The University
of Manchester

K-theory and the connection index

Kamran, Tayyab and Plymen, Roger

2009

MIMS EPrint: **2009.80**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

K-theory and the connection index

Tayyab Kamran and Roger Plymen

Abstract

Let $\mathcal{G}$ denote a split simply connected almost simple p-adic group. We study the spherical C^*-algebra of $\mathcal{G}$ and prove that the rank of the K-theory group K_0 is the connection index $f(\mathcal{G})$. We relate this result to the recent conjecture in [4, §7] and to a recent result of Solleveld [21].

1   Introduction

Let $\mathcal{G}$ denote a split simply connected almost simple p-adic group. We study the spherical C^*-algebra of $\mathcal{G}$ and prove that the rank of the K-theory group K_0 is the connection index $f(\mathcal{G})$.

The dual of the spherical C^*-algebra comprises the irreducible unitary representations of G which admit nonzero K-fixed vectors, where K is a good maximal compact subgroup of $\mathcal{G}$. The spherical C^*-algebra $C^*_{nr}(\mathcal{G})$ is a direct summand of the Iwahori C^*-algebra $\mathcal{I}$, and so contributes to the K-theory of $\mathcal{I}$.

We relate this result with the recent conjecture in [4, §7]: this is a version, adapted to the K-theory of C^*-algebras, of the geometric conjecture developed in [2, 3, 4, 5].

Quite specifically, let F be a local nonarchimedean field of characteristic 0, let $\mathcal{G}$ be the group of F-rational points in a split, almost simple, simply connected, semisimple linear algebraic group defined over F, for example SL(n). Let $\mathcal{T}$ denote a maximal split torus of $\mathcal{G}$. Let $\mathcal{G}^\vee$, $\mathcal{T}^\vee$ denote the Langlands dual groups, and let G, T denote maximal compact subgroups:

$$G \subset \mathcal{G}^\vee, \quad T \subset \mathcal{T}^\vee.$$

Then G is a compact Lie group with maximal torus T.

Let Φ be the root system for the pair (G, T) and let $\Phi^\vee$ be the coroot system associated to Φ. Let $Q(\Phi)$ be the root lattice, $P(\Phi)$ be the weight lattice; each weight takes integer values on the coroot lattice. Let $P(\Phi)/Q(\Phi)$

1

be the quotient group. The order of the group $P(\Phi)/Q(\Phi)$ is called the *connection index* $f = f(\mathcal{G})$ of $\mathcal{G}$.

The abelian groups P/Q are tabulated in [10, Plates I–X, p.265–292]. For $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ they are, respectively

$$\mathbb{Z}/(n+1)\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \ (n \ odd), \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ (n \ even)$$

$$\mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}, \quad 0, \quad 0, \quad 0$$

We will consider the *unramified* (or *spherical*) C^*-algebra $\mathfrak{A} = C^*_{nr}(\mathcal{G})$ of $\mathcal{G}$. Let

$$\pi(t) := \mathrm{Ind}^{\mathcal{G}}_{\mathcal{TU}}(t \otimes 1)$$

be the unramified unitary principal series of $\mathcal{G}$, so that $t \in \Psi(\mathcal{T})$ is an unramified unitary character of $\mathcal{T}$. Let $\mathcal{C}(t)$ be the commuting algebra of $\pi(t)$. Then

$$\mathcal{C}(t) \simeq \mathbb{C}[R(t)],$$

where $R(t)$ is the R-group of t, see [13].

The R-groups are tabulated by Keys [13]. When we compare the R-group computations in [13] with the above list, we see that *the maximal R-groups are isomorphic to the fundamental groups P/Q.* In this Note, we attempt to explain this fact, via the elementary geometry of the Lie algebra $\mathfrak{t}$: the vector space $\mathfrak{t}$ has the structure of a Euclidean apace tessellated with alcoves. In the present context, we need a refinement of this tessellation by barycentric subdivision. For example, if $\mathcal{G} = \mathrm{SL}(3)$, then the standard tiling of the Euclidean plane by equilateral triangles is refined to produce a tiling by isosceles triangles. Reducibility in the unramified unitary principal series is determined by the fundamental group P/Q.

Let K be a compact open subgroup of $\mathcal{G}$. Choose left-invariant Haar measure μ on $\mathcal{G}$. Let

$$\begin{aligned} e_K(x) &= \mu(K)^{-1} \text{ if } x \in K \\ &= 0 \qquad \text{ if } x \notin K \end{aligned}$$

Then e_K is an idempotent in the reduced C^*-algebra $C^*_r(\mathcal{G})$. Let $K_1, \ldots, K_f$ be an enumeration of the good maximal compact subgroups of $\mathcal{G}$, one from each conjugacy class in $\mathcal{G}$.

Theorem 1.1. *Let $\mathcal{G}$ be simply connected, let $\mathfrak{A} = C^*_{nr}(\mathcal{G})$ and let $f = f(\mathcal{G})$ be the connection index of $\mathcal{G}$. Then $K_0(\mathfrak{A})$ is a free abelian group on $f(\mathcal{G})$ generators, and $K_1 = 0$. The idempotents $e_{K_1}, \ldots, e_{K_f}$ are generators of K_0.*

K_0 detects the tempered L-packet $\Pi(t_0)$ in the unramified unitary principal series with the maximal number of constituents.

2 The alcoves in the Lie algebra of T

Our reference at this point is [10, IX, p.309–327]. The proof hinges on the distinction between the affine Weyl group and the extended affine Weyl group. Let $\mathfrak{t}$ denote the Lie algebra of T, and let $\exp : \mathfrak{t} \to T$ denote the exponential map. The kernel of $\exp$ is denoted $\Gamma(T)$. Denote by $N(G,T)$ the subgroup of $\Gamma(T)$ generated by the nodal vectors. The quotient $\Gamma(T)/N(G,T)$ can be identified with the fundamental group of G. Thanks to the identification of the nodal vector K_α with the coroot $\alpha^\vee$, the group $N(G,T)$ can be identified with the group $Q(\Phi^\vee)$ generated by the set $\Phi^\vee$ of coroots.

The affine Weyl group is $W_a = N(G,T) \rtimes W$ and the extended affine Weyl group is $W_a' = \Gamma(T) \rtimes W$; the subgroup W_a of W_a' is normal. The group W (resp. W_a) operates simply-transitively on the set of chambers (resp. alcoves). Let C be a chamber and A an alcove. Then $\overline{C}$ (resp. $\overline{A}$) is a fundamental domain for the operation of W on $\mathfrak{t}$ (resp. of W_a on $\mathfrak{t}$). Let H_A be the stabilizer of A in W_a'. Then W_a' is the semi-direct product

$$W_a' = W_a \rtimes H_A.$$

We have to consider the C^*-summand $\mathfrak{A}(G)$ in the reduced C^*-algebra $C_r^*(G)$ which corresponds to the unramified unitary principal series, see [13]. The algebra $C_r^*(G)$ is defined as follows. We choose a left-invariant Haar measure on G, and form a Hilbert space $L^2(G)$. The left regular representation λ of $L^1(G)$ on $L^2(G)$ is given by

$$(\lambda(f))(h) = f * h$$

where $f \in L^1(G)$, $h \in L^2(G)$ and $*$ denotes the convolution. The C^*-algebra generated by the image of λ is the reduced C^*-algebra $C_r^*(G)$. Let $\mathfrak{K}$ denote the C^*-algebra of compact operators on the standard Hilbert space H.

At this point we use *Langlands duality*. The Langlands dual of $\mathcal{G}$ is the complex reductive group $\mathcal{G}^\vee$ with maximal torus $\mathcal{T}^\vee$. Let G be a maximal compact subgroup of $\mathcal{G}^\vee$, let T be the maximal compact subgroup of $\mathcal{T}^\vee$.

The group $\Psi(\mathcal{T})$ of unramified unitary characters of $\mathcal{T}$ is isomorphic to T. The unramified C^*-algebra $C_{nr}^*(\mathcal{G})$ is given by the fixed point algebra

$$\begin{aligned}
C_{nr}^*(\mathcal{G}) : &= C(T, \mathfrak{K})^W \\
&= \{f \in C(T, \mathfrak{K}) : f(wt) = \mathfrak{c}(w : t) \cdot f(t), w \in W\}
\end{aligned}$$

where $\mathfrak{a}(w : t)$ are normalized intertwining operators, and

$$\mathfrak{c}(w : t) = \operatorname{Ad} \mathfrak{a}(w : t)$$

as in [16]. Then $\mathfrak{a} : W \to C(T, U(H))$ is a 1-cocycle:

$$\mathfrak{a}(w_2 w_1 : t) = \mathfrak{a}(w_2 : w_1 t)\mathfrak{a}(w_1 : t).$$

View $\mathfrak{t}$ as an additive group, and form the Euclidean group $\mathfrak{t} \rtimes O(\mathfrak{t})$. We have $W_a' \subset \mathfrak{t} \rtimes O(\mathfrak{t})$ and so W_a' acts as affine transformations of $\mathfrak{t}$.

We have $\overline{A} \subset \mathfrak{t}$. The group H_A is defined as the stabilizer in W_a' of $\overline{A}$:

$$H_A := \{w \in W_a' : w\overline{A} \subset \overline{A}\}$$

Then H_A is a finite abelian group which can be identified naturally with $\pi_1(G)$, see [10, IX, p.326].

Since $\mathcal{G}$ is simply connected, the dual group $\mathcal{G}^\vee$ is of adjoint type, and so $C(G) = 1$. The *special points* of W_a are the elements x of $\mathfrak{t}$ such that $\exp x \in C(G)$, see [10, IX, p.326]. In this case, $\exp x = 1$. Let S be the set of special points of $\overline{A}$. The group H_A acts freely on S, and the exponential map induces a bijection from S/H_A to $C(G)$, by [10, IX, p.327].

We will choose $\overline{A}$ such that $0 \in \overline{A}$, see [10, p.327]. We choose $s_0 \in S$ and note that the set S is a single H_A-orbit:

$$\{w \cdot s_0 : w \in H_A\} = S.$$

Define

$$x_0 := \frac{1}{f} \sum_{w \in H_A} w \cdot s_0 = \frac{1}{f} \sum_{s \in S} s$$

Then x_0 is independent of the choice of $s \in S$. Now $\overline{A}$ is convex , for it is the closure of a simplex in $\mathfrak{t}$: therefore $x_0 \in \overline{A}$. We have

$$w' \cdot x_0 = \frac{1}{f} \sum_{w \in H_A} w'(w \cdot s_0) = \frac{1}{f} \sum_{w \in H_A} w \cdot s_0 = x_0$$

for all $w' \in H_A$, so x_0 is a fixed point under the action of H_A.

Then $\overline{A}$ is equivariantly contractible to x_0:

$$r_t(x) := (1 - t)x_0 + tx \tag{1}$$

with $0 \leq t \leq 1$. This is an affine H_A-equivariant contraction from $\overline{A}$ to x_0.

Lemma 2.1. *Let $t_0 = \exp x_0$. There is an isomorphism of abelian groups:*

$$H_A \simeq W(t_0) \simeq R(t_0)$$

where $R(t_0)$ is the R-group of t_0.

4

Proof. The exponential map $\exp : \mathfrak{t} \to T$ commutes with the action of W. Let $h = \gamma w = (\gamma, w) \in \Gamma(T) \rtimes W$. Then

$$hx_0 = x_0 \implies \gamma \in \Gamma(T) \implies \exp hx_0 = \exp wx_0 = w \exp x_0 = wt_0$$

This determines the map

$$H_A \longrightarrow W(t_0), \quad \gamma w \mapsto w$$

Now

$$w(t_0) = t_0 \implies (\exists \gamma \in \Gamma(T)) \, \gamma(wx_0) = x_0 \implies \gamma w \in H_A$$

so that $H_A \to W(t_0)$ is surjective. The map is injective because

$$(\gamma_1, w) = (\gamma_2, w) \implies \gamma_1 = \gamma_2$$

This creates the canonical isomorphism of abelian groups:

$$H_A \simeq W(t_0)$$

In fact, $W(t_0)$ is the image of H_A in the canonical map $W'_a \to W$ is $W(t_0)$:

$$\Gamma(T) \rtimes W \to W, \quad H_A \simeq W(t_0)$$

In general, we have $R(t_0) \subset W(t_0)$. The calculations of Keys [13] prove that $R(t_0) = W(t_0)$. $\qquad\square$

Theorem 2.2. *The group $K_0(C^*_{nr}(G))$ is free abelian on $f(\mathcal{G})$ generators, and $K_1(C^*_{nr}(G)) = 0$.*

Proof. We have the exponential map $\exp : \mathfrak{t} \to T$. We lift f from T to a periodic function F on $\mathfrak{t}$, and lift $\mathfrak{a}$ from a 1-cocycle $\mathfrak{a} : W \to C(T, U(H))$ to a 1-cocycle $\mathfrak{b} : W'_a \to C(\mathfrak{t}, U(H))$:

$$F(x) := f(\exp x), \quad \mathfrak{b}(w' : x) := \mathfrak{a}(w : \exp x)$$

with $w' = (\gamma, w)$. The semidirect product rule is

$$w'_1 w'_2 = (\gamma_1, w_1)(\gamma_2, w_2) = (\gamma_1 w_1(\gamma_2), w_1 w_2).$$

Note that $\mathfrak{b}$ is still a 1-cocycle:

$$
\begin{aligned}
\mathfrak{b}(w'_2 w'_1 : x) &= \mathfrak{a}(w_2 w_1 : \exp x) \\
&= \mathfrak{a}(w_2 : w_1(\exp x))\mathfrak{a}(w_1 : \exp x) \\
&= \mathfrak{a}(w_2 : \exp w_1 x)\mathfrak{b}(w'_1 : x) \\
&= \mathfrak{a}(w_2 : \exp \gamma w_1 x)\mathfrak{b}(w'_1 : x) \\
&= \mathfrak{b}(w'_2 : w'_1 x)\mathfrak{b}(w'_1 : x)
\end{aligned}
$$

Now we define

$$\mathfrak{d}(w:x) := Ad\,\mathfrak{b}(w:x), \qquad w \in W_a' \tag{2}$$

The fixed algebra $C_{nr}^*(G)$ is as follows:

$$\{F \in C(\mathfrak{t}, \mathfrak{K}) : F(wx) = \mathfrak{d}(w:x) \cdot F(x), w \in W_a', \ x \in \mathfrak{t}, \ F \text{ periodic}\}$$

Now F is determined by its restriction to $\overline{A}$. Upon restriction, we obtain

$$C_{nr}^*(G) \simeq \{f \in C(\overline{A}, \mathfrak{K}) : f(wx) = \mathfrak{d}(w:x) \cdot f(x), w \in H_A, \ x \in \overline{A}\}$$

We want to make the $\mathfrak{b}(w,x)$ independent of $x \in \overline{A}$. At this point, we adapt a proof in Solleveld [20]. We will write $H = H_A$. Let

$$\mathfrak{C} := C(\overline{A}, \mathfrak{K}), \qquad \mathfrak{B} := C(\overline{A}, \mathfrak{L}(V))$$

so that $C_{nr}^*(G) = \mathfrak{C}^H$. Let $\mathfrak{C} \rtimes H$ be the crossed product of $\mathfrak{C}$ and H with respect to the action of H on $\mathfrak{C}$ via the intertwiners $\mathfrak{b}(w:x)$. Let r_t be defined as in Eq.(1). Define

$$p_t(x) := \frac{1}{|H|} \sum_{w \in H} \mathfrak{b}(w:r_tx) \cdot w \tag{3}$$

Then $p_t \in \mathfrak{B} \rtimes H$ is an idempotent by Eq.(3). By [18], we have an isomorphism of C^*-algebras:

$$\phi_1 : \mathfrak{C}^H \simeq p_1(\mathfrak{C} \rtimes H)p_1, \quad \sigma \mapsto p_1\sigma p_1$$

Clearly the idempotents p_t are all homotopic, so they are conjugate in $\mathfrak{B} \rtimes H$, see [9, Proposition 4.3.3], e.g. $p_1 = a\,p_0\,a^{-1}$. Now $\mathfrak{C} \rtimes H$ is a C^*-ideal in $\mathfrak{B} \rtimes H$, and so we have

$$\begin{aligned}
\mathfrak{C}^H &\simeq p_1(\mathfrak{C} \rtimes H)p_1 \\
&= ap_0a^{-1}(\mathfrak{C} \rtimes H)ap_0a^{-1} \\
&= ap_0(\mathfrak{C} \rtimes H)p_0a^{-1} \\
&\simeq p_0(\mathfrak{C} \rtimes H)p_0
\end{aligned}$$

as conjugate C^*-algebras in $\mathfrak{B} \rtimes H$. Define

$$\mathfrak{C}^H(0) := \{f \in C(\overline{A}, \mathfrak{K}) : f(wx) = \mathfrak{d}(w:x_0) \cdot f(x), w \in H_A, \ x \in \overline{A}\}$$

6

We have the the C^*-algebra isomorphism

$$\mathfrak{C}^H(0) \simeq p_0(\mathfrak{C} \rtimes H)p_0, \quad \sigma \mapsto p_0 \sigma p_0$$

and so

$$\mathfrak{C}^H \simeq p_0(\mathfrak{C} \rtimes H)p_0 \simeq \mathfrak{C}(0)^H$$

We have replaced the intertwiners $\mathfrak{b}(w : x)$ by the *constant* (i.e. independent of x) intertwiners $\mathfrak{b}(w : x_0)$.

To the algebra $\mathfrak{C}(0)^H$ we will apply the homotopy r_t. This shows that $\mathfrak{C}^H(0)$ is homotopy-equivalent to its fiber $\mathfrak{K}(V_0)^H$ over x_0. Each element in $\mathfrak{K}(V_0)^H$ must commute with the intertwiners $\{\mathfrak{b}(x_0 : h) : h \in H_A\}$. Now we have

$$\{\mathfrak{b}(x_0 : h) : h \in H_A\} = \{\mathfrak{a}(t_0 : w) : w \in W(t_0)\}$$

by Lemma 2.1 and Eq.(2). So each element in $\mathfrak{K}(V_0)^H$ must commute with the commuting algebra

$$\mathcal{C}(t_0) \simeq \mathbb{C}[H_A]$$

Since H_A is an abelian group of order f, we infer that

$$\mathfrak{K}(V_0)^H \simeq \mathfrak{K} \oplus \ldots \oplus \mathfrak{K}$$

with f copies of $\mathfrak{K}$. It is now immediate that $K_0(\mathfrak{A}) = \mathbb{Z}^f$, $K_1(\mathfrak{A}) = 0$. $\qquad \square$

The unramified unitary principal series is reducible at t_0, and it splits the Hilbert space V_0 into f irreducible subspaces $H_1, \ldots, H_f$. Let $E_j : V_0 \to H_j$ be a rank-one projection. Let $e_j \in \mathfrak{A}$ be such that $e_j(x_0) = E_j$. Then $\{e_1, \ldots, e_f\}$ is a set of generators in $K_0(\mathfrak{A})$. The idempotents $e_1, \ldots, e_f$ are the Fourier transforms of the idempotents $e_{K_1}, \ldots, e_{K_f}$ defined in §1. See [13] for a discussion of class 1 representations.

3 Relation to the Baum-Connes correspondence

Let G be a reductive p-adic group. The Baum-Connes correspondence is a definite isomorphism of abelian groups

$$K_j^{top}(G) \simeq K_j \, C_r^*(G) \tag{4}$$

with $j = 0, 1$, see [14]. The idea that the LHS (left-hand-side) is an *answer* for the RHS (right-hand-side) does not withstand scrutiny. For the LHS, defined in terms of K-cycles, has never been directly computed for a noncommutative

reductive p-adic group. A classical result of Higson [12] and Schneider [19] allows us to replace the LHS with the chamber homology groups. These groups have indeed been directly computed in two cases: SL(2), see [8] and GL(3), see [6]. Even here, one can be sure that representative cycles in all the homology groups have been constructed only by checking with the RHS. In other words, one always has to have an independent computation of the RHS. The present Note is a step in that direction.

We now reflect on the conjecture in §7 of [5]. This is the geometric conjecture developed in [2, 3, 4, 5] adapted to the K-theory of C^*-algebras.

We will focus on the C^*-ideal $\mathcal{I} \subset C_r^*(\mathcal{G})$ determined by the Iwahori point $\mathfrak{i} \in \mathfrak{B}(\mathcal{G})$. In this special case, the conjecture in [5, §7] asserts that

$$K_j(\mathcal{I}) = K_W^j(T)$$

with $j = 0, 1$. Here $K_W^j(T)$ is the classical topological equivariant K-theory [1, §2.3] for the finite group W acting on the compact Hausdorff space T. Applying the Chern character for discrete groups [7] gives a map

$$K_W^j(T) \to \bigoplus_l H^{j+2l}(T//W; \mathbb{C})$$

which becomes an isomorphism when $K_W^j(T)$ is tensored with $\mathbb{C}$. Hence the geometric conjecture at the level of C^*-algebra K-theory gives a much finer and more precise formula for $K_* C_r^*(G)$ than Baum-Connes alone provides.

Lemma 3.1. *The quotient space T/W is contractible.*

Proof. The exponential map $\exp : \mathfrak{t} \to T$ induces a homeomorphism of ordinary quotients:

$$\overline{A}/H_A \simeq T/W \tag{5}$$

according to Bourbaki [10, IX, §5.2]. The space $\overline{A}$ is H_A-equivariantly contractible. The map r_t in Eqn.(1) descends to the quotient $\overline{A}/H_A$ and shows that the quotient $\overline{A}/H_A$ is a contractible space. Therefore, T/W is contractible. □

It follows from the C^*-Plancherel theorem [16] that the C^*-ideal $C_{nr}^*(G)$ is a direct summand of the C^*-ideal $\mathcal{I}$. Therefore, the K-theory of $C_{nr}^*(G)$ will contribute to the K-theory of $\mathcal{I}$.

THE SPECIAL LINEAR GROUP. Let $\mathcal{G} = \mathrm{SL}(\ell)$ with ℓ a prime number. We have $\mathcal{G}^\vee = \mathrm{PGL}(\ell, \mathbb{C})$, $G = \mathrm{PU}(\ell, \mathbb{C})$ and T is a maximal torus in $\mathrm{PU}(\ell, \mathbb{C})$.

The Weyl group W is the symmetric group S_ℓ. Let γ denote the standard ℓ-cycle $(123\ldots\ell) \in S_\ell$. The fixed set of each of the permutations $\gamma, \gamma^2, \ldots, \gamma^{\ell-1}$ is

$$t_0 = \exp x_0 = (1 : \omega : \omega^2 : \cdots : \omega^{\ell-1}) \in \mathbb{T}^\ell/\mathbb{T}$$

where $\omega = \exp(2\pi i/\ell)$. The $\overline{L}$-group attached to t_0 is

$$\overline{L}(t_0) = < \omega^{\mathrm{val}_F} > = \mathbb{Z}/\ell\mathbb{Z}$$

and $X(t_0) = 0$, so that the R-group is given by

$$R(t_0) = \mathbb{Z}/\ell\mathbb{Z}.$$

The ℓ irreducible constituents of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(t_0)$ are *elliptic* representations [11]. These are the elliptic representations in the unramified unitary principal series of $\mathrm{SL}(\ell)$.

Let C_ℓ denote the cyclic group generated by the permutation γ. We have

$$T//W = \bigsqcup_{w \in C_\ell} T^w/Z(w) \bigsqcup_{w \notin C_\ell} T^w/Z(w)$$
$$= T/W \sqcup \{t_0\} \sqcup \cdots \sqcup \{t_0\} \bigsqcup_{w \notin C_\ell} T^w/Z(w)$$

By Lemma 3.1, the cohomology $H^0(T//W; \mathbb{C})$ contains the vector space $\mathbb{C}^\ell$. This is consistent with Theorem 1.1:

$$K_0 \, C_{nr}^*(\mathrm{SL}(\ell)) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^\ell.$$

Note that ℓ is the connection index of $\mathrm{SL}(\ell)$. The unramified unitary principal series of $\mathrm{SL}(\ell)$ admits a unique L-packet with ℓ elliptic constituents, and this is detected by K_0.

Let $\overline{A}//H_A$ denote the extended quotient of $\overline{A}$ by H_A. The canonical projection

$$\pi : \overline{A}//H_A \to \overline{A}/H_A$$

is a model of reducibility at the point t_0. The fibre $\pi^{-1}(t_0)$ contains ℓ points. This is very much in the spirit of the geometric conjecture in [2, 3, 4, 5].

In the special case of $\mathrm{SL}(3)$, the vector space $\mathfrak{t}$ is the Euclidean plane $\mathbb{R}^2$. The singular hyperplanes [10, IX, §5.2] tessellate $\mathbb{R}^2$ into equilateral triangles. The interior of each equilateral triangle is an *alcove*. The closure of an alcove A is denoted $\overline{A}$. The affine Weyl group W_a operates simply transitively on the set of alcoves, and $\overline{A}$ is a fundamental domain for the action of W_a on $\mathfrak{t}$, see [10, IX.§5.2]. Barycentric subdivision refines this tessellation into isosceles

triangles. The extended affine Weyl group W_a' acts simply transitively on the set of these isosceles triangles, but the closure $\overline{\Delta}$ of one such triangle is not a fundamental domain for the action of W_a'. The corresponding quotient space is [10, IX.§5.2]:

$$\mathfrak{t}/W_a' \simeq \overline{A}/H_A.$$

The abelian group H_A is the cyclic group $\mathbb{Z}/3\mathbb{Z}$ which acts on $\overline{A}$ by rotation about the barycentre of $\overline{A}$ through $2\pi/3$. The special points of $\overline{A}$ are the vertices of $\overline{A}$.

SOME EXCEPTIONAL GROUPS. If $\mathcal{G} = E_8$, F_4 or G_2 then we have $P/Q = 0$ and so the connection index $f = 1$, see [10, Plates VIII, IX, X]. Therefore

$$T/W \simeq \overline{A}.$$

This is a contractible space. There are no L-packets. The K-theory of the unramified unitary principal series of $\mathcal{G}$ is that of a point.

KAZHDAN-LUSZTIG PARAMETERS We begin by quoting a recent theorem of Solleveld:

$$K_j(\mathcal{I}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K_j(C^*(W^e)) \otimes_{\mathbb{Z}} \mathbb{C} \tag{6}$$

with $j = 0, 1$. This theorem is a special case of [21, Theorem 5.1.4] when $\Gamma = 1$. We have

$$C^*(W^e) = C^*(X \rtimes W) \simeq C(T) \rtimes W. \tag{7}$$

By the classical Green-Julg theorem, we have

$$K_j(C(T) \rtimes W) \simeq K_W^j(T). \tag{8}$$

Therefore we have

$$K_j(\mathcal{I}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq K_W^j(T) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{9}$$

The Chern character for discrete groups gives

$$K_j(C(T) \rtimes W) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_l H^{j+2l}(T//W; \mathbb{C})$$

so that, modulo torsion, the K-theory of the Iwahori C^*-ideal $\mathcal{I}$ is give by the cohomology (even or odd) of the extended quotient $T//W$.

The main result in [4] is that $T//W$ is a model of the Kazhdan-Lusztig parameters. We have to identify the KL-parameters for the unramified unitary principal series.

For the irreducible elements in the unramified unitary principal series, the KL-parameters are

$$\{(t, 1, 1) : t \in T\}. \tag{10}$$

For the f irreducible constituents of the tempered L-packet

$$\Pi(t_0) := \mathrm{Ind}_{\mathcal{T}\mathcal{U}}^{\mathcal{G}}(t_0 \otimes 1)$$

the KL-parameters will be

$$\{(t_0, 1, \rho) : \rho \in \mathrm{Irr}(H_A)\} \tag{11}$$

where ρ is a character of the component group of the centralizer in $\mathcal{G}^\vee$ of t_0. All such characters are allowed. By Lemma 2.1, this component group is H_A. The parameters in (10) and (11) will determine points in the extended quotient $T//W$. This will, in turn, determine the inclusion

$$K_j \, C_{nr}^*(\mathcal{G}) \to K_j \, \mathcal{I} \tag{12}$$

and illustrate how $\Pi(t_0)$ contributes to $K_0 \, \mathcal{I}$.

We know that

$$K_0(C_{nr}^*(\mathcal{G})) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^f \tag{13}$$

by Theorem 1.1.

For the exceptional groups $\mathcal{G} = E_8$, F_4 or G_2, the connection index $f = 1$, and so the map (12) is especially simple. There is no L-packet. The K-theory of the unramified unitary principal series of $\mathcal{G}$ is that of a point.

THE GENERAL LINEAR GROUP. Let $\mathfrak{s}$ be a point in the Bernstein spectrum of $\mathcal{G} = \mathrm{GL}(n)$, and let $\mathcal{I}_{\mathfrak{s}}$ be the corresponding C^*-ideal in $C_r^*(\mathcal{G})$. According to [17, p. 131], we have a strong Morita equivalence from $\mathcal{I}_{\mathfrak{s}}$ to a commutative unital C^*-algebra:

$$\mathcal{I}_{\mathfrak{s}} \sim C(\Delta_{\mathfrak{s}}//W_{\mathfrak{s}}).$$

Applying the Chern character for discrete groups [7] gives

$$K_j(\mathcal{I}_{\mathfrak{s}}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_l H^{j+2l}(\Delta_{\mathfrak{s}}//W_{\mathfrak{s}}; \mathbb{C})$$

Now each connected component in the compact space $\Delta_{\mathfrak{s}}//W_{\mathfrak{s}}$ is a product of symmetric products of circles. So each component is homotopy equivalent to

a compact torus. Each K-group and each cohomology group is torsion-free. Therefore, we have

$$K_j(\mathcal{I}_{\mathfrak{s}}) = K^j_{W_{\mathfrak{s}}}(\Delta_{\mathfrak{s}})$$

which *confirms the conjecture in [5, §7] for the general linear group.*

The spherical C^*-algebra $\mathfrak{A} = C^*_{nr}(\mathcal{G})$ is isomorphic to $C(T/W, \mathfrak{K})$ and the Iwahori C^*-algebra is Morita equivalent to $C(T//W)$, see [15]. The n-fold symmetric product T/W of n circles is homotopy equivalent to one circle via the product map

$$(z_1, \ldots, z_n) \mapsto z_1 \cdots z_n$$

and so $K_j(\mathfrak{A})$ contributes one generator to $K_j(\mathcal{I})$ with $j = 0, 1$.

References

[1] M.F. Atiyah, K-Theory, Benjamin, New York 1967.

[2] A.-M. Aubert, P. Baum, R.J. Plymen, Geometric structure in the representation theory of p-adic groups, C.R. Acad. Sci. Paris, Ser. I 345 (2007) 573–578.

[3] A.-M. Aubert, P. Baum, R.J. Plymen, Geometric structure in the principal series of the p-adic group G_2, Represent. Theory 15 (2011) 126 –169.

[4] A.-M. Aubert, P. Baum, R.J. Plymen, Kazhdan-Lusztig parameters and extended quotients, `http://arxiv.org/abs/1102.4172`

[5] A.-M. Aubert, P. Baum, R.J. Plymen, Geometric structure in the representation theory of p-adic groups II, Contemporary Math. 534 (2011) 71 –90. `http://eprints.ma.man.ac.uk/1553`

[6] A.-M. Aubert, S. Hasan, R.J. Plymen, Cycles in the chamber homology of GL(3), K-Theory 37 (2006) 341 – 377.

[7] A. Connes, P. Baum, The Chern character for discrete groups, A Fête of Topology, Academic Press, New York, 1988, 163 – 232.

[8] P. Baum, N. Higson, R.J. Plymen, Equivariant homology for SL(2) of a p-adic field. Contemporary Math 148 (1993) 1-18.

[9] B. Blackadar, K-Theory for operator algebras, Cambridge University Press 2002.

[10] N. Bourbaki, Lie groups and Lie algebras, chapters 4-6, Springer 2002; chapters 7-9, Springer 2005.

[11] D. Goldberg, R-groups and elliptic representations for SL_n, Pacific J. Math. 165 (1994) 77 –92.

[12] N. Higson, V. Nistor, Cyclic homology of totally disconnected groups acting on buildings, J. Functional Analysis 141 (1996) 466 –495.

[13] C. D. Keys, Reducibility of unramified unitary principal series of p-adic groups and class-1 representations, Math. Ann. 260 (1982) 397-402.

[14] V. Lafforgue, KK-theory for bivariant Banach algebras and the Baum-Connes conjecture, Invent. Math. 149 (2002) 1 – 95.

[15] R. J. Plymen, Noncommutative Geometry: Illustrations from the representation theory of $\mathrm{GL}(n)$. `http://eprints.ma.man.ac.uk/1346`

[16] R. J. Plymen, Reduced C^*-algebra for reductive p-adic groups, J. Funcional Analysis 88 (1990) 251 – 266.

[17] R.J. Plymen, Reduced C^*-algebra of the p-adic group $\mathrm{GL}(n)$ II, J. Functional Analysis 196 (2002) 119 – 134.

[18] J. Rosenberg, Appendix to: Crossed products of UHF algebras by product type actions, Duke Math. J. 46 (1979), 1-23, by O. Bratteli.

[19] P. Schneider, Equivariant homology for totally disconnected groups, J. Algebra 203 (1998) 50 – 68.

[20] M. Solleveld, Some Fréchet algebras for which the Chern character is an isomorphism, K-Theory 36 (2005) 275 – 290.

[21] M. Solleveld, On the classification of irreducible representations of affine Hecke algebras with unequal parameters.
`http://arxiv.org/abs/1008.0177`

Tayyab Kamran, Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, Peshawar Road, Rawalpindi, Pakistan

Email: tayyabkamran@gmail.com

Roger Plymen, School of Mathematics, Manchester University, Manchester M13 9PL, UK *and* School of Mathematics, Southampton University, Southampton SO17 1BJ, UK

Email: plymen@manchester.ac.uk, R.J.Plymen@soton.ac.uk