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A note on the algebraic structure of conditionals

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Abstract

We investigate the idea of representing conditional measures as simple measures (possibly with further properties) on conditional objects. To formalise this intuition we introduce a logico-algebraic framework based on the concept of *conditional algebras*. These are structures which allow us to distinguish between the logical properties of conditionals and those of conditional measures, a distinction which, we argue, helps us clarifying both concepts. We illustrate the applicability of our framework by considering three popular conditional measures in the uncertain reasoning literature, namely, plausibility, probability and possibility measures.

Keywords. Conditionals; Uncertain Reasoning; Conditional Measures; Conditional Algebra.

1 Introduction

Conditionals are widely used to represent central competencies of intelligent agents, from the ability to reason by cases to the essential mechanisms underlying belief change and learning. Given their centrality in modelling cognition and reasoning, conditionals have been studied from a very wide number of distinct perspectives in the cognitive sciences [55], artificial intelligence [6, 19], and of course logic [14].

Among non classical logics, conditionals play a fundamental role in qualitative as well as quantitative uncertain reasoning. As a consequence of the very fruitful interaction between philosophical logic and artificial intellingence, the semantic approaches to conditionals of the 1970s, mainly Stalnaker's [61] and D. Lewis's [44], paved the way for the development of comprehensive theories of non monotonic consequence relations [40, 56, 46, 60]. As to quantitative uncertain reasoning, conditionals feature prominently in the key concept of conditional probability. Indeed, the close connection between the two perspectives has been thoroughly scrutinised in many attempts either to support a thesis which sees the probability of a conditional as conditional probability[2, 1], or to reject it as ill-founded [20, 21, 22, 24]. Indeed, despite the apparent simplicity of the "ratio definition", on which more below, the

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notion of conditional probability is far from being uncontroversial. Makinson, for instance, points out in [23] how some rather undesirable behaviour can arise when conditioning on what he refers to the "critical zone". Things get inevitably more complicated if we move to non-classical probability logic, i.e. probability defined on non-classical logics, a rapidly expanding research field on which we shall come back in the concluding Section of this paper.

Our leading intution is to formalise *conditional measures as simple measures (possibly with further properties) on conditional objects.* This goes hand in hand with a general setting disclosing a two-fold perspective on conditionals. On the one hand, we characterize conditionals as objects of an algebra which we term *conditional algebra.* Within such structures conditionals are *simple* objects, a terminology whose meaning will be apparent in a short while. It is worth emphasising from the outset that our algebraic characterization has both syntactic and semantic aspects. On the other hand, since we are ultimately interested in modelling various epistemic attitudes that agents may exhibit in connection to conditional statements, assertions, events, etc., we are centrally interested investigating appropriate *measures* to be attached to conditionals. Indeed, many such epistemic measures have been introduced in the uncertain reasoning literature. Among them, plausibility measures, possibility measures and probability measures are particularly important, as we will illustrate in Section 3.1 below.

Whilst we are unaware of other proposals which separate the logical properties of conditionals from those of conditional measures, the notion of conditional algebra has been investigated in the context of the so-called *Goodman-Nguyen-van Fraassen algebras.* Since we will be in an ideal position to compare this approach with ours only after having introduced some formal details, suffice it to mention at this point that the notion of Conditional Event Algebra (CEA) introduced in [37] differs quite substantially from our notion of conditional algebra. Intuituively, a simple probability measure P defined on a Goodman-Nguyen-van Fraassen algebra A (which is in fact a boolean algebra) defines a conditional probability via the equation $P(x \mid y) = P(x \wedge y)/P(y)$. This property is also satisfied by our conditional algebras. However, conditional algebras permit us to define conditional measures (not just probability measures!) as primitive notions. In this sense, our notion of conditional algebra is substantially more general, besides being arguably simpler, as it will be pointed out in detail in Section 4.1 below.

In order to formulate the key questions addressed in this paper, let us introduce a little notation. Let $\mathcal{L} = \{p_1, \dots p_n\}$ be a set of propositional variables and let $\mathcal{SL}$ be the set of sentences (denoted by θ, ϕ ect.) built up from $\mathcal{L}$ by means of the classical propositional connectives $\{\neg, \wedge, \vee, \rightarrow\}$. Thus the formula $\theta \rightarrow \phi$ reads as the classical (material) implication which false just if $v(\theta) = 1$ and $v(\phi) = 0$ (where as usual $v : \mathcal{L} \rightarrow \{0, 1\}$ is a classical valuation on $\mathcal{L}$).

Let $\theta, \phi \in \mathcal{SL}$. We denote the conditional assertion "ϕ given θ" by $\phi \mid \theta$. It will sometimes be convenient to refer to ϕ as the *consequent* of the conditional and to θ as its *antecedent.* When presented with a conditional of this form, there are three *distinct* questions that we may ask:

1. what are the syntactic properties of $\phi \mid \theta$?

2. what are the semantic properties of $\phi \mid \theta$?

3. what properties should be satisfied by a (rational) measure of belief on $\phi \mid \theta$?

The answers put forward in this paper can be informally illustrated as follows:

1. Though it makes perfectly good sense to distinguish, in the conditional $\phi \mid \theta$, the antecedent from the consequent, we will assume that conditionals are *simple* objects which live in a conditional structure. The fundamental consequence of this approach is that the "global" properties, so to speak, of conditionals are defined for the underlying algebraic structure and not at the object level of the conditional formula. We will be in a position to clarify the import of this once we have the formal definition of a conditional algebra.

2. The semantic properties of conditionals are also given at the level of the conditional algebra. For instance, by suitably constraining the ideals of a particular freely generated boolean algebra we will be in a position to distinguish among various kinds of conditionals. As will become apparent, all the results of this paper apply to *indicative* conditionals but fail for *counterfactuals*. The reason for this lies in the adoption of a principle which we refer to as the *rejection constraint* according to which a conditional $\phi \mid \theta$ is (semanticaly) meaningless if the antecedent fails to be true (under a suitably defined valuation). This property, as we shall shortly see, is motivated by reflections on conditional events.

3. This brings us to the question of the *measure* on conditional assertions. Suppose a belief measure μ has been defined on our language, and suppose that is satisfies certain rationality principles (for present purposes it is immaterial what those principles might say in practice). One way to understand a measure of belief in a conditional $\phi \mid \theta$ is as follows. Let the agent hold that, say $\mu(\phi) = x$. We are now interested in determining the quantity y the agent is rationally entitled to equate to $\mu(\phi \mid \theta)$[1]

   This interpretation, which features in qualitative as well as quantitative measures, clearly presupposes that conditionals are "complex" objects which are constructed on the basis of simpler ones. To fix ideas, this is the intuition behind defining conditional probability by means of the ratio rule

$$P(\theta \mid \phi) = \frac{P(\theta \wedge \phi)}{P(\theta)}.$$

   In this work we propose an alternative characterization of conditionals which we take to be *simple objects* of a conditional algebra. As a consequence we can define measures on such algebras such that a conditional measure is in effect a simple measure over a conditional algebra.

[1]This interpretation of conditional measures is sometimes referred to as *revisionary* as it really says that the measure on the conditional is the revision of the simple measure on the consequent by the antecedent (see, e.g. [57]).

Remark 1.1. Before moving to the formal details, it is worth anticipating that our entire formalisation, and consequently of our answers to the above points, are based on the assumption that our underlying algebraic structures are boolean algebras. The reasons for this restriction, to which we will refer as the *boolean constraint*, will be discussed in Section 4 below.

The paper is organized as follows. In the Section 2 we introduce and discuss the *Rejection constraint* which will be central to our characterization of conditionals. In Section 3 we recall and prove some basic properties of boolean algebras which will be subsequently needed. Axiomatizations for several unconditional, as well as conditional measures on boolean algebras, are recalled in Section 3.1. Section 3.2 is devoted to comparing our approach to the existing proposals for defining operations between conditional objects. Section 4 presents the construction that allows us to define conditional boolean algebras and illustrates some of their basic properties. We then show that the structure of conditional boolean algebras, is sufficiently general to provide a semantical interpretation of a conditional $\theta \mid \varphi$ in terms of the classical material implication $\varphi \to \theta$. Section 5 tackles the central question of this paper, namely the characterization of the conditions under which a conditional measure of uncertainty on a boolean algebra is definable as a particular unconditional measure on a conditional boolean algebra of events.

2 The Rejection Constraint

The most general feature on conditionals is that they express some form of hypothetical assertion. More precisely, what they capture is the assertion of the consequent based *on the supposition* that the antecedent is satisfied (with respect, of course, to a suitably defined semantics). As Quine put it some four decades ago:

> [An] affirmation of the form 'if p then q' is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. *If, on the other hand, the antecedent turns out to have been false, our conditional affirmation is as if it had never been made* ([26] Added emphasis)

The idea here is that the semantic evaluation of a conditional (in this interpretation) amounts to a two-step procedure. We first need to check the antecedent. If this is rejected, the conditional ceases to mean anything at all. Otherwise we move on to evaluating the consequent. It seems natural enough to refer to this interpretation as based on a *rejection constraint*.

The adoption of the rejection constraint immediately implies that our framework cannot accommodate those conditionals which do not lose their meaning as a consequence of the supposed (or ascertained) rejection of the antecedent. The much investigated class of *counterfactual conditionals*, which are characterized by the presumption of the falsity of their antecedent, fall therefore beyond the scope of our investigation.

The rejection constraint is best motivated by recalling the betting framework which underlies the subjectivistic characterization of probability. According to de Finetti, probability is defined only over *events* and not over facts (see, e.g. [10], Chapters 15-16). While a fact can happen (or not) de Finetti described an event as follows:

> [T]he characteristic feature of what I refer to as an "event" is that the circumstances under which the event will turn out to be "verified" or "disproved" have been fixed in advance. ([10] p.150)

Recall that the subjective approach to probability [29, 8, 9] characterizes degrees of belief by means of a betting scheme and that the agent's degree of belief are defined in terms of the price that the agent is willing to pay to enter a well-speficied series of bets. The resulting definition of probability in terms of "fair betting quotients" then presupposes that the state of the matter concerning the relevant events –logically, the truth value of the corresponding sentences– can be ascertained *a posteriori* with the degree of precision previously (to the bet, that is) specified. This betting interpretation extends in the natural way to conditional events. In particular, with respect to a fixed possible world v,

$$\text{a bet on } \theta \mid \phi \text{ is } \begin{cases} \text{won if} & v(\phi) = v(\theta) = 1; \\ \text{lost if} & v(\phi) = 1 \text{ and } v(\theta) = 0; \\ \text{called off if} & v(\phi) = 0. \end{cases}$$

The final clause is of course the most interesing one, for it states that under the valuation which assigns 0 to the conditioning event, a conditional bet must be called off (all paid moneys are returned). As de Finetti remarks

> if one stipulates a conditional bet, one must specify exactly the conditions under which the bet will be called off. When writing $\mathbf{P}(E \mid H)$, what is actually meant by H is an *event*, not a *fact*. ([10], p. 150)

The analogy with the Quinean view recalled above is striking, but there is no coincidence in that. The rejection constraint, in our view, constitutes the deepest connection that there is to be found between conditionals and conditional measures. We now turn to introduce our formal setting, which will help us clarifying the import of this.

3 Formal prelimiaries

We begin by recalling some useful properties of boolean algebras, measures and conditional measures on boolean algebras. We assume some familiarity with the basic theory of boolean algebras and we refer the interested reader to [16], for an introductory text on the topic.

For every boolean algebra A we denote by $\delta : A \times A \to A$ the well known *symmetric difference* operator. In other words δ stands for the following abbreviation: for every $x, y \in A$,

$$\delta(x, y) = (x \vee y) \wedge \neg(x \wedge y) = \neg(x \to y) \vee \neg(y \to x). \tag{1}$$

In any boolean algebra A, the following equations hold:

(i) $\delta(x, y) = \delta(y, x)$

(ii) $\delta(x, \delta(y, z)) = \delta(\delta(x, y), z)$

(iii) $\delta(\delta(x, y), \delta(y, z)) = \delta(x, z)$

(iv) $\delta(x, \perp) = x$

(v) $\delta(x, x) = \perp$

Therefore, in particular δ is (i) commutative; (ii) associative; and (iv) has $\perp$ as neutral element.

The following proposition collects further properties of δ. In order to keep the paper self-contained, we provide proofs of all the statements, even though some of them are likely to be have appeared in the literature.

Proposition 3.1. *The following hold in any boolean algebra A:*

(a) $\delta(x, y) = \perp$ *iff* $x = y$

(b) $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$

(c) $\delta(x, y) = \delta(\neg x, \neg y)$

(d) $\delta((x \vee y), (z \vee k)) = \delta(x, z) \vee \delta(y, k)$

Proof. (a) By equation (1) write $\delta(x, y)$ as $\neg(x \to y) \vee \neg(y \to x)$. Therefore, $x = y$ iff $x \to y = \top$ and $y \to x = \top$ iff $\delta(x, y) = \perp$.

(b) If $x = z$, then the claim is trivially settled by (a). Assume without loss of generality, that $A = \{\perp, \top\}$. Then let $x = \top$ and $z = \perp$. Then $\delta(x, z) = \top$. Conversely, if either $y = \perp$ or $y = \top$, then respectively either $\delta(x, y) = \top$, or $\delta(y, z) = \top$. Therefore the claim follows.

(c) Trivially follows by order-reversing property of the negation $\neg$.

(d) This point can be easily obtained appealing to the distributivity properties of $\vee$ and $\wedge$. □

A non-empty subset $\mathfrak{i}$ of a boolean algebra A is said to be an *ideal* of A if: (1) $\perp \in \mathfrak{i}$; (2) for any $x, y \in \mathfrak{i}$, $x \vee y \in \mathfrak{i}$; (3) if $x \in \mathfrak{i}$, and $y \leq x$, then $y \in \mathfrak{i}$. If $X \subseteq A$, denote by $\mathfrak{I}(X)$, the ideal generated by X, i.e. the least ideal (w.r.t. inclusion) containing X. For every $x \in A$, we denote by $\downarrow x$ is the *principal ideal* of A generated by x, i.e. $\downarrow x = \{y \in A : y \leq x\} = \mathfrak{I}(\{x\})$. We dually denote by $\uparrow x = \{z \in A : z \geq x\}$ the principal *filter* of A generated by x[2].

Theorem 3.2. *Let A be a boolean algebra, and let $\mathfrak{i}$ be a non-empty subset of A. Then $\mathfrak{i}$ is an ideal of A iff, letting for all $x, y \in A$, $x \sim_{\mathfrak{i}} y$ iff $\delta(x, y) \in \mathfrak{i}$, $\sim_{\mathfrak{i}}$ is a congruence on A. Moreover $\mathfrak{i} = \{x \in A : x \sim_{\mathfrak{i}} \perp\}$.*

[2] Filters and ideals are dual notions for boolean algebras, see [5, IV§3]

Proof. Let us start by proving the first claim: ($\Rightarrow$). Let $\mathfrak{i}$ be an ideal. Then by Proposition 3.1(a) (see also (v)) for every $x \in A$, $\delta(x,x) = \bot \in \mathfrak{i}$, whence $x \sim_{\mathfrak{i}} x$. Moreover $\delta(x,y) = \delta(y,x)$ by (*ii*), then $x \sim_{\mathfrak{i}} y$ iff $y \sim_{\mathfrak{i}} x$ for every $x, y \in A$. Finally for $x, y, z \in A$, if $x \sim_{\mathfrak{i}} y$ and $y \sim_{\mathfrak{i}} z$, then $\delta(x,y), \delta(y,x) \in \mathfrak{i}$, whence, since $\mathfrak{i}$ is an ideal, $\delta(x,y) \vee \delta(y,z) \in \mathfrak{i}$, and hence also $\delta(x,z) \in \mathfrak{i}$ from Proposition 3.1(b). That is $x \sim_{\mathfrak{i}} z$. Therefore $\sim_{\mathfrak{i}}$ is an equivalence on A. Easily Proposition 3.1(b) and (d) show that $\sim_{\mathfrak{i}}$ is compatible with the oprations of boolean algebras, whence it is a congruence.

($\Leftarrow$). Let $\sim_{\mathfrak{i}}$ be a congruence. Then $x \sim_{\mathfrak{i}} x$ for all x, and hence $\delta(x,x) \in \mathfrak{i}$, whence, from Proposition 3.1(a), $\bot \in \mathfrak{i}$.

Moreover, let $x \in \mathfrak{i}$, and assume $y \leq x$. Then $y = x \wedge y$, and therefore

$$y \sim_{\mathfrak{i}} (x \wedge y) \tag{2}$$

From (iv), $x \in \mathfrak{i}$ implies $\delta(x, \bot) \in \mathfrak{i}$, and hence

$$x \sim_{\mathfrak{i}} \bot. \tag{3}$$

Since $\sim_{\mathfrak{i}}$ is a congruence and is therefore compatible with $\wedge$, by (2) and (3), $x \wedge y \sim_{\mathfrak{i}} (x \wedge y) \wedge \bot$, and then $y \sim_{\mathfrak{i}} \bot$. Then $y \in \mathfrak{i}$.

Finally, $x, y \in \mathfrak{i}$ implies $\delta(x, \bot), \delta(y, \bot) \in \mathfrak{i}$, whence $x \sim_{\mathfrak{i}} \bot$, and $y \sim_{\mathfrak{i}} \bot$. Since $\sim_{\mathfrak{i}}$ is compatible with $\vee$, then $x \vee y \sim_{\mathfrak{i}} \bot$ and then $\delta(x \vee y, \bot) \in \mathfrak{i}$, and by (iv) $x \vee y \in \mathfrak{i}$.

To finally settle the claim notice that by (iv), for every $x \in A$, $x \in \mathfrak{i}$ iff $\delta(x, \bot) \in \mathfrak{i}$ iff $x \sim_{\mathfrak{i}} \bot$. □

Corollary 3.3. *Let A be a boolean algebra, and let $\mathfrak{i}$ be an ideal of A. Then for every $x, y \in A$, the equation $x = y$ is valid in the quotient algebra $A/\mathfrak{i}$ iff $\delta(x,y) \in \mathfrak{i}$.*

Proof. The claim follows from Proposition 3.1. In fact it is easy to see that $x = y$ is valid in $A/\mathfrak{i}$, iff $\delta(x,y) \sim_{\mathfrak{i}} \bot$, iff $\delta(x,y) \in \mathfrak{i}$. Finally the latter holds iff $x \sim_{\mathfrak{i}} y$. □

Remark 3.4. The above Corollary 3.3 immediately implies that, whenever $\mathfrak{i}$ is a proper ideal, and $\neg\delta(x,y) \in \mathfrak{i}$, then the quotient algebra $A/\mathfrak{i}$ makes valid $\neg(x = y)$. In fact if $\neg\delta(x,y) \in \mathfrak{i}$, then $\delta(x,y) \notin \mathfrak{i}$ (otherwise $\delta(x,y) \vee \neg\delta(x,y) = \top \in \mathfrak{i}$, and hence $\mathfrak{i}$ would not be proper) iff in $A/\mathfrak{i}$, $\neg(x = y)$ holds true i.e. $x \neq y$.

3.1 Measures on boolean algebras

We now recall the basic definitions and results about measures and conditional measures on boolean algebras. We will restrict our attention to probability, possibility, and plausibility measures. For an overview of those topics we refer the interested reader to Halpern's monograph [17]. Our notation, however, diverges somewhat from Halpern's who presents an uncertainty measure m by introducing a *space* (W, A, m) based on the usual possible-world semantics: W is a non-empty set of possible-worlds, A is a boolean subalgebra of the powerset of W, and $m : A \to [0,1]$ is an uncertainty measure (in the sense that we are going to introduce). For the limited purposes of this paper, however, it will be sufficient to use a simpler presentation which introduces directly the relevant measure on boolean algebras. Note that this does not restrict the scope of the definition since for every measure m on a boolean algebra A, the triple $(Hom(A,\{0,1\}), A, m)$, where $Hom(A,\{0,1\})$ denotes the class of all the homomorphisms from A into the boolean chain $\{0,1\}$, is an *uncertainty-measure space* in the sense of [17]. The only minor modification which needs doing concerns the definition of *Popper-algebras* ([17, §3, p. 74]). For every boolean algebra A, let $\hat{A}$ be a non-empty and upward closed subset of A. Then we call a *Popper-algebra* over A, the cartesian product $A \times \hat{A}$.

Let A be a boolean algebra. A *plausibility measure* on A is a map $Pl : A \to [0,1]$ satisfying the following conditions:

(Pl1) $Pl(\top) = 1$,

(Pl2) $Pl(\bot) = 0$,

(Pl3) for all $x, y \in A$, whenever $x \leq y$, $Pl(x) \leq Pl(y)$.

Therefore a plausibility is a normalized and monotone mapping from A into $[0,1]$. Let now $A \times \hat{A}$ be a Popper-algebra over A. Then a *conditional plausibility measure* is a map $CPl : A \times \hat{A} \to [0,1]$ satisfying the following properties:

(CPl1) $CPl(\bot, x) = 0$,

(CPl2) $CPl(x, x) = 1$,

(CPl3) for every $x, y \in A$, whenever $x \leq y$, $CPl(x,z) \leq CPl(y,z)$,

(CPl4) $CPl(x,y) = CPl(x \wedge y, y)$.

Possibility measures are particular kinds of plausibility measures. In fact a possibility measure is a plausibility $\Pi : A \to [0,1]$ which also satisfies the condition

(Π) $\Pi(x \vee y) = \max\{\Pi(x), \Pi(y)\}$[3].

Notice that in fact, condition $(Pl3)$ is now redundant: if $x \leq y$, then $x \vee y = y$, and hence $\Pi(y) = \Pi(x \vee y) = \max\{\Pi(x), \Pi(y)\}$, i.e. $\Pi(y) \geq \Pi(x)$.

Consider a Popper-algebra $A \times \hat{A}$. Then a *conditional possibility* on $A \times \hat{A}$ is a conditional plausibility $C\Pi : A \times \hat{A} \to [0,1]$ satisfying:

(CΠ1) $C\Pi(x \vee y, z) = \max\{C\Pi(x,z), C\Pi(y,z)\}$[4],

(CΠ2) $C\Pi(x \wedge y, z) = \min\{C\Pi(x, y \wedge z), C\Pi(y,z)\}$, for all y, z such that $y \wedge z \in \hat{A}$.

[3] In his monograph [17], Halpern defines a (conditional) possibility measure by the apparently stronger condition $(\Pi)'$: whenever $x \wedge y = \bot$, $\Pi(x \vee y) = \max\{\Pi(x), \Pi(y)\}$ (analogously condition $(C\Pi1)$ for the conditional case). In fact (Π) and $(\Pi)'$ are equivalent, therefore we will refer to (Π) without loss of generality.

[4] See footnote 3 above.

As in the case of simple (i.e. unconditional) possibility measures, some of the properties of conditional plausibility becomes redundant with the above axiomatization of conditional possibility. In particular, ($CPl3$) can be clearly proved from ($C\Pi1$). Moreover from ($C\Pi2$) and ($CPl2$), for every $(x, y) \in A \times \hat{A}$, $C\Pi(x \wedge y, y) = \min\{C\Pi(x, y), C\Pi(y, y)\} = \min\{C\Pi(x, y), 1\} = C\Pi(x, y)$. Therefore ($CPl4$) is redundant as well.

Finally a *probability measure* is a plausibility measure $Pr : A \to [0, 1]$ satisfying additivity:

(Pr) for all $x, y \in A$, whenever $x \wedge y = \bot$, $Pr(x \vee y) = Pr(x) + Pr(y)$.

Although a probability Pr satisfies (Π) only if x and y are comparable (and hence (Π) does not hold in general), it is well known that probabilities are monotone maps. Therefore ($Pl3$) is redundant. Moreover $1 + Pr(\bot) = Pr(\top) + Pr(\bot) = Pr(\top \vee \bot) = Pr(\top)$, so $Pr(\bot) = 0$, and hence ($Pl2$) is redundant as well.

A conditional plausibility $CPr : A \times \hat{A} \to [0, 1]$ is a *conditional probability*, provided that the following holds:

(CPr1) $CPr(x \vee y, z) = CPr(x, z) + CPr(y, z)$ for all those $x, y \in A$ such that $x \wedge y = \bot$,

(CPr2) $CPr(x \wedge y, z) = CPr(x, y \wedge z) \cdot CPr(y, z)$ for all those $y, z \in A$ such that $y \wedge z \in \hat{A}$.

Remark 3.5. It is worth noticing that, when passing from unconditional to conditional maps, possibility and probability measures satisfy, beyond normalization and additivity, an extra property, namely ($C\Pi2$) in the case of conditional possibility, and ($CPr2$) in the case of conditional probability. ($CPr2$) is a generalization of the well known *Bayes theorem* for conditional probabilities, and ($C\Pi2$) is its analogue in the context of possibility (i.e. idempotent) measures. We shall come back to this peculiar property in the final Section of the paper. For the time being we limit ourselves to point out that conditional plausibility measures do not require similar properties to be satisfied.

3.2 On the conjunction between conditionals

Let A be a boolean algebra, and denote by $A \mid A$ the class $\{a \mid b : a, b \in A\}$. The problem of defining operations between the objects in $A \mid A$ has been discussed extensively discussed in the context of measure-free conditionals (see, e.g. [11, 12] and the bibliography therein contained).

Whilst widespread consensus exists about defining the negation of a conditional as $\neg(a \mid b) = \neg a \mid b$, there are at least three major proposals competing for the definition of conjunction:

(Schay, Calabrese) $(a \mid b)\&_1(c \mid d) = [(b \to a) \wedge (d \to c) \mid (b \vee d)]$ (cf. [7, 28] and see also [2] where this conjunction between conditionals is called *quasi-conjunction*).

(Goodmain and Nguyen) $(a \mid b)\&_2(c \mid d) = (a \wedge c) \mid [(\neg a \wedge b) \vee (\neg c \wedge d) \vee (b \vee d)]$ (cf. [15])

(Schay) $(a \mid b)\&_3(c \mid d) = (a \wedge c) \mid (b \wedge d)$ (cf. [28])

Disjunctions $\oplus_1, \oplus_2$ and $\oplus_3$ among conditionals, are defined by De Morgan's laws from $\&_1, \&_2$ and $\&_3$ above. Schay [28], and Calabrese [7] show that $\&_1$, and $\oplus_1$ are not distributive with respect to each other, and hence the class $A \mid A$ of conditionals, endowed with $\&_1$ and $\oplus_1$ is no longer a boolean algebra. Therefore $\&_1$ does not satisfy the boolean constraint mentioned in Remark 1.1. For this reason we reject $\&_1$ as a suitable definition of conjunction.

Similarly, the boolean constraint leads us to reject also $\&_3$, and $\oplus_3$ as candidates for defining conjunction and disjunction betwen conditionals. Indeed, if we defined the usual order relations by

1. $(a_1 \mid b_1) \leq_1 (a_2 \mid b_2)$ iff $(a_1 \mid b_1)\&_3(a_2 \mid b_2) = (a_1 \wedge a_2 \mid b_1 \wedge b_2) = (a_1 \mid b_1)$, and
2. $(a_1 \mid b_1) \leq_2 (a_2 \mid b_2)$ iff $(a_1 \mid b_1) \oplus_3 (a_2 \mid b_2) = (a_1 \vee a_2 \mid b_1 \wedge b_2) = (a_2 \mid b_2)$,

then $\leq_1 \neq \leq_2$. To see this, let a be a fixed element in A. Then $(a \mid \top)\&_3(a \mid a) = (a \mid a)$ and hence $(a \mid a) \leq_1 (a \mid \top)$. On the other hand $(a \mid \top) \oplus_3 (a \mid a) = (a \mid a)$ as well, and therefore $(a \mid \top) \leq_2 (a \mid a)$ for every $a \in A$, and in particular for a such that $a \mid a \neq a \mid \top$. Conversely, it is easy to see that, if we restrict to the class of those conditionals $a_i \mid b$ with a fixed antecedent b, then $\leq_1 = \leq_2$. Therefore $\&_3$ is suitable as a definition of conjunction only for those conditionals $a_1 \mid b_1$ and $a_2 \mid b_2$, such that $b_1 = b_2$. Interestingly enough, when when restricted to this class of conditionals, $\&_2$ and $\&_3$ do coincide.

It is worth noticing that the above conjunctions are defined in order to make the class $A \mid A$ of conditional objects closed under $\&_i$, and hence for every $a_1, b_1, a_2, b_2 \in A$, and for every $i = 1, 2, 3$, there exists $c, d \in A$ such that, $(a_1 \mid b_1)\&_i(a_2 \mid b_2) = (c \mid d)$. This leads us to introduce a further constraint:

Context constraint (CC): Let $a_1 \mid b_1$, $a_2 \mid b_2$ be conditionals in $A \mid A$. If $b_1 = b_2$, then the conjunction $(a_1 \mid b_1)AND(a_2 \mid b_2)$ is a conditional in the form $c \mid d$, and in that case $d = b_1 = b_2$.

The Context constraint is better understood by pointing out that, whenever the object $(a_1 \mid b_1)AND(a_2 \mid b_2)$ cannot be reduced to a conditional $c \mid d$, then necessarily $b_1 \neq b_2$

Note that each of the $\&_i$'s above satisfy the stronger requirement, denoted by (CC)', that for every $a_1 \mid b_1$, and $a_2 \mid b_2$, $(a_1 \mid b_1)AND(a_2 \mid b_2)$ is a conditional in the form $c \mid d$ (but in general $d \neq b_1$, and $d \neq b_2$). This stronger condition ensures in fact that $A \mid A$ is closed under $\&_i$, and hence makes $\&_i$ a total operator on $A \mid A$. On the other hand, as we are going to show in the next section, our construction of conditional algebra, defines a structure whose domain strictly contains all the elements $a \mid b$ for a in A, and b belonging to a particular subset of A guaranteeing the satisfaction of our Rejection constraint. This allows us to relax this condition of closure as stated above. Indeed, for every pair of conditionals of the form $a_1 \mid b_1$ and $a_2 \mid b_2$ belonging to the conditional algebra, their conjunction will always be an element of the algebra (i.e. the conjunction is a total, and not a partial, operation), but in general it will be not in the form $c \mid d$. Therefore we will provide a definition for conjunction between conditionals that satisfies (CC), but not, in general, (CC)'. Moreover our definition of conjunction behaves as $\&_2$, and $\&_3$ whenever restricted to those conditionals $(a_1 \mid b_1), (a_2 \mid b_2)$ with $b_1 = b_2$.

4 Conditional boolean algebras

We now show how a conditional boolean algebra can be built up from any boolean algebra A and a non-empty $\{\bot\}$-free subset of A, which we will call a *bunch* of A, and denote by A'.

Let A be any boolean algebra and let $A \times A'$ be the cartesian product of A and A' (as sets). We denote by

$$\mathcal{F}(A \times A') = (\mathcal{F}(A \times A'), \wedge^{\mathcal{F}}, \vee^{\mathcal{F}}, \neg^{\mathcal{F}}, \bot^{\mathcal{F}}, \top^{\mathcal{F}})$$

the boolean algebra freely generated by the pairs $(a, b) \in A \times A'$ (cf. [5, II §10]). Consider the following elements in $\mathcal{F}(A \times A')$: for every $x, z \in A$, $y, k \in A'$, and $x'' \in A$ and $z'' \in A'$ with $x'' \not\geq z''$,

(**t**1) $\delta((y, y), \top^{\mathcal{F}})$

(**t**2) $\delta((x, y) \wedge^{\mathcal{F}} (z, y), (x \wedge z), y)$

(**t**3) $\delta(\neg^{\mathcal{F}}(x, y), (\neg x, y))$

(**t**4) $\delta((x \wedge y, y), (x, y))$

(**t**5) $\neg\delta((x'', z''), (z'', z''))$

Consider the proper ideal $\mathfrak{I}((\mathbf{t}1)\text{-}(\mathbf{t}5))$ of $\mathcal{F}(A \times A')$ that is generated by the set of all the instances of the above introduced terms (**t**1)-(**t**5). To simplify the notation, we will henceforth denote $\mathfrak{I}((\mathbf{t}1)\text{-}(\mathbf{t}5))$ by $\mathfrak{C}$.

Definition 4.1. For every boolean algebra A and every bunch A' of A, we say that the quotient algebra $\mathcal{C}(A, A') = \mathcal{F}(A \times A')/\mathfrak{C}$ is the *conditional algebra* of A and A'.

We denote the generic element of $A \times A'$ by $a \mid b$ instead of (a, b). In a conditional algebra like $\mathcal{C}(A, A')$ we therefore have *atomic conditionals*[5] in the form $a \mid b$ for $a \in A$, and $b \in A'$, and also *compound conditionals* (or simply *conditionals* henceforth) being those elements in $\mathcal{C}(A, A')$ that are the algebraic terms definable in the language of boolean algebras, modulo the identification induced by $\mathfrak{C}$. For instance we say that the term $\neg^{\mathcal{F}}(a \mid b) \wedge^{\mathcal{F}} (c \mid d)$ is a *conditional* independently on the fact that we are able or not to reduce it to an *atomic conditional* in the form $x \mid y$.

Remark 4.2. (1) As we have already stated, for all $a_1 \mid b_1, a_2 \mid b_2 \in A \times A'$, their conjunction is the element $(a_1 \mid b_1) \wedge^{\mathcal{F}} (a_2 \mid b_2)$ that belongs to the conditional algebra by definition. Notice that $(a_1 \mid b_1) \wedge^{\mathcal{F}} (a_2 \mid b_2) = (c \mid d)$ iff, from Corollary 3.3, $\delta((a_1 \mid b_1) \wedge^{\mathcal{F}} (a_2 \mid b_2), (c \mid d)) \in \mathfrak{C}$. Therefore (**t2**) ensures that, if $b_1 = b_2 = d$, then $(a_1 \mid d) \wedge^{\mathcal{F}} (a_2 \mid d) = (c \mid d)$ (see also Proposition 4.5(**e**2) below). Therefore our Context constraint (CC) is satisfied. Also notice that (CC)' is not satisfied in general by the conjunction we have defined in $\mathcal{C}(A, A')$. In fact,

[5] We will henceforth use this notation when needed, and without risks of confusion.

when $b_1 \neq b_2$, we cannot ensure in general $(a_1 \mid b_1) \wedge^{\mathcal{F}} (a_2 \mid b_2)$ to be atomic, and hence in the form $(c \mid d)$. In any case $\mathcal{C}(A, A')$ is closed under $\wedge^{\mathcal{F}}$.

(2) The Rejection constraint introduced in Section 2, forces our construction to drop $\perp$ from the algebra intended to contain the antecedents of conditionals. For this reason we defined the *bunch* as a bottom-free subset of A. Notice that if we allowed the conditional algebra to represent counterfactual conditionals (equivalently, had we not imposed the Rejection constraint), the resulting algebraic structure would have not be boolean as shown in [25, 31]. In this sense, the Rejection constraint can be seen as being closely connected to the Boolean one.

Remark 4.3. Whenever the bunch A' is upward closed, the cartesian product $A \times A'$ is indeed a Popper-algebra, as defined above. So conditional algebras, are generalizations of Popper-algebras which are further are endowed with a precisely defined algebraic structure.

In what follows, unless differently specified, we will only be concerned with a conditional algebras $\mathcal{C}(A, A')$ in which A denotes a boolean algebra, and A' is a fixed bunch of A. Every conditional algebra $\mathcal{C}(A, A')$ is a quotient of a free boolean algebra, whence is boolean. So our Boolean constraint is satisfied.

The operations on $\mathcal{C}(A, A')$ are denoted using the following notation, which is to be interpreted in the obvious way:

$$\mathcal{C}(A, A') = (\mathcal{C}(A, A'), \cap_{\mathfrak{C}}, \cup_{\mathfrak{C}}, \neg_{\mathfrak{C}}, \perp_{\mathfrak{C}}, \top_{\mathfrak{C}}).$$

The construction of $\mathcal{C}(A, A')$, and in particular the role of the ideal $\mathfrak{C}$, is best illustrated by means of an example.

Example 4.4. Let A be the four elements boolean algebra $\{\top, a, \neg a, \perp\}$, and consider the bunch $A' = A \setminus \{\perp\}$. Then $A \times A' = \{(\top, \top), (\top, a), (\top \neg a), (a, \top), (a, a), (a, \neg a), (\neg a, \top), (\neg a, a), (\neg a, \neg a), (\perp, \top), (\perp, a), (\perp, \neg a)\}$. The cartesian product $A \times A'$ has cardinality 12, whence $\mathcal{F}(A \times A')$ is the free boolean algebra of cardinality $2^{2^{12}}$, i.e. the finite boolean algebra of 2^{12} atoms.

The conditional algebra $\mathcal{C}(A, A')$ is then obtained as the quotient of $\mathcal{F}(A \times A')$ by the ideal $\mathfrak{C}$ generated by (**t**1)-(**t**5). Having in mind Corollary 3.3, we can easily see that the ideal $\mathfrak{C}$ of $\mathcal{F}(A \times A')$ specifically *forces* the free algebra $\mathcal{F}(A \times A')$ about which elements are equal as conditionals. For instance, following Proposition 4.5, in $\mathcal{C}(A, A')$ the following equations hold: $\top \mid \top = a \mid a = (\neg a) \mid (\neg a)$; $(\top \mid \top) \cap_{\mathfrak{C}} (a \mid \top) = (\top \wedge a) \mid \top = (a \mid \top)$; $\neg_{\mathfrak{C}}(\top \mid \top) = \perp \mid \top$, $\neg_{\mathfrak{C}}(a \mid \neg a) = (\neg a) \mid (\neg a) = \top \mid \top$, and so forth.

The following Proposition collects some basic results on conditional algebras.

Proposition 4.5. *Every conditional algebra $\mathcal{C}(A, A')$ satisfies the following equations:*

(**e**1) *For all $y \in A'$, $y \mid y = \top_{\mathfrak{C}}$*

(**e**2) *For all $x, z \in A$, and $y \in A'$, $(x \mid y) \cap_{\mathfrak{C}} (z \mid y) = (x \wedge z) \mid y$*

(**e**3) *For all* $x \in A$*, and* $y \in A'$*,* $\neg_{\mathfrak{C}}(x \mid y) = (\neg x \mid y)$

(**e**4) *For all* $x \in A$*, for all* $y \in A'$*,* $(x \wedge y \mid y) = (x \mid y)$

(**e**5) *For all* $x \in A$*, and* $y \in A'$*, if* $x \not\geq y$*, then* $(x \mid y) \neq (y \mid y)$*, and in particular* $(x \mid y) < (y \mid y)$*.*

(**e**6) *For all* $x, y \in A$*, if* $(x \mid \top) = (y \mid \top)$*, then* $x = y$

(**e**7) *For all* $y \in A'$*,* $\neg y \mid y = \bot_{\mathfrak{C}}$

(**e**8) *For all* $x, z \in A$*, and* $y \in A'$*,* $(x \mid y) \cup_{\mathfrak{C}} (z \mid y) = (x \vee z \mid y)$

Proof. The properties from (**e**1), to (**e**5) directly follow from the above (**t**1) − (**t**5), and Corollary 3.3. Moreover notice that in (**e**5), if $(x \mid y) \neq (y \mid y)$, then necessarily $(x \mid y) < (y \mid y)$ from (**e**1). (**e**6) follows from (**e**5), in fact: assume $x \mid \top = y \mid \top$. Then

(1) $(x \mid \top) \vee (\neg y \mid \top) = (y \mid \top) \vee (\neg y \mid \top) = (y \vee \neg y \mid \top) = \top \mid \top$. Therefore $(\neg y \vee x \mid \top) = \top \mid \top$, and hence from (**e**5), $y \to x = \top$, that is $y \leq x$.

(2) Dually $(y \mid \top) \vee (\neg x \mid \top) = (x \mid \top) \vee (\neg x \mid \top) = \top \mid \top$. As above we get $x \leq y$.

Then $x = y$ and we are done.

Finally (**e**7) easily follows from (**e**1) and (**e**3), and (**e**8) is a trivial consequence of De Morgan laws. □

In any conditional algebra $\mathcal{C}(A, A')$ one can define the order relation $\leq$ by the letting

$$(x \mid y) \leq (z \mid k) \text{ iff } y = k, \text{ and } (x \mid y) \cap_{\mathfrak{C}} (z \mid k) = (x \mid y). \tag{4}$$

In other words $(x \mid y) \leq (z \mid k)$ iff $y = k$ and $(x \wedge z \mid y) = (x \mid y)$.

The following Proposition collects the key properties on the order of a conditional algebra.

Proposition 4.6. *In every conditional algebra* $\mathcal{C}(A, A')$ *the following hold:*

(**o**1) *For every* $x, y \in A$*, and for every* $z \in A'$*,* $(x \mid y) \leq (z \mid z)$*; moreover* $(x \mid z) \geq (z \mid z)$*, implies* $x \geq z$

(**o**2) *For every* $x, y \in A$*, and for every* $z \in A'$*, if* $x \leq y$*, then* $(x \mid z) \leq (y \mid z)$ *(where clearly* $x \leq y$ *means with respect to* A*). In particular* $x \leq y$ *iff* $(x \mid \top) \leq (y \mid \top)$

(**o**3) *For every* $x, y \in A$*, and for every* $z \in A'$*, if* $(x \mid z) \neq (y \mid z)$*, then* $x \neq y$*. In particular* $x \neq y$ *iff* $(x \mid \top) \neq (y \mid \top)$*.*

(**o**4) *For every* $x \in A'$*,* $(\top \mid x) = (x \mid x) = \top_{\mathfrak{C}}$*, and* $(\bot \mid x) = (\neg x \mid x) = \bot_{\mathfrak{C}}$

(**o**5) *For every* $x, y \in A$*, and for every* $z, k \in A'$*,* $(x \mid y) \cap_{\mathfrak{C}} (z \mid k) = (x \mid z)$ *iff* $y = k$*, and* $(x \mid z) \cup_{\mathfrak{C}} (y \mid k) = (y \mid k)$

Proof. The first part of (**o**1) follows from Proposition 4.5 (**e**1), while, if $x \mid z \geq z \mid z$, then $(x \mid z) \cap_{\mathfrak{U}} (z \mid z) = (z \mid z)$, and hence (**e**4) implies $(z \mid z) = (x \wedge z \mid z) = x \mid z$. Then from (**e**5), $x \geq z$.

If $x \leq y$, then $x \wedge y = x$, and hence $(x \wedge y \mid z) = (x \mid z)$. Therefore by Proposition 4.5 (**e**2), $(x \mid z) \cap_{\mathfrak{C}} (y \mid z) = (x \mid z)$, and $(x \mid z) \leq (y \mid z)$. Moreover if $(x \mid \top) \leq (y \mid \top)$, then easily $(x \to y \mid \top) = \top_{\mathfrak{C}} = (\top \mid \top)$. From (**e**6), $x \to y = \top$, and hence $x \leq y$ and (**o**2) holds.

(**o**3) directly follows from (**o**2), and since for every $x \in A'$, $(x \mid x) = \top_{\mathfrak{C}}$, $(\top \mid x) \leq (x \mid x)$, and from (**o**2), $x \leq \top$ implies $(x \mid x) \leq (\top \mid x)$. Moreover from Proposition 4.5 (**e**3) and (**e**7), $(\bot \mid x) = (\neg\top \mid x) = \neg_{\mathfrak{C}}(\top \mid x) = \neg_{\mathfrak{C}}(x \mid x) = (\neg x \mid x) = \bot_{\mathfrak{C}}$ and hence (**o**4) holds.

Finally (**o**5) follows from (**e**8) and the negation order-reversing property.

□

We now end this section with an easy remark about the atoms of a conditional algebra. The facts that we are going to show derive from the above proved Proposition 4.6.

Remark 4.7. As we have already observed in the Example 4.4, every conditional algebra $\mathcal{C}(A, A')$ is finite whenever A is finite. So, if A is finite, $\mathcal{C}(A, A')$ is atomic. Moreover, since the canonical homomorphism $h_{\mathfrak{C}} : \mathcal{F}(A \times A') \to \mathcal{C}(A, A')$ is onto, we conclude that

$$2^{2^{|A \times A'|}} = |\mathcal{F}(A \times A')| \geq |\mathcal{C}(A, A')|.$$

So, let A be a finite Boolean algebra, let $\mathcal{AT}(A)$ be the class of its atoms, and let A' be a bunch of A. It is clear that for every $a \in \mathcal{AT}(A)$, and for every $b \in A'$ with $a \neq b$, $a \mid b$ is an atom of $\mathcal{C}(A, A')$. In fact, assume that $a \mid b$ is not an atom of $\mathcal{C}(A, A')$, then let $x \in \mathcal{C}(A, A')$ such that $\bot_{\mathfrak{C}} < x < a \mid b$. Recalling how we defined the order in (4), this means that there exists $c \in A$ such that $c \neq \bot$, $x = c \mid b$, and

$$(a \mid b) \cap_{\mathfrak{C}} (c \mid b) = (a \wedge c) \mid b = c \mid b.$$

That is $\bot < c < a$, and hence a is not an atom of A which is a contradiction.

4.1 Comparison with Goodman-Nguyen-van Fraassen algebra

As briefly anticipated in the Introduction, Goodman, Nguyen and van Frassen (see [3, §2.3]) have put forward a characterization of Conditional Event Algebras which is to some extent comparable to conditional algebras introduced in this paper, a comparison to which we now turn.

Let A be a boolean algebra of subsets of a set W. A Goodman-Nguyen-van Frassen algebra is a boolean algebra A^* that includes A, and whose elements are the conditional events that can be defined from A. This is entirely analogous to conditional algebra defined in Section 4. A^* is defined as follows. Let W^* be the infinite product $W \times W \times \ldots$, and then let A^* be the σ-algebra generated by the infinite product of A. Probability measures can be defined on A^* by letting, for every probability measure P on A, P^* be the infinite product measure generated by P on the space (W^*, A^*).

For every $x, y \in A$ define the *conditional* $y \mid x$, as an element of A^*, by the following stipulation:

$$\bigvee_{k \geq 0} (\neg x)^k \times (x \wedge y) \times \top \times \top \ldots, \tag{5}$$

where of course, for every k, $(\neg x)^k$ is a shorthand for $\underbrace{\neg x \times \ldots \times \neg x}_{k\text{-times}}$. Therefore, any $w^* = (w_1, w_2, \ldots) \in W^*$ is such that $w^* \in y \mid x$ iff:

(i) there exists an index $i \geq 1$ such that $w_i \in x$, and

(ii) letting j the smallest index such that $w_j \in x$, then $w_j \in y$.

Then the set of all conditionals of the form $x \mid \top$ is a subalgebra of A^* which is isomorphic to A. Moreover, for every probability measure P on (W, A), $P^*(x \mid \top) = P(x)$, and if P is such that $P(x) > 0$, then

$$P^*(y \mid x) = P(y \wedge x)/P(x). \tag{6}$$

Summing up, both our approach and the one followed by Goodman, Nguyen and van Frassen aim at the introduction of a formal structure to study conditional measures (in their case only probability) as measures of conditionals. To this end we both require the underlying algebraic structure to be boolean as we anticipated in Remark 1.1 above. On the other hand, and this marks a first technical difference, whenever the starting boolean algebra A is finite, $\mathcal{C}(A, A')$ is always finite (in fact, its cardinality is bounded above by $2^{2^{|A \times A'|}}$, remind Remark 4.7), while A^* is infinite (and indeed each conditional in A^* as an infinite expression, see (5)) above.

A further important difference concerns the definition of conditional objects, which, as we stressed in Section 4, in A^* are always *atomic*. In fact, although in our approach the algebra $\mathcal{C}(A, A')$ itself defines the conditionals and characterizes their properties at the level of its formal algebraic definition, in A^* conditionals are identified with those (particular) elements that can be expressed via (5) above. This difference owes mainly to the fact that the two approaches really capture distinct intuitions about conditional probability. A probability on a Goodman-Nguyen-van Frassen algebra *is* a conditional probability because it satisfies (6). As we will illustrate in Section 5 below, we define conditional probability axiomatically (i.e. the primitive approach to conditional probability mentioned in Section 3.1 above).

Finally, it must be emphatised that whilst the Goodman-Nguyen-van Frassen construction only captures the specific conditional measure provided by conditional probability (and indeed the construction of A^* is driven by this intuition), our approach, being motivated by the understanding of conditionals in general, allows to investigate various kinds of conditional measures, of which conditional probability is a special case. This level of mathematical generality will allow us to capture, in Section 5 below, the distiction between the properties of the conditional and those of the conditional measure, which we anticipated in the introduction.

4.2 Conditionals and deduction

A common way to look at conditional assertions consists in interpreting a conditional $\theta \mid \varphi$ in terms of the classical material implication $\varphi \to \theta$. Modulo the Deduction theorem of classical logic, this interpretation actually allows us to relate (object level) conditionals to (meta level) deduction by noting that $\theta \mid \varphi$ holds iff $\varphi \to \theta$ holds, iff $\varphi \vdash \theta$.

In this section we are going to show that this intuition can be formally represented in our setting, as a very particular case. In what follows, and without loss of generality, we are going to restrict our attention to those conditionals expressions which are defined from the elements of (a finitely generated) Lindenbaum-Tarski algebra $\mathcal{L}$ of classical propositional calculus, and particularly suited bunches of $\mathcal{L}$.

For every $[\varphi] \in \mathcal{L} \setminus \{[\bot]\}$, consider the principal filter $\uparrow[\varphi]$ of $\mathcal{L}$ generated by $[\varphi]$. Since $[\bot] \notin \uparrow[\varphi]$, $\uparrow[\varphi]$ is a bunch of $\mathcal{L}$. Consider then the conditional algebra $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$.

Also consider the subset of $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$, defined by $\mathcal{L} \mid \{[\varphi]\} = \{[\theta] \mid [\varphi] : [\theta] \in \mathcal{L}\}$. First of all we prove the following easy result.

Proposition 4.8. *$\mathcal{L} \mid \{[\varphi]\}$ is the domain of a boolean subalgebra of $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$. Moreover $\mathcal{C}(\mathcal{L}, \uparrow[\top])$ is isomorphic to $\mathcal{L}$*

Proof. Since $[\varphi]$ is fixed in $\mathcal{L} \setminus \{[\bot]\}$, the operations of $\mathcal{L}$ define the operations between the elements in $\mathcal{L} \mid \{[\varphi]\}$. In other words, the operations on $\mathcal{L} \mid \{[\varphi]\}$ are the restriction of the operations on $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$. Notice that the top and the bottom elements of $\mathcal{L} \mid \{[\varphi]\}$ respectively coincide with $[\varphi] \mid [\varphi]$ and $[\neg\varphi] \mid [\varphi]$.

Consider the map $f : \mathcal{C}(\mathcal{L}, \uparrow[\top]) \to \mathcal{L}$ associating $[\varphi] \mid [\top] \mapsto [\varphi]$. For every $[\varphi], [\psi] \in \mathcal{L}$, such that $[\varphi] \neq [\psi]$, then $[\varphi] \mid [\top] \neq [\psi] \mid [\top]$ from Proposition 4.6, (**o**3), and hence f is well defined. Moreover it is clear that f is a homomorphism because, for instance,

$$\begin{aligned} f(([\varphi] \mid [\top]) \sqcap_{\mathfrak{C}} ([\psi] \mid \top)) &= f([\varphi] \wedge [\psi] \mid [\top]) \\ &= f([\varphi \wedge \psi] \mid [\top]) \\ &= [\varphi \wedge \psi] \\ &= [\varphi] \wedge [\psi] \\ &= f([\varphi] \mid [\top]) \wedge f([\psi] \mid [\top]) \end{aligned}$$

f is clearly surjective, and f is injective again from Proposition 4.6 (**o**3). In fact we proved in Proposition 4.6 (**o**3) that, if $([\varphi_1] \mid [\top]) \neq ([\varphi_2] \mid [\top])$, then $[\varphi_1] \neq [\varphi_2]$ and hence f is injective. $\square$

Remark 4.9. Notice that the above Proposition 4.8 can be proved in the more general framework of any conditional algebras $\mathcal{C}(A, A')$ defined by a generic bunch A' of A. In fact the following holds: for every $b \in A'$ the set $A \mid \{b\}$ is the domain of a boolean subalgebra of $\mathcal{C}(A, \uparrow b)$.

Since the bunch $\uparrow[\varphi]$ is a principal filter of $\mathcal{L}$, it makes sense to consider the quotient algebra $\mathcal{L}/\uparrow[\varphi]$. Denote by $[\theta]/\uparrow[\varphi]$ the generic element of $\mathcal{L}/\uparrow[\varphi]$.

Theorem 4.10. *For every $[\varphi] \in \mathcal{L} \setminus \{[\bot]\}$, $\mathcal{L}/\uparrow[\varphi]$ is isomorphic to a boolean subalgebra of $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$. In particular $\mathcal{L}/\uparrow[\top]$ is isomorphic to $\mathcal{L}$.*

Proof. Consider the map $F : \mathcal{L}/\uparrow[\varphi] \to \mathcal{C}(\mathcal{L}, \uparrow[\varphi])$ so defined: for every $[\theta]/\uparrow[\varphi] \in \mathcal{L}/\uparrow[\varphi]$,

$$F\left([\theta]/\uparrow[\varphi]\right) = [\theta] \mid [\varphi]. \tag{7}$$

Let $[\theta] \mid [\varphi] \neq [\theta'] \mid [\varphi]$, and hence $[\theta] \leftrightarrow [\theta'] \mid [\varphi] \neq \top_{\mathfrak{C}} = [\varphi] \mid [\varphi]$. In particular $[\theta \leftrightarrow \theta'] \mid [\varphi] \not\geq [\varphi] \mid [\varphi]$, and hence $[\theta \leftrightarrow \theta'] \not\geq [\varphi]$ because of Proposition 4.6 (**o**1). Therefore $[\theta \leftrightarrow \theta'] \notin \uparrow [\varphi]$, whence $[\theta \leftrightarrow \theta']/\uparrow[\varphi] \neq [\top]/\uparrow[\varphi]$, and hence $[\theta]/\uparrow[\varphi] \neq [\theta']/\uparrow[\varphi]$, i.e. F is well defined.

By definition of $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$, F is a homomorphism, in fact

$$\begin{array}{rcl} F([\theta]/\uparrow[\varphi] \wedge [\theta']/\uparrow[\varphi]) & = & F([\theta \wedge \theta']/\uparrow[\varphi]) \\ & = & [\theta \wedge \theta'] \mid [\varphi] \\ & = & ([\theta] \wedge [\theta']) \mid [\varphi] \\ & = & ([\theta] \mid [\varphi]) \cap_{\mathfrak{C}} ([\theta'] \mid [\varphi]) \\ & = & F\left([\theta]/\uparrow[\varphi]\right) \cap_{\mathfrak{C}} F\left([\theta']/\uparrow[\varphi]\right). \end{array}$$

The case of $\neg$ is analogous and omitted.

Finally F is obviously injective. In fact $F\left([\theta]/\uparrow[\varphi]\right) = \top_{\mathfrak{C}}$ iff $F\left([\theta]/\uparrow[\varphi]\right) = [\varphi] \mid [\varphi]$ iff $[\theta] \mid [\varphi] = [\varphi] \mid [\varphi]$. Proposition 4.6 (**o**1) then implies that $[\theta] \geq [\varphi]$, and hence $[\theta] \in \uparrow[\varphi]$ iff $[\theta]/\uparrow[\varphi] = \top$. Therefore F is an embedding of $\mathcal{L}/\uparrow[\varphi]$ into $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$, and the first part of our claim is settled. Notice that the map F defined as in (7) is not surjective in general. In fact let $[\gamma] > [\varphi]$. Then for every $[\theta] \in \mathcal{L}$, $[\theta] \mid [\gamma] \in \mathcal{C}(\mathcal{L}, \uparrow[\varphi])$, but for every $[\theta'] \in \mathcal{L}$,

$$F([\theta']/\uparrow [\varphi]) \neq [\theta] \mid [\gamma].$$

Conversely, in the particular case of $[\varphi] = [\top]$, $\mathcal{L}/\uparrow[\top]$ is isomorphic to $\mathcal{L}$, because, in this case F is also surjective. In fact vvfor every $[\varphi] \in \mathcal{L}$, clearly $F([\varphi]/\uparrow[\top]) = [\varphi] \mid [\top]$. Therefore the claim follows from Proposition 4.8. $\square$

Notice that, for every $[\varphi] \in \mathcal{L} \setminus \{[\bot]\}$, $\mathcal{L}/\uparrow [\varphi]$ is in fact isomorphic to the boolean subalgebra $\mathcal{L} \mid \{[\varphi]\}$ of $\mathcal{C}(\mathcal{L}, \uparrow [\varphi])$ we studied in Proposition 4.8. The following result clearly shows that conditional algebras allows to interpret conditional assertions by the classical deduction operator via the paradigm we commented at the beginning of this section. In what follows, for every pair of formula φ, θ, we say that $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$ *satisfies* $\theta \mid \varphi$ iff $[\theta] \mid [\varphi] = \top_{\mathfrak{C}}$.

Corollary 4.11. *For every propositional formulas θ and φ, $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$ satisfies $\theta \mid \varphi$ iff $\varphi \vdash \theta$.*

Proof. Since $[\theta] \mid [\varphi] \in \mathcal{L} \mid \{[\varphi]\}$, from Theorem 4.10, $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$ satisfies $\theta \mid \varphi$ iff $[\theta] \mid [\varphi] = \top_{\mathfrak{C}}$ iff $\mathcal{L}/\uparrow [\varphi]$ satisfies $F^{-1}([\theta] \mid [\varphi]) = F^{-1}(\top_{\mathfrak{C}}) = \top$ iff $[\theta]/\uparrow[\varphi] = \top$ iff $[\theta] \geq [\varphi]$ iff $[\varphi \to \theta] = [\top]$ iff $\varphi \vdash \theta$. $\square$

It is worth noticing that conditional algebra $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$, has a limited expressive power, and the atomic conditional object that are here definable all the form $[\psi] \mid [\gamma]$ for $[\psi] \in \mathcal{L}$, but $[\gamma]$ is required to satisfy $[\gamma] \geq [\varphi]$. In other words we are dealing, in $\mathcal{C}(\mathcal{L}, \uparrow[\varphi])$ with those atomic conditionals that can express deductions from the fixed theory $[\varphi]$. Moreover those conditional algebras which allow us to interpret conditionals as deductions, are particular cases of our general construction. This means that our construction of conditional algebras defines structures that are immune to the complications which arise when conditionals are defined via material implication, chief among them the well-known Triviality result proved by David Lewis [20]

5 Measures on conditional algebras

By construction, whenever we fix a boolean algebra A and a bunch A' of A, the conditional algebra $\mathcal{C}(A, A')$ built up over A and A' is a boolean algebra, and therefore it makes sense to study simple (i.e. unconditional) belief measures on it. In this section we are going to analyze which properties of conditional objects are satisfied by various conditional measures. Following [17] we will analyze the cases of probability, possibility, plausibility measures.

Let us start with probability measures, and hence let $\mu : \mathcal{C}(A, A') \to [0, 1]$ be a finitely additive probability on $\mathcal{C}(A, A')$.

Proposition 5.1. *Let μ be a probability function on a conditional algebra $\mathcal{C}(A, A')$ defined by A and its bunch A'. Then the following hold:*

(μ1) *for all $y \in A'$, $\mu(y \mid y) = 1$,*

(μ2) *for every $(x_1 \mid y), (x_2 \mid y) \in \mathcal{C}(A, A')$ with $x_1 \wedge x_2 = \bot$, then $\mu((x_1 \vee x_2) \mid y) = \mu(x_1 \mid y) + \mu(x_2 \mid y)$,*

(μ3) *$\mu(x \wedge y \mid y) = \mu(x \mid y)$.*

Proof. The properties (μ1) and (μ3) respectively follow from Proposition 4.5 (**e**1), (**e**4) together, with the normalization property for probability measures: $\mu(\top) = 1$. In order to show (μ2), notice that whenever $x_1 \wedge x_2 = \bot$, then from Proposition 4.5 (**e**2), for every $y \in A'$, $(x_1 \mid y) \cap_{\mathfrak{C}} (x_2 \mid y) = (x_1 \wedge x_2 \mid y) = (\bot \mid y) = \bot_{\mathfrak{C}}$. Therefore, since μ is additive,

$$\mu((x_1 \mid y) \cup_{\mathfrak{C}} (x_2 \mid y)) = \mu(x_1 \mid y) + \mu(x_2 \mid y).$$

Therefore the claim follows because by Proposition 4.5 (**e**8), $(x_1 \mid y) \cup_{\mathfrak{C}} (x_2 \mid y) = (x_1 \vee x_2) \mid y$. □

The above Proposition 5.1 shows that a simple probability μ on a conditional algebra $\mathcal{C}(A, A')$ satisfies almost all the properties of a conditional probability with the exception of the following.

(CP) For all $(x \mid z), (x \mid y), (y \mid z) \in \mathcal{C}(A, A')$ with $x \leq y \leq z$,

$$\mu(x \mid z) = \mu(x \mid y) \cdot \mu(y \mid z).$$

The following shows that there exist examples of probability measures on conditional algebras that actually do not satisfy (CP).

Example 5.2. Let W be the set $\{a, b, c\}$, let $A = 2^W$, and let A' be its bunch $\uparrow\{b\}$. Then consider the atoms (recall Remark 4.7) $\{b\} \mid \{a, b, c\}$, $\{b\} \mid \{b, c\}$, and $\{c\} \mid \{a, b, c\}$ of $\mathcal{C}(A, A')$ and let μ be the probability measure on on $\mathcal{C}(A, A')$ given by the distribution that assign $\{b\} \mid \{a, b, c\} \mapsto 1/7$; $\{b\} \mid \{b, c\} \mapsto 3/7$; $\{c\} \mid \{a, b, c\} \mapsto 2/7$. Then we have: $\mu(\{b\} \mid \{a, b, c\}) = 1/7$; $\mu(\{b\} \mid \{b, c\}) = 3/7$; $\mu(\{b, c\} \mid \{a, b, c\}) = \mu((\{b\} \vee \{c\}) \mid \{a, b, c\}) = \mu((\{b\} \mid \{a, b, c\}) \vee (\{c\} \mid \{a, b, c\})) = \mu(\{b\} \mid \{a, b, c\}) + \mu(\{c\} \mid \{a, b, c\}) = 3/7$.

Therefore, letting $x = \{b\}$, $y = \{b, c\}$ and $z = \{a, b, c\}$ we get a counterexample to (CP). In fact $\mu(\{b\} \mid \{a, b, c\}) = 1/7$, and $\mu(\{b\} \mid \{b, c\}) \cdot \mu(\{b, c\} \mid \{a, b, c\}) = 3/7 \cdot 3/7 = 9/49$.

It can be shown (see [17, §3.2]) that under $(\mu 1)$, $(\mu 3)$ and (CP) are the same as the more familiar general version of Bayes Theorem: $\mu(x \wedge y \mid z) = \mu(x \mid y \wedge z) \cdot \mu(y \mid z)$.

This simple remark leads to the following definition:

Definition 5.3. Let A be a boolean algebra, and let A' be a bunch of A. Then let $\mathcal{C}(A, A')$ be the conditional algebra built up over A and A'. A finitely additive probability measure $\mu : \mathcal{C}(A, A') \to [0, 1]$ is said to be a *Bayes probability* if μ further satisfies (CP).

The following characterization is immediate from Proposition 5.1, and Definition 5.3, whence the proof is omitted.

Theorem 5.4. *Let A be a boolean algebra, and let A' be any bunch of A. Then the following are equivalent:*

(i) $\mu : \mathcal{C}(A, A') \to [0, 1]$ *is a Bayes measure.*

(ii) μ *is a conditional probability measure on* A.

Simple measures on conditional algebras fail to satisfy all the properties of a conditional measure also in the case of *possibility* measures. Similarly to what happens with probability, conditional possibility measures must satisfy a further axiom which captures the essential features of Bayes Theorem in the context of idempotent measures.

Proposition 5.5. *Let* Π *be a possibility measure on the conditional algebra* $\mathcal{C}(A, A')$ *defined by* A *and its bunch* A'. *Then* Π *satisfies the following properties:*

$(\Pi 1)$ $\Pi(\perp \mid y) = 0$, *and* $\Pi(y \mid y) = 1$ *for all* $y \in A'$,

$(\Pi 2)$ $\Pi(x_1 \vee x_2 \mid y) = \max\{\Pi(x_1 \mid y), \Pi(x_2 \mid y)\}$.

Proof. The property (Π1) is easy and almost directly follows from Proposition 4.5 (**e**1), and observing that $(\bot \mid y) = \bot_{\mathfrak{C}}$ for every $y \in A'$. Also (Π2) is easy. In fact by Proposition 4.5 it easily follows that $(x_1 \vee x_2 \mid y) = (x_1 \mid y) \cup_{\mathfrak{C}} (x_2 \mid y)$, and hence

$$\Pi(x_1 \vee x_1 \mid y) = \Pi((x_1 \mid y) \cup_{\mathfrak{C}} (x_2 \mid y)) = \max\{\Pi(x_1 \mid y), \Pi(x_2 \mid y)\}.$$

Therefore the claim is settled. □

In analogy with probability measures, we call *Bayes* any possibility measure on $\mathcal{C}(A, A')$ satisfying

($C\Pi$) $\Pi(x \wedge y \mid z) = \min\{\Pi(x \mid y \wedge z), \Pi(y \mid z)\}$.

The following is easily proved.

Theorem 5.6. *A possibility measure* Π *on* $\mathcal{C}(A, A')$ *is a Bayes possibility iff* Π *is a conditional possibility measure on* A.

As recalled in Section 3 above, plausibility measures capture the fundamental aspects which are shared by all belief measures. It should not come as a surprise then the fact that conditiona plausibility measures, unlike probability and possibility, do behave as simple measures on conditional algebras, as our next result shows.

Theorem 5.7. *Let* $\mathcal{C}(A, A')$ *be the conditional algebra built up over the boolean algebra* A *and its bunch* A'. *Then a map* $Pl : \mathcal{C}(A, A') \to [0, 1]$ *is a plausibility measure iff* Pl *is a conditional plausibility measure on* A.

Proof. The right-to-left direction is trivial. Therefore consider the following: for every $y \in A'$, $Pl(y \mid y) = Pl(\top_{\mathfrak{C}}) = 1$, and analogously to the proof of Proposition 5.5, it is easy to see that $Pl(\bot \mid y) = 0$. Moreover Proposition 4.5 (**e**4) implies that $Pl(x \wedge y \mid y) = Pl(x \mid y)$. Finally, let $x_1, x_2 \in A$ and $x_1 \leq x_2$. Then $x_1 \wedge x_2 = x_1$, and therefore $(x_1 \mid y) \cap_{\mathfrak{C}} (x_2 \mid y) = (x_1 \wedge x_2 \mid y) = (x_1 \mid y)$, whence $(x_1 \mid y) \leq (x_2 \mid y)$. Hence $Pl(x_1 \mid y) \leq Pl(x_2 \mid y)$ as required. □

6 Conclusions and Further work

Bayes Theorem is a very natural property of conditional probability, so natural that, as de Finetti puts it, "theorem" is quite an overstatement for it, since it is an immediate algebraic consequence of the (ratio) definition of conditional probability (see, e.g. chapter 4 in [10]).

As Theorem 5.4 clearly shows, Bayes Theorem is a property of the (conditional) measure, i.e. probability, and not, as one might be led to think, of the conditional *tout court*. This point of view can be intuitively reaffirmed by referring to Cox's Theorem, a result to the effect that any conditional measure of belief satisfying (what is necessary to prove)

Bayes Theorem is in fact isomorphic to a (scaled) conditional probability. What Cox's result shows[6], in this respect, is the fact that conditional belief measures must be equipped with sufficient properties in order to be adequate for Bayes Theorem, and those properties essentially amount to being a probability measure. The results of this paper can therefore be seen as leading to a very similar conclusion albeit from a very different point of view.

In general, the results reported in this paper constitute a first step towards providing a rather flexible framework for conditionals which builds on the distinction between the properties of a conditional and those of a conditional measure. We have investigated our problem in the limited cases of boolean, indicative conditionals. Our next step will involve relaxing the boolean constraint, a relaxation which as we now briefly outline, implies a substantial generalization of the Rejection constraint as well. It is expected that this will have a significant impact on our understanding of conditional *many-valued* probability, a topic to which considerable research effort has been devoted for the past decade or so (see, e.g. Gerla's [36] and Kroupa's [41, 42] and the more recent characterizations by Mundici [53, 54] and Montagna [50]).

The framework introduced in this paper promises to be extremely helpful in providing an alternative, much more flexible, characterization of many-valued conditional probability. By constructing a suitable conditional algebra we can restate the characterization of *conditional probability* in terms of the *unconditional probability on conditional events*. The key advantage in taking this perspective lies in the flexibility of the resulting framework to accommodate a variety of conditionals which, by their very nature, can hardly be accounted for in a framework which overlooks the distinct interpretations which can be meaningfully associated to conditionals. Moreover, taking conditional probabilities as unconditional probabilities over conditional algebras will allow us to "move" the conditional operation outside the scope of the probability function. As we pointed out at the end of Section 4.2, this will make the framework immune to the unpleasant consequences of taking the conditioning operation as an object-level connective, chiefly Lewis's Triviality result.

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