202055330100593 \times 10^{-5}$$

$$S_5 = 6.868591224429734 \times 10^{-5}$$

$$S_6 = 7.640973549617473 \times 10^{-7}$$

Now we use slightly different, more complex MATLAB commands to compute $S_2$ and $S_3$.

```
eta = 0.00002283333334;
alpha = 6e12;
sym S2;
a = sym(alpha);
e = sym(eta);
digits(683);
zero=0;i=1;
while (zero==0)
    i=i+1;
    vpa(sym((2*exp((-a*e^2)/2))/(sqrt(2*pi*a)*e))
    *sym(1000000000000)^sym(i))
    zero=double(vpa(sym((2*exp((-a*e^2)/2))/(sqrt(2*pi*a)*e))
    *sym(1000000000000)^sym(i)));
    %vpa()
end;
zero
i
vpa(sym((2*exp((-a*e^2)/2))/(sqrt(2*pi*a)*e)))
```

Therefore, we obtain the following

$$\begin{aligned} S_2 &= \frac{7.608610945955197 \times 10^{-322}}{10^{(12^{30})}} \\ &= \frac{7.608610945955197 \times 10^{-322}}{10^{360}} \\ &= 7.608610945955197 \times 10^{(-322-360)} \\ \Longrightarrow S_2 &= 7.608610945955197 \times 10^{-682} \end{aligned}$$

The following MATLAB commands were used to compute $S_3$:

```
eta = 0.00002283333334;

alpha = 6e12;

sym S3;

a = sym(alpha);

e = sym(eta);

digits(675);

zero=0;i=1;

while (zero==0)

    i=i+1;

    vpa(sym(0.08*sqrt(a)*exp((-a*e^2)/2))*sym(1000000000000)^sym(i))

    zero=double(vpa(sym(0.08*sqrt(a)*exp((-a*e^2)/2))

    *sym(1000000000000)^sym(i)));

    %vpa()

end;

zero

i

vpa(sym(0.08*sqrt(a)*exp((-a*e^2)/2)))
```

Therefore, we have

$$\begin{aligned} S_3 &= \frac{1.045693267943254 \times 10^{-314}}{10^{(12^{30})}} \\ &= \frac{1.045693267943254 \times 10^{-314}}{10^{360}} \\ &= 1.045693267943254 \times 10^{-314-360} \\ \Longrightarrow S_3 &= 1.045693267943254 \times 10^{-674} \end{aligned}$$

Hence using,

$$I(\omega,\eta) \geq I^*(\omega,\eta) - \Delta I - S_1' - S_2 - S_3 - S_4 - S_5 - S_6$$

I obtain the following

$$I(\omega,\eta) \geq 0.00003753342649673460$$

However, Saouter-Demichel found the following

$$I(\omega,\eta) \geq 0.00003751696746 \tag{5.1}$$

Hence my result is close enough to the above, so we continue using Saouter-Demichel's result.

Thus they prove that there exists a value $x$ in $[\omega-\eta,\omega+\eta] = [e^{727.9513130}, e^{727.9513586}]$ for which $\pi(x) > li(x)$ holds. More precisely, there are some values $u$ in the interval such that

$$\begin{aligned} \pi(e^u) - li(e^u) &> 0.00003751696746 \times \frac{e^u}{u} = 6.091784870469674 \times 10^{150} \\ &> 6.091784490 \times 10^{150} \end{aligned}$$

where $u = 727.9513130$.

The reason for the above basically comes from Lehman's theorem,

$$\begin{aligned} I(\omega,\eta) &= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{\frac{u}{2}} \{\pi(e^u) - li(e^u)\}.du \\ I(\omega,\eta) &> u e^{\frac{u}{2}} \{\pi(e^u) - li(e^u)\} \\ \Longrightarrow \pi(e^u) - li(e^u) &> \frac{I(\omega,\eta)}{u e^{\frac{-u}{2}}} \\ &> I(\omega,\eta) \times \frac{e^{\frac{u}{2}}}{u} \\ &> 0.00003751696746 \times \frac{e^{\frac{u}{2}}}{u} \\ &> 6.091784490 \times 10^{150} \end{aligned}$$

This now leads them to their theorem.

**Theorem 5.0.8.** *There exists at least one value $x$ in the interval*

$[exp(727.9513130), exp(727.9513586)]$ *for which $\pi(x) > li(x)$ holds. Moreover, there are more than $6.09 \times 10^{150}$ successive integers in the vicinity of $exp(727.951335792)$ where the inequality holds.*

# Chapter 6

# Sharpening the interval

The theorem in the previous chapter gives us an upper bound of $exp(727.9513586)$ for the first crossover. Saouter-Demichel suggest it is better than the value obtained by Chao-Plymen. Nevertheless, its possible to reduce the length of the interval again. We are aware that the integrand function decays very fast to 0 around its centre and thus the meaningful part of the integral is in fact around $\omega$. In order to reduce the interval we need some information about the growth of $\pi(x) - li(x)$. Saouter-Demichel decide to split the study into two cases from here. First, they consider the general case and second, they suppose the Riemann hypothesis holds.

In the general case, they prove the following theorem.

**Theorem 6.0.9.** *If $x \geq exp(8)$, we have*

$$0 \leq li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \leq \frac{12x}{\log^4 x} + C_2$$

*where*

$$C_1 = li(2) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)$$
$$C_2 = \int_2^{e^8} \frac{48}{\log^5 t}.dt - \frac{24}{\log^4 2}$$

*Proof.* From the definition of $li(x)$, we have for $x \geq 2$

$$li(x) = li(2) + \int_2^x \frac{dt}{\log t}$$

Using integration by parts, we work out $\int_2^x \frac{dt}{\log t}$

The formula I will use for integration by parts is,

$$\int u\frac{dv}{dt}.dt = uv - \int v\frac{du}{dt}.dt$$

I shall take the following substitutions:

$$u = \frac{1}{\log t}$$
$$\frac{dv}{dt} = 1$$
$$\frac{du}{dt} = \frac{-1}{t\log^2 t}$$
$$v = t$$

$$\Longrightarrow \int \frac{1}{\log t}.dt = \frac{t}{\log t} - \int t\left(\frac{-1}{t\log^2 t}\right).dt$$
$$\int_2^x \frac{1}{\log t}.dt = \left[\frac{t}{\log t}\right]_2^x + \int_2^x \frac{1}{\log^2 t}.dt$$

Now using integration by parts the same way as before, we compute $\int \frac{1}{\log^2 t}.dt$. Let

$$u = \frac{1}{\log^2 t}$$
$$\frac{dv}{dt} = 1$$
$$\frac{du}{dt} = \frac{-2}{t\log^3 t}$$
$$v = t$$

$$\Longrightarrow \int \frac{1}{\log^2 t}.dt = \frac{t}{\log^2 t} - \int t\left(\frac{-2}{t\log^3 t}\right).dt$$
$$= \frac{t}{\log^2 t} + \int \frac{2}{\log^3 t}.dt$$
$$\int_2^x \frac{1}{\log^2 t}.dt = \left[\frac{t}{\log^2 t}\right]_2^x + \int_2^x \frac{2}{\log^3 t}.dt$$

Hence we have

$$\int_2^x \frac{1}{\log t}.dt = \left[\frac{t}{\log t}\right]_2^x + \left[\frac{t}{\log^2 t}\right]_2^x + \int_2^x \frac{2}{\log^3 t}.dt$$

Using integration by parts once again we compute $\int \frac{1}{\log^3 t}.dt$. Let

$$\begin{aligned} u &= \frac{1}{\log^3 t} \\ \frac{dv}{dt} &= 1 \\ \frac{du}{dt} &= \frac{-3}{t\log^4 t} \\ v &= t \end{aligned}$$

$$\begin{aligned} \implies \int \frac{1}{\log^3 t}.dt &= \frac{t}{\log^3 t} - \int t\left(\frac{-3}{t\log^4 t}\right).dt \\ &= \frac{t}{\log^3 t} + \int \frac{3}{\log^4 t}.dt \\ \int_2^x \frac{1}{\log^3 t}.dt &= \left[\frac{t}{\log^3 t}\right]_2^x + \int_2^x \frac{3}{\log^4 t}.dt \end{aligned}$$

Hence we now have,

$$\begin{aligned} \int_2^x \frac{1}{\log t}.dt &= \left[\frac{t}{\log t}\right]_2^x + \left[\frac{t}{\log^2 t}\right]_2^x + \left[\frac{2t}{\log^3 t}\right]_2^x + \int_2^x \frac{6}{\log^4 t}.dt \\ \int_2^x \frac{1}{\log t}.dt &= \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x + \int_2^x \frac{6}{\log^4 t}.dt \end{aligned}$$

So we see that

$$li(x) = li(2) + \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x + \int_2^x \frac{6}{\log^4 t}.dt$$

$$li(x) = li(2) + \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right) + \int_2^x \frac{6}{\log^4 t}.dt$$

$$li(x) - li(2) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) + \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right) = \int_2^x \frac{6}{\log^4 t}.dt$$

$$\implies li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 = \int_2^x \frac{6}{\log^4 t}.dt$$

where

$$C_1 = li(2) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)$$

Now we look at $\int_2^x \frac{6}{\log^4 t}.dt$. Using integration by parts, let

$$\begin{aligned} u &= \frac{1}{\log^4 t} \\ \frac{dv}{dt} &= 1 \\ \frac{du}{dt} &= \frac{-4}{t\log^5 t} \\ v &= t \end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\log^4 t}.dt &= \frac{t}{\log^4 t} - \int t\left(\frac{-4}{t\log^5 t}\right).dt \\
&= \frac{t}{\log^4 t} + \int \frac{4}{\log^5 t}.dt \\
6\int_2^x \frac{1}{\log^4 t}.dt &= \left[\frac{6t}{\log^4 t}\right]_2^x + \int_2^x \frac{24}{\log^5 t}.dt \\
\int_2^x \frac{6}{\log^4 t}.dt - \left[\frac{6t}{\log^4 t}\right]_2^x &= \int_2^x \frac{24}{\log^5 t}.dt
\end{aligned}$$

Now we see for, $x \geq e^8$

$$\int_2^x \frac{6}{\log^4 t}.dt - \left[\frac{6t}{\log^4 t}\right]_2^x = \int_2^{e^8} \frac{24}{\log^5 t}.dt + \int_{e^8}^x \frac{24}{\log^5 t}.dt$$

$$\int_2^x \frac{6}{\log^4 t}.dt - \left[\frac{6t}{\log^4 t}\right]_2^x - \int_2^{e^8} \frac{24}{\log^5 t}.dt = \int_{e^8}^x \frac{24}{\log^5 t}.dt$$

For $t \geq e^8$, we have

$$\frac{24}{\log^5 t} \leq \frac{1}{2}\frac{6}{\log^4 t}$$

This is because

$$\begin{aligned}
&\frac{24}{\log^5 t} \leq \frac{1}{2}\frac{6}{\log^4 t} \\
\iff &\frac{24}{\log t} \leq 3 \\
\iff &\frac{24}{3} \leq \log t \\
\iff &8 \leq \log t \\
\iff &e^8 \leq e^{\log t} \\
\iff &e^8 \leq t
\end{aligned}$$

So we note here that,

$$\frac{24}{\log^5 t} = \frac{1}{2}\frac{6}{\log^4 t}$$

when $t = e^8$ and

$$\frac{24}{\log^5 t} < \frac{1}{2}\frac{6}{\log^4 t}$$

when $t > e^8$.

Hence, for $x \geq e^8$ we see the following

$$\int_2^x \frac{6}{\log^4 t}.dt - \left[\frac{6t}{\log^4 t}\right]_2^x - \int_2^{e^8} \frac{24}{\log^5 t}.dt = \int_{e^8}^x \frac{24}{\log^5 t}.dt$$

$$\implies \int_2^x \frac{6}{\log^4 t}.dt - \left[\frac{6t}{\log^4 t}\right]_2^x - \int_2^{e^8} \frac{24}{\log^5 t}.dt \le \frac{1}{2}\int_{e^8}^x \frac{6}{\log^4 t}.dt \le \frac{1}{2}\int_2^x \frac{6}{\log^4 t}.dt$$

Rearranging the above, we have the following

$$\int_2^x \frac{6}{\log^4 t}.dt \le \frac{1}{2}\int_2^x \frac{6}{\log^4 t}.dt + \left[\frac{6t}{\log^4 t}\right]_2^x + \int_2^{e^8} \frac{24}{\log^5 t}.dt$$

$$\int_2^x \frac{6}{\log^4 t}.dt - \frac{1}{2}\int_2^x \frac{6}{\log^4 t}.dt \le \frac{6x}{\log^4 x} - \frac{12}{\log^4 2} + \int_2^{e^8} \frac{24}{\log^5 t}.dt$$

$$\frac{1}{2}\int_2^x \frac{6}{\log^4 t}.dt \le \frac{6x}{\log^4 x} - \frac{12}{\log^4 2} + \int_2^{e^8} \frac{24}{\log^5 t}.dt$$

$$\implies \int_2^x \frac{6}{\log^4 t}.dt \le \frac{12x}{\log^4 x} - \frac{24}{\log^4 2} + \int_2^{e^8} \frac{48}{\log^5 t}.dt$$

Hence we see that we have proved the theorem:

$$li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 = \int_2^x \frac{6}{\log^4 t}.dt$$

$$\implies li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \le \frac{12x}{\log^4 x} - \frac{24}{\log^4 2} + \int_2^{e^8} \frac{48}{\log^5 t}.dt$$

Therefore,

$$li(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \le \frac{12x}{\log^4 x} + C_2$$

where

$$C_1 = li(2) - \frac{2}{\log 2}\left(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2}\right)$$

$$C_2 = \int_2^{e^8} \frac{48}{\log^5 t}.dt - \frac{24}{\log^4 2}$$

□

The latter theorem can be further optimised but Saouter-Demichel suggest that it will suffice for their purpose.

Now we look at Theorem 4.0.6 and Theorem 6.0.9 to obtain the following theorem.

**Theorem 6.0.10.** *If* $x \ge 355991$ *we have*

$$\frac{-0.2x}{\log^3 x} - \frac{-12x}{\log^4 x} - 44.53131 \le \pi(x) - li(x) \le \frac{0.51x}{\log^3 x} + 1.80141$$

*Moreover, if* $x \ge e^{40}$ *then*

$$|\pi(x) - li(x)| \le \frac{0.51x}{\log^3 x} + 1.80141$$

Below is a brief outline of how the above theorem was obtained. We know Theorem 4.0.6 states:

If $x \geq 32299$ and $x \geq 355991$ then

$$\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)$$

and Theorem 6.0.9 states:

If $x \geq e^8$, we have

$$0 \leq li(x) - \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) - C_1 \leq \frac{12x}{\log^4 x} + C_2$$

From Theorem 6.0.9, we look at

$$\begin{aligned} 0 &\leq li(x) - \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) - C_1 \\ \Longrightarrow \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) &+ C_1 \leq li(x) \end{aligned} \tag{6.1}$$

Now we look at the following in Theorem 6.0.9:

$$li(x) - \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) - C_1 \leq \frac{12x}{\log^4 x} + C_2$$

We then see

$$li(x) \leq \frac{12x}{\log^4 x} + \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) + C_1 + C_2 \tag{6.2}$$

We look at Theorem 4.0.6 and subtract $li(x)$ from both sides:

$$\begin{aligned} \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) - li(x) &\leq \pi(x) - li(x) \\ &\leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right) - li(x) \end{aligned}$$

Observing the LHS and using (6.2) we have,

$$\begin{aligned} \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) &- \left[\frac{12x}{\log^4 x} + \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) + C_1 + C_2\right] \\ &\leq \pi(x) - li(x) \end{aligned}$$

$$\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right)-\frac{12x}{\log^4 x} - \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right)-C_1-C_2 \leq \pi(x)-li(x)$$

$$\frac{x}{\log x}+\frac{x}{\log^2 x}+\frac{1.8x}{\log^3 x}-\frac{12x}{\log^4 x}-\frac{x}{\log x} - \frac{x}{\log^2 x}-\frac{2x}{\log^3 x}-C_1-C_2 \leq \pi(x)-li(x)$$

$$\frac{-0.2x}{\log^3 x}-\frac{12x}{\log^4 x}-C_1-C_2 \leq \pi(x)-li(x) \tag{6.3}$$

where

$$\begin{aligned}-C_1-C_2 &= -li(2)+\frac{2}{\log 2}\left(1+\frac{1}{\log 2}+\frac{2}{\log^2 2}\right)-\int_2^{e^8}\frac{48}{\log^5 t}.dt+\frac{24}{\log^4 2}\\ &= -44.53131\end{aligned}$$

Now we look at the RHS and use (6) to see

$$\begin{aligned}\pi(x)-li(x) &\leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)-li(x)\\ \Longrightarrow \pi(x)-li(x) &\leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)-\left[\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right)-C_1\right]\\ \pi(x)-li(x) &\leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)-\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right)+C_1\\ \pi(x)-li(x) &\leq \frac{x}{\log x}+\frac{x}{\log^2 x}+\frac{2.51x}{\log^3 x}-\frac{x}{\log x}-\frac{x}{\log^2 x}-\frac{2x}{\log^3 x}+C_1\end{aligned}$$

$$\Longrightarrow \pi(x)-li(x) \leq \frac{0.51x}{\log^3 x}+C_1 \tag{6.4}$$

Putting (6.3) and (6.4) together we obtain the following

$$\frac{-0.2x}{\log^3 x}-\frac{12x}{\log^4 x}-C_1-C_2 \leq \pi(x)-li(x) \leq \frac{0.51x}{\log^3 x}+C_1 \tag{6.5}$$

Reason for the latter statement of Theorem 6.0.10 is because we compare both sides of (6.5).

Saouter-Demichel use Theorem 6.0.10 to study the tail parts of the following integral:

$$I(\omega,\eta)=\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)ue^{\frac{-u}{2}}\{\pi(e^u)-li(e^u)\}.du$$

Now let $\eta_0$ be a real positive number such that $\eta_0 < \eta$. We then see, since $\omega > 40$,

$$
\begin{aligned}
&|\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\{\pi(e^u) - li(e^u)\}.du| \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\{|\pi(e^u) - li(e^u)|\}.du \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\left\{\frac{0.51e^u}{u^3} + 1.80141\right\}.du \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\left\{\frac{0.51e^u}{u^3}ue^{\frac{-u}{2}} + 1.80141ue^{\frac{-u}{2}}\right\}.du \\
&\leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\left\{\frac{0.51e^{\frac{u}{2}}}{u^2} + 1.80141ue^{\frac{-u}{2}}\right\}.du \\
&\leq (\eta-\eta_0)K(\eta_0)\left\{\frac{0.51e^{\frac{(\omega+\eta)}{2}}}{(\omega+\eta_0)^2} + 1.80141(\omega+\eta)e^{\frac{-(\omega+\eta_0)}{2}}\right\}
\end{aligned}
$$

We have $u = \omega+\eta$ since some of the terms above are increasing and we have $u = \omega+\eta_0$ when some of the terms are decreasing. Similarly, for $\omega - \eta > 40$. We have

$$
\begin{aligned}
&|\int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\{\pi(e^u) - li(e^u)\}.du| \\
&\leq \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\{|\pi(e^u) - li(e^u)|\}.du \\
&\leq \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\left\{\frac{0.51e^u}{u^3} + 1.80141\right\}.du \\
&\leq \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)\left\{\frac{0.51e^{\frac{u}{2}}}{u^2} + 1.80141ue^{\frac{-u}{2}}\right\}.du \\
&\leq (\eta-\eta_0)K(-\eta_0)\left\{\frac{0.51e^{\frac{(\omega-\eta_0)}{2}}}{(\omega-\eta)^2} + 1.80141(\omega-\eta_0)e^{\frac{-(\omega-\eta)}{2}}\right\}
\end{aligned}
$$

We have $u = \omega-\eta$ since some of the terms above are decreasing and we have $u = \omega-\eta_0$ when some of the terms are increasing.

Now we denote $T_1$ and $T_2$,

$$
T_1 = (\eta-\eta_0)K(\eta_0)\left\{0.51\frac{e^{\frac{(\omega+\eta)}{2}}}{(\omega+\eta_0)^2} + 1.80141(\omega+\eta)e^{\frac{-(\omega+\eta_0)}{2}}\right\}
$$

$$
T_2 = (\eta-\eta_0)K(-\eta_0)\left\{0.51\frac{e^{\frac{(\omega-\eta_0)}{2}}}{(\omega-\eta)^2} + 1.80141(\omega-\eta_0)e^{\frac{-(\omega-\eta)}{2}}\right\}
$$

We see that the two tail integrals are then bounded above by $T_1 + T_2$. By further numerical computations, Saouter-Demichel set $\eta_0 = \frac{\eta}{2.074}$. Therefore, we obtain the following values for $T_1$ and $T_2$, with the use of the following values

$$
\begin{aligned}
\eta &= 0.00002283333334 \\
\eta_0 &= \frac{\eta}{2.074} = \frac{0.00002283333334}{2.074} = 1.100932176470588 \times 10^{-5} \\
&= 0.000011009321765 \\
\omega &= 727.951335792 \\
K(\eta_0) &= \sqrt{\frac{\alpha}{2\pi}} e^{\frac{-\alpha\eta_0{}^2}{2}} = 4.9930428736 \times 10^{-11} \\
\Longrightarrow T_1 &= (\eta - \eta_0)K(\eta_0)\left\{0.51\frac{e^{\frac{(\omega+\eta)}{2}}}{(\omega+\eta_0)^2} + 1.80141(\omega+\eta)e^{\frac{-(\omega+\eta_0)}{2}}\right\} \\
&= 0.00001594193847 \\
T_2 &= (\eta - \eta_0)K(-\eta_0)\left\{0.51\frac{e^{\frac{(\omega-\eta_0)}{2}}}{(\omega-\eta)^2} + 1.80141(\omega-\eta_0)e^{\frac{-(\omega-\eta)}{2}}\right\} \\
&= 0.00001594167019
\end{aligned}
$$

where $K(-\eta_0) = \sqrt{\frac{\alpha}{2\pi}} e^{\frac{-\alpha(-\eta_0)^2}{2}} = 4.9930428736 \times 10^{-11}$.

We should note here that Saouter-Demichel's values for $T_1$ and $T_2$ differ slightly from the values I have obtained above. They found that $T_1 = 0.00001594194397$ and $T_2 = 0.00001594167602$. So using Saouter-Demichel's result they obtain the following value for $I(\omega, \eta_0)$:

$$I(\omega, \eta_0) \geq 0.00000563334747$$

They did this by subtracting $T_1$ and $T_2$ from (5.1) i.e. $I(\omega, \eta_0) \geq I(\omega, \eta) - T_1 - T_2$.

However, I obtain the following result for $I(\omega, \eta_0)$ using my results for $T_1$ and $T_2$:

$$I(\omega, \eta_0) \geq 0.00000563335880$$

But since there's not much difference in the result, we continue to use Saouter-Demichel's value. This result then enables us to obtain a result finer than the one obtained in Theorem 5.0.8, namely $6.09 \times 10^{150}$. Using $I(\omega, \eta_0)$ we now state the following theorem.

**Theorem 6.0.11.** *There exists an* $x \in [exp(727.95132478), exp(727.95134681)]$ *such that* $\pi(x) - li(x) > 9.1472 \times 10^{149}$.

We see that the above comes from the following,

$$\begin{aligned}[exp(\omega-\eta_0), exp(\omega+\eta_0)] &= [exp(727.951335792 - 0.000011009321765),\\ &\quad exp(727.951335792 + 0.000011009321765)]\\ &= [exp(727.95132478), exp(727.951346801)]\end{aligned}$$

and we also see that $\pi(x) - li(x) > 9.1472 \times 10^{149}$ comes from

$$\begin{aligned}\pi(e^u) - li(e^u) &> I(\omega, \eta_0) \times \frac{e^{\frac{u}{2}}}{u}\\ &= 0.00000563334747 \times \frac{e^{\frac{727.951335792}{2}}}{727.951335792}\\ &= 9.1472 \times 10^{149}\\ \implies \pi(e^u) - li(e^u) &> 9.1472 \times 10^{149}\end{aligned}$$

Moving on, now from this point forward, we will assume Riemann Hypothesis holds. So assuming the Riemann Hypothesis we state the following theorem given by Schoenfeld [11].

**Theorem 6.0.12.** *If the Riemann Hypothesis holds, then for $x \geq 2657$, we have*

$$|\pi(x) - li(x)| < \frac{1}{8\pi}\sqrt{x}\log x$$

Using the expressions of $T_1$ and $T_2$, we use these to obtain their upper bounds as follows,

$T_1'$:

$$\begin{aligned}
&\left|\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\left\{\pi(e^u)-li(e^u)\right\}.du\right| \\
&< \int_{\omega+\eta_0}^{\omega+\eta} \left|K(u-\omega)ue^{\frac{-u}{2}}\left\{\pi(e^u)-li(e^u)\right\}\right|.du \\
&< \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\left\{|\pi(e^u)-li(e^u)|\right\}.du \\
&< \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\left\{\frac{1}{8\pi}\sqrt{e^u}\log e^u\right\}.du \\
&< \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{\frac{-u}{2}}\frac{1}{8\pi}e^{\frac{u}{2}}u.du \\
&< \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)u^2\frac{1}{8\pi}.du \\
&< \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega).du\frac{1}{8\pi}u^2 \\
&< K(\eta_0)(\eta-\eta_0)\frac{1}{8\pi}(\omega+\eta_0)^2
\end{aligned}$$

where $u=\omega+\eta_0$.

$$\Longrightarrow T_1' = \frac{1}{8\pi}K(\eta_0)(\omega+\eta_0)^2(\eta-\eta_0)$$

$T_2'$:

$$\begin{aligned}
&\left|\int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\left\{\pi(e^u)-li(e^u)\right\}.du\right| \\
&< \int_{\omega-\eta}^{\omega-\eta_0} \left|K(u-\omega)ue^{\frac{-u}{2}}\left\{\pi(e^u)-li(e^u)\right\}\right|.du \\
&< \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\left\{|\pi(e^u)-li(e^u)|\right\}.du \\
&< \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\left\{\frac{1}{8\pi}\sqrt{e^u}\log e^u\right\}.du \\
&< \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)ue^{\frac{-u}{2}}\frac{1}{8\pi}e^{\frac{u}{2}}u.du \\
&< \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega)u^2\frac{1}{8\pi}.du \\
&< \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega).du\frac{1}{8\pi}u^2 \\
&< K(-\eta_0)(\eta-\eta_0)\frac{1}{8\pi}(\omega-\eta_0)^2
\end{aligned}$$

where $u=\omega-\eta_0$.

$$\Longrightarrow T_2' = \frac{1}{8\pi}K(-\eta_0)(\omega-\eta_0)^2(\eta-\eta_0)$$

However, we note here that Saouter-Demichel state a slightly different result to what I've obtained above. They happen to be missing the term $(\eta - \eta_0)$ in both expressions for $T_1'$ and $T_2'$.

Numerically, now Saouter-Demichel set $\eta_0 = \frac{\eta}{6.72}$ through trial and error. Using this result, we find the following values for $T_1'$ and $T_2'$. We know that,

$$
\begin{aligned}
\eta &= 0.00002283333334 \\
\omega &= 727.951335792 \\
\eta_0 &= \frac{0.00002283333334}{6.72} = 0.000003397817461 \\
\alpha &= 6 \times 10^{12} \\
\implies T_1' &= \frac{1}{8\pi} K(\eta_0)(\omega + \eta_0)^2(\eta - \eta_0) \\
&= 3.635333032694410 \times 10^{-10} \\
&= 3.6353330327 \times 10^{-10} \\
T_2' &= \frac{1}{8\pi} K(-\eta_0)(\omega - \eta_0)^2(\eta - \eta_0) \\
&= 3.635332964820654 \times 10^{-10} \\
&= 3.6353329648 \times 10^{-10}
\end{aligned}
$$

In Saouter-Demichel's paper, they obtain the following result for $T_1'$ and $T_2'$:

$$
\begin{aligned}
T_1' &= 0.00001870458817 \\
T_2' &= 0.00001870458683
\end{aligned}
$$

Clearly, these values differ from my values since the expressions for $T_1'$ and $T_2'$ are missing $(\eta - \eta_0)$ in Saouter-Demichel's paper. However, if we carry on looking at their paper then they obtain the following for $I(\omega, \eta_0)$:

$$I(\omega, \eta_0) \geq 0.000000107793$$

by

$$
\begin{aligned}
I(\omega, \eta_0) &\geq I(\omega, \eta) - T_1' - T_2' \\
&\geq 0.00003751696746 - 0.00001870458817 - 0.00001870458683 \\
&= 0.000000107793
\end{aligned}
$$

However, using the values I obtained, I get the following value for $I(\omega, \eta_0)$:

$$\begin{aligned} I(\omega, \eta_0) &\geq I(\omega, \eta_0) - T_1' - T_2' \\ &\geq 0.00003751624039 \end{aligned}$$

Now the next theorem uses the information and values obtained for $T_1', T_2'$ and $I(\omega, \eta_0)$ as it is stated in Saouter-Demichel's paper. Which means according to my results this theorem could be much improved. Saying this, it does not mean that their theorem is incorrect, it could be improved more.

**Theorem 6.0.13.** *If the Riemann Hypothesis holds then there exists one value $x$ in the interval $[exp(727.95133239), exp(727.95133919)]$ such that $\pi(x) - li(x) > 1.7503 \times 10^{148}$.*

We see the above interval comes from

$$\begin{aligned} [exp(\omega - \eta_0), exp(\omega + \eta_0)] &= [exp(727.951335792 - 0.000003397817461), \\ &\quad exp(727.951335792 + 0.000003397817461)] \\ &= [exp(727.95133239), exp(727.95133919)] \end{aligned}$$

We also see that the value $1.7503 \times 10^{148}$ comes from

$$\begin{aligned} \pi(e^u) - li(e^u) &> I(\omega, \eta_0) \times \frac{e^{\frac{u}{2}}}{u} \\ &= 0.000000107793 \times \frac{e^{\frac{727.951335792}{2}}}{727.951335792} \\ &= 1.7503 \times 10^{148} \end{aligned}$$

According to my results, I would improve the previous theorem to the following.

**Theorem 6.0.14.** *If the Riemann Hypothesis holds, then there exists one value $x$ in the interval $[exp(727.9513324), exp(727.9513392)]$ such that $\pi(x) - li(x) > 6.09 \times 10^{150}$*

Given the Riemann Hypothesis, this is the best known result (cf. Theorem 5.0.8).

# Chapter 7

# Interval of Positivity

We have in Theorem 5.0.8 that there is an interval of $6.09 \times 10^{150}$ consecutive integers where $\pi(x) - li(x)$ is positive. We also have that

$$\begin{aligned} \pi(e^u) - li(e^u) &> 0.00003751696746 \times \frac{e^{\frac{u}{2}}}{u} \\ &> 6.091784490 \times 10^{150} \end{aligned}$$

which states that there exists a point $x$ such that $\pi(x) - li(x) > 6.09 \times 10^{150}$.

Now let $b$ be a positive integer. Then we state the obvious that $li(x - b) \le li(x)$. Moreover, for any $x \ge 1$ we have

$$\pi(x - 1) \ge \pi(x) - 1$$

The above is obvious since

$$\begin{aligned} &\pi(x - 1) \ge \pi(x) - 1 \\ &\Longleftrightarrow 1 \ge \pi(x) - \pi(x - 1) \\ &\Longleftrightarrow \pi(x) - \pi(x - 1) \le 1 \end{aligned}$$

Hence, by recurrence and with the previous inequality we can deduce the following

$$\pi(x - b) - li(x - b) > 6.09 \times 10^{150} - b$$

Since

$$\begin{aligned} \pi(x - b) - li(x - b) &\ge \pi(x) - b - li(x) \\ &> \pi(x) - li(x) - b \\ &> 6.09 \times 10^{150} - b \end{aligned}$$

So we conclude that the $6.09 \times 10^{150}$ successive integers preceding $x$ belong to the interval of positivity. Saouter-Demichel mention that this result was obtained by considering integers inferior to $x$. However, we see that this result can be much improved by considering integers greater than $x$. So now Saouter-Demichel state the following theorem.

**Theorem 7.0.15.** *Let $x > 1$ and $y > 0$, then we have*

$$li(x+y) - li(x) = \int_x^{x+y} \frac{dt}{\log t} < \frac{y}{\log t}$$

We see this comes from the basic definition of $li$,

$$\begin{aligned} li(x+y) - li(x) &= \int_0^{x+y} \frac{dt}{\log t} - \int_0^{x} \frac{dt}{\log t} \\ &= \int_x^{x+y} \frac{dt}{\log t} \end{aligned}$$

Drawing up the graph of $\int_x^{x+y} \frac{dt}{\log t}$, it is obvious that we see this is less than $\frac{y}{\log t}$, i.e. base multiplied by height.

With the use of the theorem above Saouter-Demichel then go on to state another theorem.

**Theorem 7.0.16.** *Let $x$ be a real positive number such that $\pi(x) - li(x) = A > 0$. Then if $y$ is a real number such that $0 < y < A \log x$ we have*

$$\pi(x+y) - li(x+y) > 0$$

*Proof.* Let $y > 0$, since the function $\pi(x)$ is increasing we have the following

$$\begin{aligned} \pi(x+y) - li(x+y) &= \{\pi(x+y) - \pi(x)\} + \{\pi(x) - li(x)\} + \{li(x) - li(x+y)\} \\ &= \{\pi(x+y) - \pi(x)\} + A - \frac{y}{\log x} \\ &> A - \frac{y}{\log x} > 0 \end{aligned}$$

since $\pi(x+y) - \pi(x) > 0$. □

Now looking back, Theorem 6.0.11 enables us to state the following theorem.

**Theorem 7.0.17.** *There are at least* $6.6587 \times 10^{152}$ *consecutive integers* $x$ *in the interval* $[exp(727.95132478), exp(727.95134682)]$ *such that* $\pi(x) - li(x) > 0$.

We see this value, $6.6587 \times 10^{152}$ comes from the direct application of the above theorem. We let $A = 9.1472 \times 10^{149}$ then $A \log x = A \log e^u = Au = A\omega = 9.1472 \times 10^{149} \times 727.951335792 = 6.6587 \times 10^{152}$.

However, here we do not know where the first $x$ lies in the interval of $[exp(727.95132478), exp(727.95134681)]$. We only know its maximal value is $exp(727.95134681)$. We also have that

$$\begin{aligned} & exp(727.95134682) - exp(727.95134681) \\ & \simeq 1.397 \times 10^{308} \\ & > 6.6587 \times 10^{152} \end{aligned}$$

So we conclude that the $6.6587 \times 10^{152}$ integers following $x$ belong to the interval $[exp(727.95132478), exp(727.95134682)]$.

In a similar way, Theorem 6.0.13. gives us the following result by Saouter-Demichel.

**Theorem 7.0.18.** *If the Riemann Hypothesis holds, then there are at least* $1.2741 \times 10^{151}$ *consecutive integers* $x$ *in the interval* $[exp(727.95133239), exp(727.95133920)]$ *such that* $\pi(x) - li(x) > 0$

We see that in the above theorem we let $A = 1.7503 \times 10^{148}$ then $A \log x = A \log e^u = Au = A\omega = 1.7503 \times 10^{148} \times 727.951335792 = 1.2741 \times 10^{151}$.

However, since my results were slightly different to Saouter-Demichel's in the theorem above, I will rephrase it.

**Theorem 7.0.19.** *If the Riemann Hypothesis holds, then there are at least* $4.4344 \times 10^{153}$ *consecutive integers* $x$ *in the interval* $[exp(727.95133239), exp(727.95133920)]$ *such that* $\pi(x) - li(x) > 0$.

**Remark.** Regarding Theorem 7.0.18, the number of consecutive integers satisfying $\pi(x) - li(x) > 0$ is approximately 50 times smaller than in Theorem 7.0.17. This theorem might then appear weaker, which would then make assuming the Riemann hypothesis pointless. In fact this theorem is stronger than Theorem 7.0.17, since the length of the interval is 3 times shorter. The difference in terms of consecutive integers comes from the fact that the estimate for $I(\omega, \eta_0)$ is much sharper when the Riemann hypothesis holds.

# Chapter 8

# Conclusions

We have looked carefully at two papers published very recently in July 2010. These papers compute new upper bounds for the least $x$ for which $\pi(x) > li(x)$. This has been a challenge since 1914 when Littlewood's theorem was announced. We have examined the numerical aspects and found a number of numerical errors in both papers, with the help of the symbolic toolbox in MATLAB. We have also found some mistakes in a few formulae/expressions in both papers. One particular correction that should be mentioned here is te Riele's formula (1987), where a mistake had been made in using the product rule. Fortunately, these numerical errors that were found in both the papers do not make much difference to the results. There is also one new much improved Theorem 6.0.14. at the end of my dissertation.

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