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Fractals

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ABSTRACT: The idea of a fractal as a mathematical object and as a model of natural phenomena is introduced by way of simple examples. The characterization of fractals both mathematically and experimentally is then considered. Finally a model exemplifying the generation of fractal geometries in nature is discussed.

1 INTRODUCTION

The word "fractal" was coined by Mandelbrot (see Mandelbrot 1982), an act which has subsequently given voice to many. The present talk is intended to motivate interest in, rather than to review exhaustively, this large area of research.

Let us motivate the motivation by looking at a classic piece of work by Lovejoy (1982) on the morphology of equatorial cloud formations. Fig. 1 (taken from this reference) summarizes the results of studies of cloud images obtained using satellite-born infra-red and radar techniques. The images were divided into resolution cells, or "pixels", whose size was determined by the resolution limit of the imaging technique employed. The area, A, and the perimeter, P, of each cloud image were then estimated by counting, respectively, the number of pixels covering the whole image and the boundary of the image. The log-log plot of A versus P clearly indicates a power law relationship. This is not of itself surprising, since one expects generally for closed curves drawn on a surface that $P \sim \sqrt{A}$. The surprise is that this is not the power law implied by Fig. 1. For cloud images one finds $P \sim \sqrt{A^{d_f}}$ where $d_f \sim 1.35$.

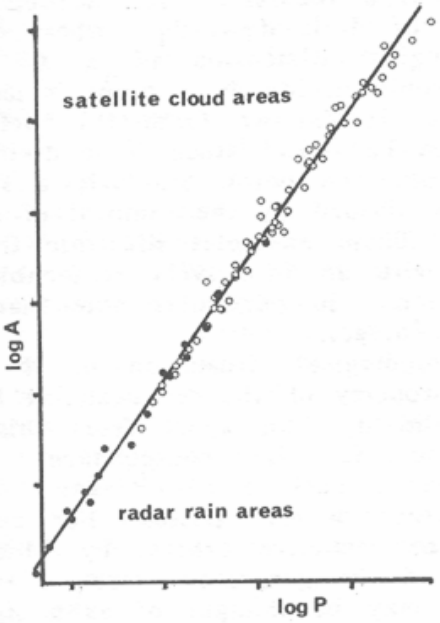

Fig. 1
The perimeter-area scaling of cloud images

The conclusion to be drawn from this work is that there is something unusual about the perimeter of clouds. Fig. 2 shows another shape, a Koch island as illustrated in Mandelbrot (1982), which, although a manifestly poor model of a cloud, does have again an unusual perimeter. This comment can be made more precise by considering the relationship between the area and the perimeter of the Koch island. Consider the recursive generation process indicated in Fig. 2. Initially one begins with a square of side L and rearranges its perimeter as shown. The resulting figure, which can be thought of as being constructed of squares of side L/6, has the same area; its perimeter, however, has increased. The recursion is to treat the unshared edges of the smaller squares in the same way as the edges of the original large square. One now assumes that the observation of the resulting object has limited resolution. For simplicity let us say that the smallest resolvable length is $\delta l = L/6^n$, where n is a

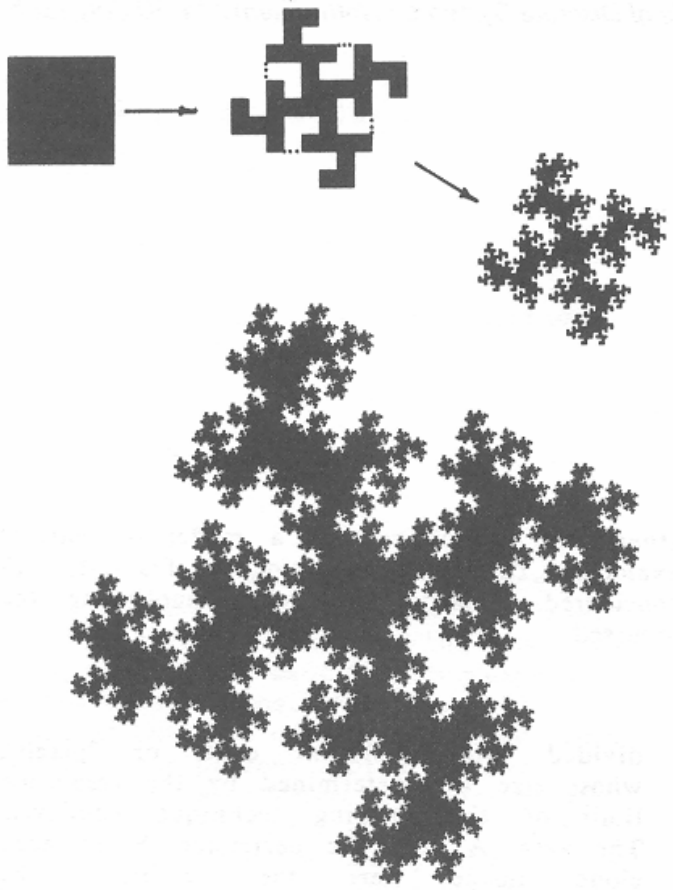

Fig. 2
A Koch island

positive integer. In this case the Koch island is indistinguishable from the nth iterate of the generation procedure, where $n=\ln(L/\delta l)/\ln 6$. In particular, P, appears to consist of 18^n pixels and is estimated to be $P=18^n \delta l = \delta l(L/\delta l)^{\ln 18/\ln 6}$. Since $A=L^2$, one obtains a scaling law analogous to that found in Fig. 1: $P \sim /A^{d_f}$, where $d_f=\ln 18/\ln 6 \simeq 1.61$.

Despite the manifest differences, therefore, both the Koch island and the clouds in Lovejoy's study have perimeters which appear to be too large for the area that they enclose - one is reminded of the skin of a dried fruit such as a prune. In consequence, rather than appearing as a clear boundary between inside and outside the perimeter is strongly folded to the extent that it appears to form a diffuse "boundary layer". The following section will contain useful generalisations of the concept of dimension which can be used to characterize this kind of property. It will be shown that the quantity d_f used above may be thought of as the "dimension" of the perimeter. Intuitively one thinks of a boundary curve as being a 1-dimensional object. The fact that, for both the Koch island and the cloud images, $d_f>1$ is interpreted as being indicative of the tendency of their perimeters to fill a 2-dimensional region.An object for which a generalised dimension such as d_f exceeds the intuitive dimension (defined precisely in the next section) is termed a fractal. It will emerge that this property results from the fractal having structure on all length scales. As such fractals make good models for many naturally occurring objects - cloud formations, for example. Fractal models have several nice features - in particular, as the calculation of d_f for the Koch island shows, they have symmetry. Thus, in the same way that a perfect lattice can be used as a basic model of a crystal and is
useful because it transforms in a simple way under the action of a suitable space group, so a fractal usefully models multiscaled objects in nature because it has simple properties under the action of a scaling group.

2 DIMENSION, METRIC AND MEASURE

The previous section referred to an "intuitive" notion of dimension. A more precise statement of this basic idea will now be given. The objective is to define the dimension of a set of points, Φ, found in a suitable d_u-dimensional space, U. Assuming that the latter has the structure of a linear vector space one can assert that its dimension is the largest number of mutually independent vectors that it can support. This is, however, too restrictive a definition to be applied to the general set Φ.

Topolgical dimension, d_T, requires only that continuity has meaning on the set. The definition is a recursive one: the topological dimension of Φ is $d_T=1+d_T'$ where d_T' is the topological dimension of a set whose removal would divide Φ. A point is taken to have $d_T=0$. It follows from this definition that a line has $d_T=1$ since it is divided by the removal of a point. Similarly a surface, since it is divided by the removal of a line, has $d_T=2$. These examples illustrate that d_T coincides with an intuitively reasonable idea of dimension - in particular note that it is always an integer.

The topological dimension of Φ is an intrinsic property of the set itself. It has an upper bound d_u but, apart from this owes nothing to U. In consequence, it is completely insensitive to any fractal character that Φ may possess. For example, consider the recursive process by which the Koch island is generated from a square. Since this may be thought of as a sequence of continuity preserving distortions, it has no effect on the topology of the square. The topological dimension of the perimeter of the

Koch island is therefore just that of the perimeter of the square i.e. $d_T=1$. The fractal structure of the perimeter of the Koch island is to do with the separations and relative positions of all points in the set. It follows that the definition of a dimension which takes account of this structure must employ metric properties on the space U. With this in mind consider the following generalisation of dimension.

Hausdorff dimension, d_H, is defined in terms of coverings of Φ with sets of d_u-dimensional cubes (for a good survey see Farmer, Ott, Yorke 1983). Initially the sizes of the cubes $\{\epsilon_i\}$ are allowed to vary subject to the constraint that all $\epsilon_i \leqslant \epsilon$, where ϵ is an arbitary constant. The quantity:

$$l_d(\epsilon) = \inf \sum_i \epsilon_i^d$$

($d \geqslant 0$ is a real parameter), is defined to have the form of a measure of the volume of the whole set. The summation is taken over a given choice of cover, while the infimum is taken over all possible covers subject to the constraint on the $\{\epsilon_i\}$. The effect of the infimum is to select a cover which is most efficient in the sense that it uses the smallest possible volume to cover Φ. Taking the limit

$$l_d = \lim_{\epsilon \to 0} l_d(\epsilon)$$

essentially gives the volume of the set assuming that the volume of an ϵ_i-cube is ϵ_i^d. Hausdorff proved the existence of a critical value of d ($=d_H$) such that: for $d>d_H$, $l_d=0$ and for $d<d_H$, $l_d=\infty$. The quantity d_H is the Hausdorff dimension of Φ, while l_{d_H} is its Hausdorff measure or volume.

As an example consider Φ to be the perimeter of the Koch island. Having chosen $\epsilon=L/6^n$, where n is an arbitrary positive integer, a particular choice of cover is to take 18^n identical cubes of size $\epsilon_i=\epsilon$. In this case $\Sigma\epsilon_i^d=18^n(L/6^n)^d$. A new cover may be generated by replacing one the cubes with a set of smaller cubes, say of size $L/6^{n+1}$. This requires 18 of the smaller cubes and causes the value of the summation to change by an amount: $(18-6^d)(L/6^{n+1})^d$. There are two cases to be dealt with:

1.For $18>6^d$ the effect of changing the cover is to increase the value of the sum. Therefore, the infimum corresponds to the original choice. thus $l_d(\epsilon)=18^n(L/6^n)^d$. It follows that

$$l_d = \lim_{n\to\infty} L^d(18/6^d)^n = \infty$$

2.For $18<6^d$ the value of the summation decreases with the $\{\epsilon_i\}$. Therefore $l_d(\epsilon)=0$ which implies $l_d=0$.

The Hausdorff dimension of the perimeter of the Koch island is the critical value of d which separates these two cases: $d_H=\ln 18/\ln 6$. Note that the scaling arguments of the previous section also gave this value for the exponent d_f.

It is an easy matter to show that the Hausdorff and topological dimensions coincide for smooth objects such as surfaces and lines. The above example shows that this is not necessarily the case, indeed d_H need not be an integer. Intuitively, the value of d_H characterises the way in which the set Φ fills the space U. Thus $1<d_H<2$ suggests that the perimeter of the Koch island is a curve which is in some sense tending to fill an area of the plane.

Mandelbrot used this fundamental distinction between Hausdorff and topological dimension to **define** a fractal as a set for which $d_H>d_T$. Unfortunately, it is often a difficult task to calculate the Hausdorff dimension of a set, generally because of the need to obtain the infimum over the range of possible covers. Consequently, other related but more convenient definitions of dimension are often employed.

Capacity, d_c, is also defined via a cover of Φ using d_u-dimensional cubes. Here, however, the cover is uniform, all cubes having the size ϵ.

$$d_c = \lim_{\epsilon\to 0} \frac{\ln N(\epsilon)}{\ln \epsilon^{-1}} \quad ,$$

where $N(\epsilon)$ is the number of cubes required to cover Φ. Since the uniform cover is not necessarily the most efficient, d_c is strictly an upper bound to d_H.

Returning to the example of the Koch island perimeter, if one sets $\epsilon=L/6^n$ then the recursive generation algorithm implies $N(\epsilon)=18^n$, from which follows:

$$d_c = \lim_{n\to\infty} \frac{\ln 18^n}{\ln 6^n} = \frac{\ln 18}{\ln 6} \quad .$$

As in this case it is often true that $d_H=d_c$, for this reason, when the context is not clear (see the introduction) the general term "fractal dimension", d_f, has been employed. It should be noted, however that this terminology also appears in the literature with more specific meanings. An example of a set for which $d_c \neq d_H$ is $\Phi=\{1, 1/2, 1/3, 1/4,....\} \subset [0,1]$. It is an instructive exercise to show that $d_H=0$ while $d_c=1/2$, since the effect of the infimum becomes clear in the analysis.

Pointwise dimension, d_p, is closely related to capacity, although its definition uses a new generalisation - a probability measure $d\mu(x)$ defined on the set Φ. There are a variety of contexts in which this is useful - the most obvious is to think of the measure as giving the mass density of the material from which the fractal has been formed. Thus if one writes:

$$\mu(B_\epsilon(x)) = \int_{B_\epsilon(x)} d\mu(x') \quad ,$$

as the total probability contained in the ball $B_\epsilon(x)=\{x_i \mid \|x-x_i\|<\epsilon, x_i \in \Phi\}$, it may also be interpreted as the mass of a ball of radius ϵ cut from the fractal at a point x. The definition proceeds as follows. If $d_p(x)$ defined by:

$$d_p(x) = \lim_{\epsilon \to 0} \frac{\ln \mu(B_\epsilon(x))}{\ln \epsilon} \quad ,$$

is independent of x for almost all x with respect to $\mu(x)$, then $d_p=d_p(x)$ is defined as the pointwise dimension of Φ (with respect to $\mu(x)$).

There is a definition of fractal dimension, commonly used in work on the statistical mechanics of aggregation and growth, which is similar to the above with the interpretation of $d\mu(x)$ as a mass density. In this case the scaling relation between the radius of gyration, R, and the mass, M, of an aggregating cluster is written $M \sim R^{d_f}$. For small radii the definition of pointwise dimension leads to the same form: $\mu(B_\epsilon(x)) \sim \epsilon^{d_p(x)}$, and may be thought of as an idealisation of the mass scaling exponent. In practice, however, one cannot allow $R \to 0$, nor can the x-independence of $d_p(x)$ be established.

A survey of the statistical mechanics literature reveals a bewildering variety of fractal dimensions (see contributions to Family, Landau 1984 and Stanley, Ostrowsky 1985). Much can be done to unify these conceptually, by considering them as pointwise dimensions associated with appropriate measures. For example, a measure corresponding to the density of material being transported through a percolating cluster will give a dimension associated with its, so-called, backbone. Similarly, a measure giving the flux of new material onto a growing surface will give a fractal dimension for the set of active growth sites.

By way of a finale to this section we shall now introduce an uncountable number of dimensions! This will be referred to as the **Hentschel-Procaccia continuum**, $\{d_q \mid q \in \mathbb{R}\}$ (Hentschel, Procaccia 1983) and is defined as follows:

$$d_q = \frac{1}{q-1} \lim_{\epsilon \to 0} \left[\frac{\ln \left[\int d\mu(x) \; \mu(B_\epsilon(x))^{q-1} \right]}{\ln \epsilon} \right] ,$$

where q is any real number. The measure of the ϵ-ball, $\mu(B_\epsilon(x))$, is again used, in contrast with the pointwise dimension, however, it (or some power of it) is averaged over the measure $d\mu(x)$. There is, therefore, no analogue of the requirement that $d_p(x)$ be constant almost everywhere. Indeed, if the fractal is homogeneous in the sense that $\mu(B_\epsilon(x))=\mu(B_\epsilon)$, a constant, almost everywhere, then the $\{d_q\}$ reduce to a single quantity:

$$\lim_{\epsilon \to 0} \left[\ln \mu(B_\epsilon) / \ln \epsilon \right] ,$$

which is, in fact, the pointwise dimension.

The nature of the information carried by the $\{d_q\}$ can be illustrated with some particular examples. For integer values of $q \geqslant 2$, d_q gives the scaling with radius of the probability of finding q points within ϵ-balls constructed on the fractal. In particular, d_2 gives the scaling at small separations of the two point correlation function of points in Φ with respect to $d\mu(x)$. d_2 is often referred to as the **correlation dimension** and is particularly useful since experimental probes often provide information about the pairwise correlation function.

Other cases of interest are the limit $q \to 0$, wich gives the capacity, d_c, defined earlier, and the limit $q \to 1$ which yields the **information dimension**, d_s, defined by:

$$d_s = \lim_{\epsilon \to 0} \left[\int d\mu(x) \ln \mu(B_\epsilon(x)) / \ln \epsilon \right] .$$

The latter has been discussed in some detail recently (Farmer 1982) and may be thought of as the exponent for the scaling of the information content of a measurement made on a fractal with the resolution of the measurement.

The set $\{d_q\}$ forms an ordered sequence since it can be shown that $q' > q$ implies that $d_{q'} \leqslant d_q$. The equality holds in the case of a homogeneous fractal where, as we have discussed, the whole heirachy reduces to a single quantity. Any deviation from this degenerate situation is a measure of the degree to which the scaling properties differ from place to place within the fractal set.

From the point of view of the following discussion the dimensions described in this section will be sufficient. It is probably not true to say that they characterise all the interesting properties of fractals. One can certainly think of other quantities which are potentially useful. For example there is the

"thickness" of fractal sets defined by Newhouse (see discussion in Guckenheimer, Holmes 1983) which characterises the degree to which such sets can occupy the same volume without intersecting each other.

3 THE PHYSICAL PROPERTIES OF FRACTAL OBJECTS

The scattering of short wavelength radiation from fractals and "subfractals" (which are objects with a fractal surface gradient) gives rise to scintillation effects which are commonly observed in nature. The glittering of a disturbed water surface, the twinkling of starlight passing through the turbulent atmosphere (Berry 1977, Berry 1979, Jakeman 1982) are examples. In fact our ability to perceive **texture** visually is indicative of correlations in the scattering medium which extend over many length scales (E. Jakeman: private communication), and points to the ubiquity of fractals in nature.

The general question of what physical properties are implied by an object having a fractal geometry is as large as it is fascinating. In keeping with the basic philosophy of this article we shall be content with a limited approach. The discussion to follow will be concerned with some experimental methods aimed at providing evidence for the existence of fractal structures and obtaining estimates of fractal dimensions. Broader physical implications of the effect of a fractal structure on mechanical properties (see for example:Turcotte, Smalley, Sola 1985, Alexander, Orbach 1982) or on transport properties such as conductivities or diffusion (see various articles in Family, Landau 1984 and Stanley, Ostrowskii 1985) will not be dealt with here.

In many cases it is possible to make observations on a system which give directly a characteristic scaling law of a fractal. Observations which deal directly with an image of the fractal, such as Lovejoy's study, are most direct and least vulnerable to ambiguous interpretation. An example of this approach is work done on metallic films sputtered onto inert substrates (Voss, Laibowitz, Alessandrini 1982, Kapitulnik, Deutscher 1982). This work employed digital image processing techniques applied to electron micrographs to extract not only the scaling relations but also impressive visualisations of the fractals themselves.

It was remarked in section 2 that scaling laws which are easy to measure are the mass-radius relation, giving something like the pointwise dimension, d_p, and the small scale behaviour of the particle-particle correlation function, which gives d_2, the correlation dimension. In a recent experiment Schaefer, Martin, Wiltzius, Cannell (1984) demonstrated that using light scattering techniques both can be obtained independently for the same system. Light scattering measures the scattered intensity as a function of angle of scatter, this relates directly to the structure factor S(k), where k is the magnitude of the scattering vector. S(k) is the Fourier transform of the pair distribution function, $g_2(R)$. One loosely interprets k as being the inverse separation of the scatterers contributing to the observed intensity. The experiment of Schaefer et al was concerned with the coagulation of colloidal suspensions of silica spheres. The observed S(k) has several distinct regions:

1. At small k the separation of the scatterers is large enough for the dominant contribution to the scattering to arise from scatterers on distinct aggregates within the fluid. The experiment sees the system as a suspension of aggregate particles which when sufficiently dilute can be modelled as a perfect gas. In this limit leading terms in the expansion of the structure factor give the following Lorenzian form:

$$S^{-1}(k) \simeq [cM_w]^{-1}[1 + k^2\langle R\rangle^2/3 + \ldots]$$

where c is the solution concentration by weight, M_w is the weight averaged molecular weight of the aggregates while $\langle R\rangle$ is the mean aggregate radius. thus a study of the behaviour of S(k) at small k gives both the mass and radius of aggregates, which for a range of coagulation times can be plotted on a log-log plot to give $M_w \sim \langle R\rangle^{d_f}$. For their system Schaefer et al find $d_f \simeq 2.12$. In three dimensions one expects $M \sim R^3$ for solid bodies, which suggests that these aggregates have rather an open structure.

2. The second region of interest arises when the separation of the scatterers is such that scattering is dominated by scatterers on different spheres within the same aggregate. For fractal aggregates at small interparticle separations the pair distribution function scales as $g_2(R) \sim R^{d_2}$, which implies that $S(k) \sim k^{-d_2}$ when the scatterers come from the same fractal. Schaefer et al, give a log-log plot of S(k) in this region. The data used was from both light scattering and small angle X-ray scattering. The plot shows a linear region extending over about two orders of magnitude in k which has a slope $-d_2 = -2.12 \pm 0.05$. Note that this value agrees with the value obtained for d_f using the small k limit.

These observations lend confidence to the interpretation that the aggregates are

(homogeneously) fractal and provide more evidence for the sponaneous generation of fractals in nature.

4 A FRACTAL GROWTH MODEL

We have commented on the ubiquity of fractals in nature, but have said nothing of the reasons for this. The present section is concerned with a particular model for an aggregation process which suggests a general (though not, of course, unique) criterion responsible for the growth of fractals. The model in question is called Diffusion Limited Aggregation (DLA) and has generated considerable interest in the literature since its description by Witten, Sander (1981). The form of this section owes much to Ball (1985).

The model describes the slow growth of a single aggregate which is seeded by a single fixed particle placed at the centre of a large sphere (or, in 2-dimensions, a circle). The sphere is an isotropic source of material which is able to diffuse and adhere to the aggregate. Mathematically, one has a diffusion problem:

$$\partial_t u(r,t) = D\nabla^2 u(r,t) \quad ,$$

(where D is the diffusion constant and u(r,t) is the concentration field of the material) subject to the following boundary conditions:

1. $u(r,t)=0$ as the surface of the aggregate is approached from the outside
2. $u(r,t)=u_{source}$ at large radius
3. $\rho\langle v\rangle=D\nabla u(r,t)$ at the growing surface, where ρ is the density of the aggregate, $\langle v\rangle$ is the mean velocity of the surface, and ∇u is the concentration gradient as the aggregate surface is approached from the outside.

The important feature of this model is that the growing interface is unstable. To see this, imagine that the interface is initially planar but becomes distorted by a fluctuation which generates a small localised peak. The distortion generates a locally large value of ∇u and hence, in this region of the aggregate surface, the growing interface will advance more rapidly, thereby amplifying the original fluctuation. One expects, therefore in a realistic, noisy environment that the growing surface of the aggregate will be very complicated, reflecting the history of fluctuations that it has experienced. Of course, a realistic environment will also provide stabilising processes which will limit the effect of the instability. At small scales surface tension, crystallinity effects and so on restrict the development of highly curved regions of the interface thus limiting the growth of "needle-like" fluctuations. At large scales the diffusion field cannot respond over lengths larger than the diffusion length and hence the growth of coherent,large "mountain-like" distortions will be limited by the diffusion process itself. However, between "needles" and "mountains" there are many length scales at which distortions can grow in an unconstrained manner. Thus the resulting aggregate will grow with structure on many length scales and will appear fractal to observations not able to probe the scales at which the stabilsing mechanisms dominate.

The DLA model usually employed is a modification of the one described above. An additional assumption is made that the size of the aggregate is less than diffusion length. The outer scale limit does not therefore arise. Moreover the diffusing field in the neighbourhood of the aggregate can be assumed to respond adiabatically to fluctuations so that the explicit time dependence in the diffusion can be dropped. Paradoxically, this results in a model of rather general form. One now has a field $\varphi(r)$ which satisfies Laplace's equation: $\nabla^2\varphi=0$, and is subject to the boundary conditions given above. The moving boundary is represented by: $\langle v\rangle\propto\nabla\varphi(r)$+noise, where the explicit noise is included to drive the instability. Formulated in this way the model clearly has strong connections with electrostatics. Indeed it has been used to describe the morphology of dielectric breakdown at point electrodes buried in insulating material (Niemeyer, Pietronero, Wiesmann 1984) . However, Laplace's equation can occur more generally, for example the model has recently been used to describe viscous fingering in interpenetrating fluids (Nittmann, Daccord, Stanley 1985). In this case $\varphi(r)$ is a pressure field and Laplace's equation arises from the application of d'Arcy's law to the mass continuity equation for the fluid.

Much of the interest in DLA has arisen from the ease with which it may be implemented as a computer model. Fig. 3 shows a DLA cluster of 100,000 sites grown numerically by Meakin (1985). The model consists of assuming a square lattice on which the aggregate can grow from an initial seed. The diffusion process is represented by a single random walker which is started a large distance from the aggregate. Depending upon the implementation the random walker may travel on or off the lattice. On arrival at a lattice site adjacent to the aggregate that site is incorporated into the aggregate and a new walker initiated. Clearly the inner scale here is given by the lattice spacing, while the single

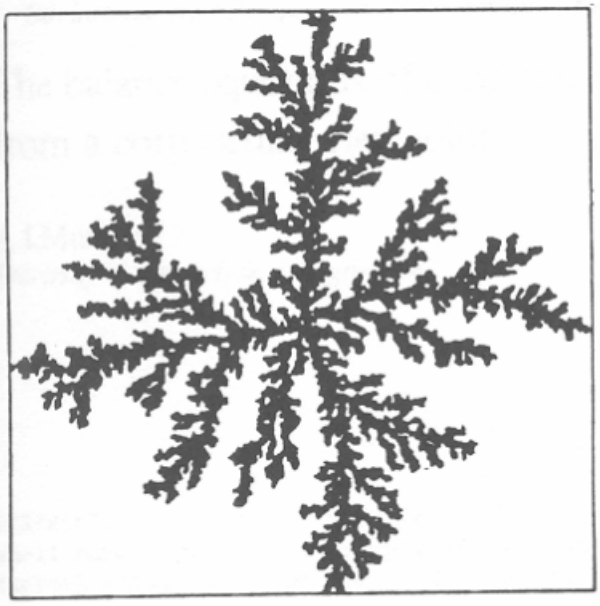

Fig. 3
A DLA aggregate (off-lattice walks, on-lattice growth, 100,000 sites)

random walker constraint is equivalent to the slow growth/adiabatic assumption. The noise enters the problem as shot noise when the walker decides to which lattice it belongs as it attaches to the aggregate.

Apparently from Fig. 3 the aggregate resulting from DLA simulations is a fractal - this is in accord with the stability considerations above. The fractal dimension of these aggregates generally converges to a value which is independent of the particular realization of the random walk used, and of whether or not the walk was on or off lattice. The general method of calculation is to relate the radius of gyration to the number of sites on the aggregate - this is the mass radius scaling relation which has already been discussed. Generally it is found that $d_f=2.495\pm.06$ in 3-dimensions, and $d_f=1.67$ on a plane. These values are consistent with the fairly open structure manifest in Fig. 3. The openess results from the aggregate shielding its own inner structure from the diffusional flux.

Recent work on the morpholgy of DLA clusters provides an example of the use of measures defined on fractal set (Turkevich, Scher 1985, and Halsey, Meakin, Procaccia 1985). In this case the relevant measure is harmonic measure on the aggregate. This is the analogue of ∇u for the computer model and is calculated by studying the incidence and disposition of contacts of random walkers with an aggregate of fixed size. The harmonic measure is generally very singular with the singularities corresponding to growing tips. Halsey et al have analysed it in some detail using the Henschel-Procaccia hierachy, and developed a scaling theory to describe their results.

Several experimental systems for which DLA should be good model have been studied. These involve viscous fingering (Nittmann et al 1985), dielectric breakdown (Pietronero, Wiesmann 1984), and electrodeposition of zinc metal in a chemically engineered 2-dimensional environment (Matsushita, Sano, Hayakawa, Honjo, Sawada 1984). Brady and Ball (1984) looked at the electrodeposition of copper at a point electrode under circumstances carefully controlled to ensure that the process is diffusion limited. They note that the diffusion current onto a spherical shell is proportional to the radius of the shell, while the mass of copper deposited is proportional to the total charge passed by the cell. Thus by monitoring the current through the cell and its time integral, continuous measurement of mass of copper deposited and effective radius of the deposite could be made. In this way an experimental mass radius scaling exponent was obtained. The value found, $d_f\sim2.43$, compares well with Meakin's computer result.

The general question of the origin of fractals in nature has yet to be fully answered, however unstable growth processes must surely be important. In particular, the lack of "detailed balance" between the instability and corresponding limiting processes in the DLA model results in a broad band of unstable modes. Recent molecular dynamics work on phase separation by the spinodal decomposition mechanism (Desai 1985) provides another good example of this. In the situation that the homogeneous phase is linearly unstable to fluctuations having a broad range of wavelengths it is to be expected that the separating inhomogeneous structure will appear to be fractal. This is indeed found to be the case in Desai's computer experiments. Finally, a fractal description of strong turbulence in fluids (Mandelbrot 1982) is suggested by a similar lack of "detailed balance". Here, instabilities occurring over a range of length scales feed energy to spacial modes with shorter and shorter wavelength until finally at small scales dissipation becomes dominant. The Kolmogorov power law spectrum is seen to support this viewpoint.

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