Standard Triples of Structured Matrix Polynomials

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Standard Triples of Structured Matrix Polynomials ☆

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Abstract

The notion of standard triples plays a central role in the theory of matrix polynomials. We study such triples for matrix polynomials $P(\lambda)$ with structure $\mathcal{S}$, where $\mathcal{S}$ is the Hermitian, symmetric, $\star$-even, $\star$-odd, $\star$-palindromic or $\star$-antipalindromic structure (with $\star = *, T$). We introduce the notion of $\mathcal{S}$-structured standard triple. With the exception of T-(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, we show that $P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, and moreover that every standard triple of a matrix polynomial with structure $\mathcal{S}$ is $\mathcal{S}$-structured. We investigate the important special case of $\mathcal{S}$-structured Jordan triples.

Keywords: standard triple, Jordan triple, structured matrix polynomial, Hermitian matrix polynomial, symmetric matrix polynomial, palindromic matrix polynomial, even matrix polynomial, odd matrix polynomial
2000 MSC: 15A18, 65F15

1. Introduction

Standard and Jordan triples for matrix polynomials were introduced and developed by Gohberg, Lancaster and Rodman (see for example [4], [5], [6]). Jordan triples extend to matrix polynomials of degree m

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \quad A_j \in \mathbb{F}^{n\times n}, \quad \det(A_m) \neq 0, \tag{1}$$

the notion of Jordan pair (X, J) for a single matrix $A \in \mathbb{C}^{n\times n}$, where $X \in \mathbb{C}^{n\times n}$ is nonsingular, J is a Jordan canonical form for A, and $A = XJX^{-1}$. The matrix X in a Jordan triple (X, J, Y) for $P(\lambda)$ is $n \times mn$ and, as for the single matrix case, it contains

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Table 1: Matrix polynomials $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ with structure $\mathcal{S} \in \mathbb{S}$.

Structure $\mathcal{S}$	**Definition**	**Coefficients property**
Hermitian	$P(\lambda) = P^*(\lambda)$	$A_j = A_j^*$
symmetric	$P(\lambda) = P^T(\lambda)$	$A_j = A_j^T$
$\star$-even	$P(\lambda) = P^\star(-\lambda)$	$A_j = (-1)^j A_j^\star$
$\star$-odd	$P(\lambda) = -P^\star(-\lambda)$	$A_j = (-1)^{j+1} A_j^\star$
$\star$-palindromic	$P(\lambda) = \lambda^m P^\star(\frac{1}{\lambda})$	$A_j = A_{m-j}^\star$
$\star$-antipalindromic	$P(\lambda) = -\lambda^m P^\star(\frac{1}{\lambda})$	$A_j = -A_{m-j}^\star$

the right eigenvectors and generalized eigenvectors of $P(\lambda)$. The matrix $J \in \mathbb{C}^{mn\times mn}$ is in Jordan canonical form, displaying the elementary divisors of $P(\lambda)$, and the matrix $Y \in \mathbb{C}^{mn\times n}$ plays the role of X^{-1} for a single matrix, i.e., the columns of Y^* determine left eigenvectors and generalized eigenvectors of $P(\lambda)$. A Jordan triple is a particular standard triple $(U, \mathcal{T}, V)$ in which the matrix $\mathcal{T}$ is in canonical form. Standard and Jordan triples are defined precisely in section 2.2.

Our objective is to study the standard and Jordan triples of structured matrix polynomials $P(\lambda)$ of the types listed in Table 1, where we use $\star$ to denote the transpose T for real matrices and either the transpose T or the conjugate transpose $*$ for matrices with complex entries. The structure of standard and Jordan triples are well understood for Hermitian matrix polynomials [4], [5] and more recently real symmetric matrix polynomials [2], [11]. With no assumption on the sizes of the Jordan blocks, Gohberg, Lancaster and Rodman [4] show that if (X, J, Y) is a Jordan triple for a Hermitian matrix polynomial then $Y = SX^*$ for some nonsingular $mn \times mn$ matrix S such that $S = S^*$ and $JS = (JS)^*$. We show in section 3 that results of this type also hold for the structures in $\mathbb{S}$, where

$$\begin{aligned}\mathbb{S} = \{&\text{Hermitian, symmetric, } *\text{-even, } *\text{-odd, } T\text{-even, } T\text{-odd,} \\ &*\text{-palindromic, } *\text{-antipalindromic, } T\text{-palindromic, } T\text{-antipalindromic}\}.\end{aligned} \tag{2}$$

For $\mathcal{S} \in \mathbb{S}$, we introduce the notion of $\mathcal{S}$-structured standard triples. With the exception of T-(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, we show that $P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, and that for any $P(\lambda)$ with structure $\mathcal{S}$, all standard triples for $P(\lambda)$ are $\mathcal{S}$-structured. Finally, we study in section 4 the special case of $\mathcal{S}$-structured Jordan triples.

Two important features of this work are (a) a distinction, when necessary, between triples and matrix polynomials defined over the complex ($\mathbb{C}$) or real ($\mathbb{R}$) fields, and (b) a unified presentation of the results, except in section 4, where we provide explicit expressions for the S-matrix of $\mathcal{S}$-structured Jordan triples that are structure-dependent.

2. Preliminaries

The set of all matrix polynomials with coefficient matrices in $\mathbb{F}^{n\times n}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is denoted by $\mathcal{P}(\mathbb{F}^n)$. When the polynomials are structured with structure $\mathcal{S}$, the corresponding set is denoted by $\mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ (see Table 1). Throughout this paper we assume that $P(\lambda)$ has a nonsingular leading coefficient matrix as in (1). Recall that λ is an eigenvalue of $P(\lambda)$ with corresponding right eigenvector $x \neq 0$ and left eigenvector $y \neq 0$ if $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$. We denote by $\Lambda(P)$ the set of eigenvalues of $P(\lambda)$.

2.1. Structured linearizations

Linearizations play a major role in the theory of matrix polynomials. They are $mn \times mn$ linear matrix polynomials $L(\lambda) = \lambda\mathcal{A} + \mathcal{B}$ related to $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ of degree m by

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

for some matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants. For example, the companion form

$$\mathcal{C} = -\begin{bmatrix} A_m^{-1}A_{m-1} & A_m^{-1}A_{m-2} & \dots & A_m^{-1}A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} \tag{3}$$

of $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ defines a linearization $\lambda I - \mathcal{C}$ of $P(\lambda)$.

Some of the results in section 3 and all the results in section 4 rely on the construction of linearizations that preserve the structure of $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. The vector space of pencils

$$\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{F}^m \,\},$$

introduced in [15], provides a rich source of such linearizations. Here $\Lambda = [\,\lambda^{m-1} \quad \dots \quad \lambda \quad 1\,]^T$. It is shown in [7], [12], [14] that for some $v \in \mathbb{F}^m$ satisfying the admissible constraint

(i) $v \in \mathbb{R}^m$ if $\mathcal{S} =$ Hermitian,
(ii) $v = \Sigma_m v$ if $\mathcal{S} \in \{T\text{-even}, T\text{-odd}\}$ or $v = \Sigma_m \bar{v}$ if $\mathcal{S} \in \{*\text{-even}, *\text{-odd}\}$,
(iii) $v = F_m v$ if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ or $v = F_m \bar{v}$ if $\mathcal{S} \in \{*\text{-palindromic}, *\text{-antipalindromic}\}$,

where

$$\Sigma_m = \operatorname{diag}((-1)^{m-1}, \dots, (-1)^0), \qquad F_m = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix},$$

there exists a unique pencil $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} \in \mathbb{L}_1(P)$ with structure $\mathcal{S} \in \mathbb{S}$. This pencil is a linearization of $P(\lambda)$ if the roots of the v-polynomial

$$\mathsf{p}(x; v) = v_1 x^{m-1} + v_2 x^{m-2} + \dots + v_{m-1}x + v_m$$

are not eigenvalues of P [14, Thm. 6.3 & Thm. 6.5]. The vector $v = e_m$, where e_m is the mth column of the $m \times m$ identity matrix, is an admissible vector for $\mathcal{S} \in \{$Hermitian, symmetric, $\star$-even, $\star$-odd} since $e_m \in \mathbb{R}^m$ and $\Sigma_m e_m = e_m$. Also, the roots of $\mathsf{p}(x; e_m)$ are all equal to ∞ and since $\det(A_m) \neq 0$ then $\infty \notin \Lambda(P)$. Hence the structured pencils $\lambda \mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} \in \mathbb{L}_1(P)$ with vector e_m are linearizations of P. They are given by (see [7] and [14] for the construction)

$$\lambda \mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \begin{cases} \lambda \mathcal{A}(1) + \mathcal{B}(1) \text{ when } \mathcal{S} \in \{\text{Hermitian, symmetric}\}, \\ \lambda \mathcal{A}(-1) + \mathcal{B}(-1) \text{ when } \mathcal{S} \in \{\star\text{-even}, \star\text{-odd}\}, \end{cases} \tag{4}$$

where

$$\mathcal{A}(\varepsilon) = \begin{bmatrix} 0 & \cdots & 0 & \varepsilon^{m-1} A_m \\ \vdots & & \iddots & \varepsilon^{m-2} A_{m-1} \\ \vdots & \iddots & \iddots & \vdots \\ \varepsilon^0 A_m & \varepsilon^0 A_{m-1} & \cdots & \varepsilon^0 A_1 \end{bmatrix},$$

and

$$\mathcal{B}(\varepsilon) = -\begin{bmatrix} 0 & \cdots & 0 & \varepsilon^{m-1} A_m & 0 \\ \vdots & \iddots & \varepsilon^{m-2} A_m & \varepsilon^{m-2} A_{m-1} & \vdots \\ 0 & \iddots & \iddots & \vdots & \vdots \\ \varepsilon A_m & \varepsilon A_{m-1} & \cdots & \varepsilon A_2 & 0 \\ 0 & \cdots & \cdots & 0 & -A_0 \end{bmatrix}.$$

Note that for $\star$-(anti)palindromic $P(\lambda)$, we have $0 \notin \Lambda(P)$ since $\infty \notin \Lambda(P)$. When $m = 2k+1$, $v = e_{k+1}$ satisfies $v = F_m v = F_m \bar{v}$ and $0, \infty$ are the only roots of the v-polynomial. The corresponding $\star$-(anti)palindromic pencils in $\mathbb{L}_1(P)$ are linearizations. They are given by (see [14] for the construction)

$$\lambda \mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \begin{cases} \lambda \mathcal{A}^{odd} + (\mathcal{A}^{odd})^\star & \text{when } \mathcal{S} = \star\text{-palindromic with } m = 2k+1, \\ \lambda \mathcal{A}^{odd} - (\mathcal{A}^{odd})^\star & \text{when } \mathcal{S} = \star\text{-antipalindromic with } m = 2k+1, \end{cases} \tag{5}$$

where

$$\mathcal{A}^{odd} = \begin{bmatrix} \mathcal{A}^{odd}_{11} & \mathcal{A}^{odd}_{12} \\ \mathcal{A}^{odd}_{21} & \mathcal{A}^{odd}_{22} \end{bmatrix}, \tag{6}$$

with $\mathcal{A}^{odd}_{11} = (\mathcal{A}^{odd}_{22})^T = 0_{nk \times n(k+1)}$ and

$$\mathcal{A}^{odd}_{12} = \begin{bmatrix} -A^\star_m & 0 & \cdots & 0 \\ -A^\star_{m-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -A^\star_{k+2} & \cdots & -A^\star_{m-1} & -A^\star_m \end{bmatrix}, \quad \mathcal{A}^{odd}_{21} = \begin{bmatrix} A_m & A_{m-1} & \cdots & A_{k+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{m-1} \\ 0 & \cdots & 0 & A_m \end{bmatrix}.$$

For $\star$-(anti)palindromic polynomials of even degree $m = 2k$, a nonzero vector v satisfying $F_m v = v$ when $\star = T$ or $F_m v = \bar{v}$ when $\star = *$ can be taken of the form $v = z e_k + z^\star e_{k+1}$. The corresponding $\star$-(anti)palindromic pencil in $\mathbb{L}_1(P)$ is a linearization of $P(\lambda)$ if $-z/z^\star$ is not an eigenvalue of P and is given by (see [14])

$$\lambda \mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \begin{cases} \lambda \mathcal{A}^{even}_-(z) + (\mathcal{A}^{even}_-(z))^\star & \text{when } \mathcal{S} = \star\text{-palindromic}, m = 2k, \\ \lambda \mathcal{A}^{even}_-(z) - (\mathcal{A}^{even}_-(z))^\star & \text{when } \mathcal{S} = \star\text{-antipalindromic}, m = 2k, \end{cases} \tag{7}$$

where

$$\mathcal{A}_{-}^{even}(z) = \begin{bmatrix} \mathcal{A}_{11}^{even}(z) & \mathcal{A}_{12}^{even}(z) \\ \mathcal{A}_{21}^{even}(z) & \mathcal{A}_{22}^{even}(z) \end{bmatrix}, \tag{8}$$

with

$$\mathcal{A}_{11}^{even}(z) = z \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ A_m & A_{m-1} & \dots & A_{k+1} \end{bmatrix}, \quad \mathcal{A}_{22}^{even}(z) = z \begin{bmatrix} A_{k+1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m-1} & 0 & \dots & 0 \\ A_m & 0 & \dots & 0 \end{bmatrix},$$

$$\mathcal{A}_{12}^{even}(z) = - \begin{bmatrix} z^\star A_0 & zA_0 & 0 & \cdots & \cdots & 0 \\ z^\star A_1 & z^\star A_0 + zA_1 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ z^\star A_{k-2} & z^\star A_{k-2} + zA_{k-1} & \cdots & z^\star A_1 + zA_2 & z^\star A_0 + zA_1 & zA_0 \\ -zA_k + z^\star A_{k-1} & z^\star A_{k-2} & & \cdots & z^\star A_1 & z^\star A_0 \end{bmatrix},$$

$$\mathcal{A}_{21}^{even}(z) = \begin{bmatrix} z^\star A_m & zA_m + z^\star A_{m-1} & zA_{m-1} + z^\star A_{m-2} & \cdots & \cdots & zA_{k+2} + z^\star A_{k+1} \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & zA_{m-1} + z^\star A_{m-2} \\ \vdots & & & \ddots & z^\star A_m & zA_m + z^\star A_{m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & z^\star A_m \end{bmatrix}.$$

Note that when $\star = *$, we can always pick a $z \in \mathbb{F}$ such that $-z/z^\star \notin \Lambda(P)$. But when $\star = T$, $-z/z^\star = -1$ so if $-1 \in \Lambda(P)$ the corresponding $\star$-(anti)palindromic pencil in $\mathbb{L}_1(P)$ is not a linearization of $P(\lambda)$. In fact it is shown in [14] that some T-(anti)palindromic matrix polynomials of even degree do not have T-(anti)palindromic linearizations. Instead, we allow a linearization with "anti" structure: palindromic becomes antipalindromic and vice versa. For this, let $v = e_{k+1} - e_k$ satisfying $v = -F_m v$. If $P(\lambda)$ is T-palindromic then there is a unique T-antipalindromic pencil in $\mathbb{L}_1(P)$ with vector v. Similarly if $P(\lambda)$ is T-antipalindromic then there is a unique T-palindromic pencil in $\mathbb{L}_1(P)$ with vector v. Such pencils are linearizations of P if $1 \notin \Lambda(P)$ and are given by

$$\lambda \mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \begin{cases} \lambda \mathcal{A}_{+}^{even} - (\mathcal{A}_{+}^{even})^T & \text{when } \mathcal{S} = T\text{-palindromic with } m = 2k, \\ \lambda \mathcal{A}_{+}^{even} + (\mathcal{A}_{+}^{even})^T & \text{when } \mathcal{S} = T\text{-antipalindromic when } m = 2k, \end{cases} \tag{9}$$

where $\mathcal{A}_{+}^{even}(z)$ has a block structure similar to that of $\mathcal{A}_{-}^{even}(z)$ in (7) with z replaced by -1 and $z^\star$ replaced by 1. In particular, when $m = 2$,

$$\mathcal{A}_{+}^{even} = \begin{bmatrix} -A_2 & -A_1 - A_0 \\ A_2 & -A_2 \end{bmatrix}.$$

The next result, useful later, shows that the linearizations (4)–(9) share a property.

Lemma 2.1 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}_\mathcal{S}(\mathbb{F}^n)$ with nonsingular leading coefficient. If $\lambda \mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S}$ is a structured linearization of $P(\lambda)$ as in* (4)–(9) *then $\mathcal{C} = -\mathcal{A}_\mathcal{S}^{-1} \mathcal{B}_\mathcal{S}$, where $\mathcal{C}$ is the companion form of $P(\lambda)$ given in* (3).

Proof. Some easy calculations show that $-\mathcal{A}_{\mathcal{S}}\mathcal{C} = \mathcal{B}_{\mathcal{S}}$. $\square$

Hence, with the exception of T-(anti)palindromic $P(\lambda)$ of even degree with both -1 and 1 as eigenvalues, the companion form of $P(\lambda)$ can be factorized as $\mathcal{C} = -\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{B}_{\mathcal{S}}$, where $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \mathcal{A}_{\mathcal{S}}(\lambda I - \mathcal{C})$ is a structured linearization of $P(\lambda)$.

2.2. Standard triples

Recall that $(U, \mathcal{T})$ is an (m, n)-*standard pair* over $\mathbb{F}$ if $\mathcal{T} \in \mathbb{F}^{mn\times mn}$ and $U \in \mathbb{F}^{n\times mn}$ are such that

$$Q = Q(U, \mathcal{T}) := \begin{bmatrix} U\mathcal{T}^{m-1} \\ \vdots \\ U\mathcal{T} \\ U \end{bmatrix} \tag{10}$$

is nonsingular [11, Def. 2.1]. The triple $(U, \mathcal{T}, V)$ forms an (m, n)-*standard triple* over $\mathbb{F}$ if $(U, \mathcal{T})$ is an (m, n)-standard pair over $\mathbb{F}$ and $V \in \mathbb{F}^{mn\times n}$ is such that $U\mathcal{T}^{m-1}V$ is nonsingular and, if $m \geq 2$,

$$U\mathcal{T}^{j}V = 0, \quad j = 0\colon m-2, \tag{11}$$

or equivalently,

$$QV = e_1 \otimes N \tag{12}$$

for some nonsingular $n \times n$ matrix N, where e_1 is the first column of the $m \times m$ identity matrix [11, Def. 2.3]. Note that the definitions of standard pairs and triples make no reference to matrix polynomials.

An (m, n)-standard pair $(U, \mathcal{T})$ over $\mathbb{F}$ is a *standard pair for* $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ if

$$A_m U\mathcal{T}^m + A_{m-1}U\mathcal{T}^{m-1} + \cdots + A_1 U\mathcal{T} + A_0 U = 0 \tag{13}$$

[6, p. 46]. A standard triple $(U, \mathcal{T}, V)$ is a *standard triple for* $P(\lambda)$ if (13) holds and $A_m = (U\mathcal{T}^{m-1}V)^{-1}$ (i.e., $N = A_m^{-1}$ in (12)). Any $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient admits a standard triple. For example, it is easy to check that

$$(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1}) \tag{14}$$

with $\mathcal{C}$ as in (3) is a standard triple for $P(\lambda)$. We refer to (14) as the *primitive standard triple* for $P(\lambda)$.

Let $U_i \in \mathbb{F}^{n\times mn}$, $\mathcal{T}_i \in \mathbb{F}^{mn\times mn}$ and $V_i \in \mathbb{F}^{mn\times n}$, $i = 1, 2$. Then $(U_1, \mathcal{T}_1, V_1)$ is *similar* to $(U_2, \mathcal{T}_2, V_2)$ if there exists a nonsingular $G \in \mathbb{F}^{mn\times mn}$ such that

$$U_2 = U_1 G, \qquad \mathcal{T}_2 = G^{-1}\mathcal{T}_1 G, \qquad V_2 = G^{-1}V_1. \tag{15}$$

It is easy to check that $Q(U_1, \mathcal{T}_1)G = Q(U_2, \mathcal{T}_2)$. Hence G is uniquely defined by $(U_1, \mathcal{T}_1)$, $(U_1, \mathcal{T}_2)$ and is given by

$$G = Q(U_1, \mathcal{T}_1)^{-1}Q(U_2, \mathcal{T}_2). \tag{16}$$

Also, $(U_2, \mathcal{T}_2, V_2)$ defined in (15) is a standard triple if $(U_1, \mathcal{T}_1, V_1)$ is a standard triple [5, Prop. 12.1.3]. Moreover if $(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ then, with $Q = Q(U, \mathcal{T})$ as in (10), we find that

$$(e_m^T \otimes I_n)Q = U, \quad Q^{-1}\mathcal{C}Q = \mathcal{T}, \quad Q^{-1}(e_1 \otimes A_m^{-1}) = V. \tag{17}$$

Table 2: Definition of $u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})$ for some $\mathcal{T} \in \mathbb{F}^{mn\times mn}$ satisfying assumption (b), where α is some scalar in $\mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \Lambda(\mathcal{T})$.

Structure $\mathcal{S}$	$u_{\mathcal{S}}(\mathcal{T})$	$t_{\mathcal{S}}(\mathcal{T})$	$v_{\mathcal{S}}(\mathcal{T})$
Hermitian/symmetric	I	$\mathcal{T}^\star$	I
$\star$-even	$-I$	$-\mathcal{T}^\star$	I
$\star$-odd	I	$-\mathcal{T}^\star$	I
$\star$-palindromic, $m = 2k+1$	$-\mathcal{T}^{\star(k-1)}$	$\mathcal{T}^{-\star}$	$\mathcal{T}^{\star k}$
$\star$-palindromic, $m = 2k$	$-\mathcal{T}^{\star(k-1)}(I+\alpha\mathcal{T}^\star)^{-1}$	$\mathcal{T}^{-\star}$	$(I+\alpha\mathcal{T}^\star)\mathcal{T}^{\star(k-1)}$
$\star$-antipalindromic, $m = 2k+1$	$\mathcal{T}^{\star(k-1)}$	$\mathcal{T}^{-\star}$	$\mathcal{T}^{\star k}$
$\star$-antipalindromic, $m = 2k$	$\mathcal{T}^{\star(k-1)}(I+\alpha\mathcal{T}^\star)^{-1}$	$\mathcal{T}^{-\star}$	$(I+\alpha\mathcal{T}^\star)\mathcal{T}^{\star(k-1)}$

Hence any standard triple $(U, \mathcal{T}, V)$ for $P(\lambda)$ is similar to the primitive standard triple $(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$. Note that because $\mathcal{T}$ is similar to $\mathcal{C}$, $\lambda I - \mathcal{T}$ is a linearization of $P(\lambda)$ and $\Lambda(P) = \Lambda(\mathcal{T})$. The following result [5, Thm. 12.1.4] will be needed.

Lemma 2.2 *Let $U \in \mathbb{F}^{n\times mn}$, $\mathcal{T} \in \mathbb{F}^{mn\times mn}$, $V \in \mathbb{F}^{mn\times n}$ and let $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ be of degree m with nonsingular leading coefficient. Then $(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ if and only if $P(\lambda)^{-1} = U(\lambda I - \mathcal{T})^{-1}V$ for $\lambda \in \mathbb{C} \setminus \Lambda(P)$.*

A Jordan triple (X, J, Y) over $\mathbb{F}$ for $P(\lambda)$ is a standard triple for $P(\lambda)$ for which the matrix J is in Jordan form or real Jordan form if $\mathbb{F} = \mathbb{R}$. By (13) and [6, Prop. 2.1], we have that $\sum_{j=0}^m A_j X J^j = 0$ and $\sum_{j=0}^m J^j Y A_j = 0$. The columns of X and Y^* determine right and left eigenvectors and generalized eigenvectors of $P(\lambda)$. The matrix J is the Jordan form of the companion form $\mathcal{C}$ of $P(\lambda)$.

3. $\mathcal{S}$-structured standard triples

We now consider standard triples in the context of structured matrix polynomials. We start by listing two assumptions used in our analysis. Let $\mathcal{S} \in \mathbb{S}$, $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ have degree m with nonsingular leading coefficient and let $\mathcal{T} \in \mathbb{F}^{mn\times mn}$.

> **Assumption (a)**: if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ and $P(\lambda)$ has degree $m = 2k$ then either $-1 \notin \Lambda(P)$ or $1 \notin \Lambda(P)$.
>
> **Assumption (b)**: if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ and $m = 2k$ then either $-1 \notin \Lambda(\mathcal{T})$ or $1 \notin \Lambda(\mathcal{T})$.

Assumption (a) ensures the existence of a structured linearization. Assumption (b) ensures the existence of $\alpha \in \mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \Lambda(\mathcal{T})$. Also, for $\star$-(anti)palindromic structures, the eigenvalues of $\mathcal{T}$ come in pairs $(\lambda, \lambda^{-\star})$. Hence $0 \notin \Lambda(\mathcal{T})$ since $\infty \notin \Lambda(\mathcal{T})$ and $\mathcal{T}^{-\star}$ is well defined. So for some $\mathcal{T}$ satisfying assumption (b) we define $u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})$ as in Table 2. We note that assumptions (a) and (b) are equivalent when $\lambda I - \mathcal{T}$ is a linearization of $P(\lambda)$.

Before stating our main result in Theorem 3.4, we provide a few lemmas and introduce the notion of $\mathcal{S}$-structured standard triple. The first lemma of this section extends to all structures in $\mathbb{S}$ a result in [6, Thm. 10.1] for Hermitian structure.

Lemma 3.1 *Let $(U, \mathcal{T}, V)$ be an (m, n)-standard triple for $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient and let $\mathcal{S} \in \mathbb{S}$. Assume that $\mathcal{T}$ satisfies assumption (b). Then $P(\lambda)$ has structure $\mathcal{S}$ if and only if $(V^\star u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})U^\star)$ is a standard triple for $P(\lambda)$.*

Proof. ($\Rightarrow$) Assume that $P(\lambda)$ is structured with structure $\mathcal{S}$. Since any standard triple for $P(\lambda)$ is similar to the primitive standard triple $(U_0, \mathcal{C}, V_0) := (e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$ (see comment before Lemma 2.2), it suffices to show that $(U_0, \mathcal{C}, V_0)$ is similar to $(V_0^\star u_{\mathcal{S}}(\mathcal{C}), t_{\mathcal{S}}(\mathcal{C}), v_{\mathcal{S}}(\mathcal{C})U_0^\star)$. Note that under assumption (b), $P(\lambda)$ has a structured linearization $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$, which is one of (4)–(9) and by Lemma 2.1, $\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{B}_{\mathcal{S}} = -\mathcal{C}$. Define

$$G^{-1} := \begin{cases} z^{-\star}\mathcal{A}_-^{even}(z) & \text{if } P \text{ is } \star\text{-(anti)palindromic, } m = 2k,\ -z/z^\star \notin \Lambda(P), \\ \mathcal{A}_{\mathcal{S}} & \text{otherwise,} \end{cases} \tag{18}$$

with $\mathcal{A}_-^{even}(z)$ as in (8). We aim to show that

$$V_0^\star u_{\mathcal{S}}(\mathcal{C}) = U_0 G, \qquad G^{-1}\mathcal{C}G = t_{\mathcal{S}}(\mathcal{C}), \qquad v_{\mathcal{S}}(\mathcal{C})U_0^\star = G^{-1}V_0, \tag{19}$$

that is, $(U_0, \mathcal{C}, V_0)$ is similar to $(V_0^\star u_{\mathcal{S}}(\mathcal{C}), t_{\mathcal{S}}(\mathcal{C}), v_{\mathcal{S}}(\mathcal{C})U_0^\star)$ for all $\mathcal{S} \in \mathbb{S}$. That (19) holds for $\mathcal{S} \in \{$Hermitian, symmetric, $\star$-even, $\star$-odd$\}$ is easy to check.

For $\mathcal{S} \in \{\star$-palindromic, $\star$-antipalindromic$\}$, the proof that $G^{-1}\mathcal{C}G = \mathcal{C}^{-\star} = t_{\mathcal{S}}(\mathcal{C})$ follows from the definition of G and $\mathcal{C} = \varepsilon\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{A}_{\mathcal{S}}^\star$, where $\varepsilon = \pm 1$ depends on whether $\mathcal{B}_{\mathcal{S}} = \mathcal{A}_{\mathcal{S}}^\star$ or $\mathcal{B}_{\mathcal{S}} = -\mathcal{A}_{\mathcal{S}}^\star$. To prove that the first and third equalities in (19) hold for palindromic structures, we consider three cases.

(i) $m = 2k+1$. In that case, $G^{-1} = \mathcal{A}^{odd}$, with $\mathcal{A}^{odd}$ as in (6). Then

$$G^{-1}V_0 = G^{-1}(e_1 \otimes A_m^{-1}) = e_{k+1} \otimes I = (\mathcal{C}^\star)^k(e_m \otimes I) = v_{\mathcal{S}}(\mathcal{C})U_0^\star,$$

from which it follows that $V_0^\star = (e_m^T \otimes I)\mathcal{C}^k G^\star$ so that, on using $G^{-1}\mathcal{C}G = \mathcal{C}^{-\star}$,

$$\begin{aligned} V_0^\star u_{\mathcal{S}}(\mathcal{C})G^{-1} &= (e_m^T \otimes I)\mathcal{C}^k G^\star(-\mathcal{C}^{\star(k-1)})G^{-1} \\ &= (e_m^T \otimes I)\mathcal{C}^k\mathcal{C}^{(1-k)}(-G^\star G^{-1}) \\ &= (e_m^T \otimes I) = U_0. \end{aligned}$$

(ii) $m = 2k$, $\star = T$ and $-1 \in \Lambda(\mathcal{T})$. In that case, $G^{-1} = \mathcal{A}_+^{even}$ with $\mathcal{A}_+^{even}$ as in (9). Then

$$v_{\mathcal{S}}(\mathcal{C})U_0^T = (I - \mathcal{C}^T)\mathcal{C}^{T(k-1)}(e_m \otimes I_n) = e_{k+1} \otimes I - e_k \otimes I = G^{-1}(e_1 \otimes I)A_m^{-1} = G^{-1}V_0.$$

From $V_0 = Gv_{\mathcal{S}}(\mathcal{C})U_0^T$ it follows that $V_0^T = U_0\mathcal{C}^{(k-1)}(I - \mathcal{C})G^T$, so that

$$\begin{aligned} V_0^T u_{\mathcal{S}}(\mathcal{C}) &= -U_0\mathcal{C}^{(k-1)}(I - \mathcal{C})G^T\mathcal{C}^{T(k-1)}(I - \mathcal{C}^T)^{-1} \\ &= -U_0\mathcal{C}^{(k-1)}(I - \mathcal{C})\mathcal{C}^{(1-k)}G^T(I - \mathcal{C}^T)^{-1} \\ &= U_0 G(I - \mathcal{C}^T)(I - \mathcal{C}^T)^{-1} = U_0 G, \end{aligned}$$

where we used $\mathcal{C}G^T = G$ and $G^T\mathcal{C}^{T(k-1)}G^{-T} = \mathcal{C}^{-(k-1)}$.

(iii) $m = 2k$, $\star = *, T$ and if $\star = T$ then $-1 \notin \Lambda(\mathcal{T})$. The proof is similar to that in (ii) with $\alpha = z/z^\star$ in the definition of $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$, and $G^{-1} = z^{-\star}\mathcal{A}_-^{even}(z)$ with $\mathcal{A}_-^{even}(z)$ as in (8).

The case of antipalindromic structures is proved similarly.

($\Leftarrow$) Suppose that $(U, \mathcal{T}, V)$ and $(V^\star u_\mathcal{S}(\mathcal{T}), t_\mathcal{S}(\mathcal{T}), v_\mathcal{S}(\mathcal{T})U^\star)$ are standard triples for $P(\lambda)$. By Lemma 2.2, we have that

$$U(\lambda I - \mathcal{T})^{-1}V = P(\lambda)^{-1} = V^\star u_\mathcal{S}(\mathcal{T})\big(\lambda I - t_\mathcal{S}(\mathcal{T})\big)^{-1} v_\mathcal{S}(\mathcal{T})U^\star. \tag{20}$$

As shown in the proof of [6, Thm. 10.1] for Hermitian structure, (20) implies that

$$\big(P^*(\lambda)\big)^{-1} = \big(P(\bar{\lambda})\big)^{-*} = (U(\bar{\lambda}I - \mathcal{T})^{-1}V)^* = V^*(\lambda I - \mathcal{T}^*)^{-1}U^* = P(\lambda)^{-1}$$

showing that $P(\lambda)$ is Hermitian. This proof extends easily to structures $\mathcal{S} \in$ {symmetric, $\star$-even, $\star$-odd}.

We now concentrate on palindromic structures. Using the left hand side of (20) we find that

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = \lambda^{-m}(U(\lambda^{-\star}I - \mathcal{T})^{-1}V)^\star = \lambda^{1-m}V^\star(I - \lambda\mathcal{T}^\star)^{-1}U^\star.$$

If $\|\lambda\mathcal{T}^\star\| < 1$ for some subordinate matrix norm $\|\cdot\|$ then

$$(I - \lambda\mathcal{T}^\star)^{-1} = I + \lambda\mathcal{T}^\star + \lambda^2\mathcal{T}^{\star 2} + \cdots. \tag{21}$$

Using (21) and the fact that $V^\star\mathcal{T}^{\star j}U^\star = 0$, $j = 0{:}\, m-2$ (see (11)), we obtain

$$\begin{aligned}\lambda^{-m}(P(\lambda^{-\star}))^{-\star} &= V^\star\mathcal{T}^{\star(m-1)}(I + \lambda\mathcal{T}^\star + \lambda^2\mathcal{T}^{\star 2} + \cdots)U^\star \\ &= V^\star\mathcal{T}^{\star(k-1)}(I - \lambda\mathcal{T}^\star)^{-1}\mathcal{T}^{\star(m-k)}U^\star \\ &= -V^\star\mathcal{T}^{\star(k-1)}(\lambda I - \mathcal{T}^{-\star})^{-1}\mathcal{T}^{\star(m-k-1)}U^\star\end{aligned} \tag{22}$$

for all $|\lambda| < \|\mathcal{T}^\star\|^{-1}$. When $m = 2k+1$, (22) and the right hand side of (20) yield

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = V^\star u_\mathcal{S}(\mathcal{T})(\lambda I - \mathcal{T}^{-\star})^{-1}v_\mathcal{S}(\mathcal{T})U^\star = P(\lambda)^{-1}. \tag{23}$$

Note that $(\lambda I - \mathcal{T}^{-\star})^{-1}$ commutes with $\mathcal{T}^{\star k-1}$, $(I + \alpha\mathcal{T}^\star)$ and $(I + \alpha\mathcal{T}^\star)^{-1}$ so when $m = 2k$, (22) can be rewritten to yield (23). Since $\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = P(\lambda)^{-1}$ holds for many values of λ, $P(\lambda) = \lambda^m P^\star(\lambda^{-1})$ for all λ, that is, $P(\lambda)$ is $\star$-palindromic.

That $P(\lambda) = -\lambda^m P^\star(\lambda^{-1})$ for the $\star$-antipalindromic structure is proved in a similar way. □

Lemma 3.1 naturally leads to the following definition.

Definition 3.2 ($\mathcal{S}$-structured standard triple) *Let $\mathcal{S} \in \mathbb{S}$. An (m,n)-standard triple $(U, \mathcal{T}, V)$ with $\mathcal{T}$ satisfying assumption* (b) *is said to be* $\mathcal{S}$-structured *if it is similar to $(V^\star u_\mathcal{S}(\mathcal{T}), t_\mathcal{S}(\mathcal{T}), v_\mathcal{S}(\mathcal{T})U^\star)$.*

If $(U, \mathcal{T}, V)$ is an $\mathcal{S}$-structured standard triple then there is a nonsingular $S \in \mathbb{F}^{mn \times mn}$ such that

$$US = V^\star u_\mathcal{S}(\mathcal{T}), \quad S^{-1}\mathcal{T}S = t_\mathcal{S}(\mathcal{T}), \quad S^{-1}V = v_\mathcal{S}(\mathcal{T})U^\star. \tag{24}$$

The matrix S is unique and is given by (see (16))

$$S = Q(U, \mathcal{T})^{-1}Q\big(V^\star u_\mathcal{S}(\mathcal{T}), t_\mathcal{S}(\mathcal{T})\big).$$

We refer to S as the the S-matrix of the $\mathcal{S}$-structured standard triple $(U, \mathcal{T}, V)$.

The next lemma shows that any standard triple that is similar to an $\mathcal{S}$-structured standard triple is itself $\mathcal{S}$-structured.

Lemma 3.3 *Let $(U, \mathcal{T}, V)$ be a standard triple similar to $(U_1, \mathcal{T}_1, V_1)$, that is, $(U_1, \mathcal{T}_1, V_1) = (UG, G^{-1}\mathcal{T}G, G^{-1}V)$ for some nonsingular matrix G. Let $\mathcal{S} \in \mathbb{S}$ and assume $\mathcal{T}$ satisfies assumption (b). If $(U, \mathcal{T}, V)$ is $\mathcal{S}$-structured with S-matrix S then $(U_1, \mathcal{T}_1, V_1)$ is $\mathcal{S}$-structured with S-matrix $S_1 = G^{-1}SG^{-\star}$.*

Proof. If $(U_1, \mathcal{T}_1, V_1) = (UG, G^{-1}\mathcal{T}G, G^{-1}V)$ with $(U, \mathcal{T}, V)$ $\mathcal{S}$-structured then

$$\begin{aligned}(V_1^\star G^\star u_{\mathcal{S}}(G\mathcal{T}_1G^{-1}), t_{\mathcal{S}}(G\mathcal{T}_1G^{-1}), v_{\mathcal{S}}(G\mathcal{T}_1G^{-1})G^{-\star}U_1^\star) &= (V^\star u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})U^\star)\\ &= (US, S^{-1}\mathcal{T}S, S^{-1}V)\\ &= (U_1G^{-1}S, S^{-1}G\mathcal{T}_1G^{-1}S, S^{-1}GV_1).\end{aligned}$$

Since $u_{\mathcal{S}}(G\mathcal{T}_1G^{-1}) = G^{-\star}u_{\mathcal{S}}(\mathcal{T}_1)G^\star$, $t_{\mathcal{S}}(G\mathcal{T}_1G^{-1}) = G^{-\star}u_{\mathcal{S}}(\mathcal{T}_1)G^\star$, and $v_{\mathcal{S}}(G\mathcal{T}_1G^{-1}) = G^{-\star}v_{\mathcal{S}}(\mathcal{T}_1)G^\star$, it follows that $(U_1, \mathcal{T}_1, V_1)$ is $\mathcal{S}$-structured with S-matrix $G^{-1}SG^{-\star}$. □

We can now state our main result, which is a direct consequence of Lemma 3.1 and Lemma 3.3. It extends a result for Hermitian structure [5, Thm. 12.2.2] to all structures in $\mathbb{S}$.

Theorem 3.4 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient satisfying assumption (a). Then $P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, in which case every standard triple for $P(\lambda)$ is $\mathcal{S}$-structured.*

The relations in (24) imply certain properties of S, as shown in the next theorem.

Theorem 3.5 *Let $\mathcal{S} \in \mathbb{S}$. An (m, n)-standard triple $(U, \mathcal{T}, V)$ with $\mathcal{T}$ satisfying assumption (b) is $\mathcal{S}$-structured with matrix S if and only if $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ and S satisfies the following properties:*

- *$S = S^\star$, $\mathcal{T}S = (\mathcal{T}S)^\star$ when $\mathcal{S} \in \{$Hermitian, symmetric$\}$,*
- *$S = -S^\star$, $\mathcal{T}S = (\mathcal{T}S)^\star$ when $\mathcal{S} = \star$-even,*
- *$S = S^\star$, $\mathcal{T}S = -(\mathcal{T}S)^\star$ when $\mathcal{S} = \star$-odd,*
- *$\mathcal{T}S^\star = -S$ when $\mathcal{S} = \star$-palindromic and $m = 2k+1$ or $\mathcal{T}S^\star = -\alpha S$ when $\mathcal{S} = \star$-palindromic and $m = 2k$,*
- *$\mathcal{T}S^\star = S$ when $\mathcal{S} = \star$-antipalindromic and $m = 2k+1$ or $\mathcal{T}S^\star = \alpha S$ when $\mathcal{S} = \star$-antipalindromic and $m = 2k$,*

for some $\alpha \in \mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \Lambda(\mathcal{T})$.

Proof. ($\Leftarrow$) Assume that $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ and that S satisfies the properties listed in the theorem. We show that (24) holds. The last equality follows from $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ and the second equality follows from the properties of S. Now from $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ we have that $V^\star u_{\mathcal{S}}(\mathcal{T}) = U(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T})$. That $(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T}) = S$ for $\mathcal{S} \in \{$Hermitian, symmetric, $\star$-even, $\star$-odd$\}$ follows from the definition of $u_{\mathcal{S}}, v_{\mathcal{S}}$ and the properties of S. For palindromic structures, $S^{-1}\mathcal{T}S = t_{\mathcal{S}}(\mathcal{T})$ implies that

$$S^\star(\mathcal{T}^\star)^{(k-1)} = \mathcal{T}^{-(k-1)}S^\star. \tag{25}$$

Hence, when $m = 2k + 1$,

$$(v_{\mathcal{S}}(\mathcal{T}))^{\star} S^{\star} u_{\mathcal{S}}(\mathcal{T}) = -\mathcal{T}^k S^{\star} \mathcal{T}^{\star(k-1)} = -\mathcal{T}^k \mathcal{T}^{-(k-1)} S^{\star} = -\mathcal{T} S^{\star} = S,$$

where we used (25) and the assumption that $\mathcal{T} S^{\star} = -S$. When $m = 2k$,

$$\begin{aligned}(v_{\mathcal{S}}(\mathcal{T}))^{\star} S^{\star} u_{\mathcal{S}}(\mathcal{T}) &= -\mathcal{T}^{(k-1)}(I + \alpha^{\star}\mathcal{T}) S^{\star} \mathcal{T}^{\star(k-1)} (I + \alpha \mathcal{T}^{\star})^{-1} \\ &= -(I + \alpha^{\star}\mathcal{T}) S^{\star} (I + \alpha \mathcal{T}^{\star})^{-1} \\ &= (S - S^{\star})(I + \alpha \mathcal{T}^{\star})^{-1} = S(I + \alpha \mathcal{T}^{\star})(I + \alpha \mathcal{T}^{\star})^{-1} = S.\end{aligned}$$

In a similar way we can show that $(v_{\mathcal{S}}(\mathcal{T}))^{\star} S^{\star} u_{\mathcal{S}}(\mathcal{T}) = S$ for antipalindromic structures. Hence $V^{\star} u_{\mathcal{S}}(\mathcal{T}) = US$.

($\Rightarrow$) Assume that $(U, \mathcal{T}, V)$ is $\mathcal{S}$-structured with S-matrix S so that (24) holds and hence $V = S v_{\mathcal{S}}(\mathcal{T}) U^{\star}$. By [11, Thm. 2.4] there exists a unique matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ for which $(U, \mathcal{T}, V)$ is a standard triple. This triple is similar to the primitive triple $(U_0, \mathcal{T}_0, V_0) = (e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$, where $A_m^{-1} = U \mathcal{T}^{m-1} V$. The proof of Lemma 3.1 shows that $(U_0, \mathcal{T}_0, V_0)$ is $\mathcal{S}$-structured with S-matrix $S_0 = G$ defined in (18). It is easy to check that $S_0 = G$ and $\mathcal{T}_0 = \mathcal{C}$ satisfy the properties displayed in the the bullet points of the theorem. By Lemma 3.3, $S = Q^{-1} S_0 Q^{-\star}$ and since $\mathcal{T} = Q^{-1} \mathcal{T}_0 Q$ (see (17)), we have that $\mathcal{T} S = Q^{-1} \mathcal{T}_0 S_0 Q^{-\star}$, $\mathcal{T} S^{\star} = Q^{-1} \mathcal{T}_0 S_0^{\star} Q^{-\star}$. This completes the proof since the properties of S_0 and $\mathcal{T}_0 S_0$ are preserved by $\star$-congruences and it is easy to check that $\mathcal{T} S^{\star}$ is the appropriate multiple of S for the (anti)palindromic structures. □

We point out that Hermitian and symmetric structured standard triples are called *self-adjoint standard triples* in the literature (see for example [5, p. 244]). For (anti)palindromic structures, the matrix $\mathcal{T}$ of an $\mathcal{S}$-structured standard triple $(U, \mathcal{T}, V)$ with S-matrix S is S^{-1}-unitary, that is, $\mathcal{T}^{\star} S^{-1} \mathcal{T} = S^{-1}$. With additional constraints on $\mathcal{T}$'s structure, Lancaster, Prells and Rodman refer to $(U, \mathcal{T}, V)$ as a *unitary standard triple* [8, Def. 4]. Hence a unitary standard triple is $\mathcal{S}$-structured but the converse is not true in general.

The S-matrix of an $\mathcal{S}$-structured standard triple $(U, \mathcal{T}, V)$ for $P(\lambda)$ can be expressed in terms of $U, \mathcal{T}$ and the matrix coefficients of $P(\lambda)$ as the next result shows.

Proposition 3.6 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ be of degree m with nonsingular leading coefficient and satisfying assumption* (a)*. If $(U, \mathcal{T})$ is a standard pair for $P(\lambda)$ then $(U, \mathcal{T}, S v_{\mathcal{S}}(\mathcal{T}) U^{\star})$ is an $\mathcal{S}$-structured standard triple for $P(\lambda)$ with S-matrix S given by*

$$S^{-1} = \begin{cases} z^{-\star} Q^{\star} \mathcal{A}_{-}^{even}(z) Q & \text{if } P \text{ is } \star\text{-(anti)palindromic, } m = 2k,\ -z/z^{\star} \notin \Lambda(P), \\ Q^{\star} \mathcal{A}_{\mathcal{S}} Q & \text{otherwise,} \end{cases}$$

where $Q := Q(U, \mathcal{T})$ is as in (10)*, and $\mathcal{A}_{\mathcal{S}}$ and $\mathcal{A}_{-}^{even}(z)$ are as in* (4)*–*(9)*.*

Proof. The primitive standard triple $(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$ is $\mathcal{S}$-structured with matrix G defined in (18). Since $(U, \mathcal{T})$ is a standard pair of $P(\lambda)$, we easily check that $Q^{-1} \mathcal{C} Q = \mathcal{T}$ and $\left(e_m^T \otimes I_n\right) Q = U$. Define $V = Q^{-1}\left(e_1 \otimes A_m^{-1}\right)$. Then $(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ similar to $\left(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1}\right)$. By Lemma 3.3, $(U, \mathcal{T}, V)$ is $\mathcal{S}$-structured with matrix $S = Q^{-1} G Q^{-\star}$ and $V = S v_{\mathcal{S}}(\mathcal{T}) U^{\star}$. □

4. $\mathcal{S}$-structured Jordan triples

We now explain how to obtain explicit expressions for the Jordan matrix and S-matrix of $\mathcal{S}$-structured Jordan triples $(X, J, S_J v_{\mathcal{S}}(J)X^\star)$ of $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. We note that the matrix S_J displays the sign characteristic of $P(\lambda)$, whose definition we now give.

Let $(U, \mathcal{T}, S_{\mathcal{T}} v_{\mathcal{S}}(\mathcal{T})U^\star)$ be a standard triple for $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. The *sign characteristic* of $P(\lambda)$ is defined as the sign characteristic of the pair $(\mathcal{T}, S_{\mathcal{T}}^{-1})$, which is a list of signs, with a sign ($+1$ or -1) attached to each partial multiplicity of

- real eigenvalues of Hermitian or real symmetric matrix polynomials,
- purely imaginary eigenvalues of $*$-even, $*$-odd, real T-even and real T-odd matrix polynomials,
- eigenvalues with unit modulus of $*$-(anti)palindromic and real T-(anti)palindromic matrix polynomials.

These signs can be read off the canonical decomposition of $\lambda S_{\mathcal{T}}^{-1} - S_{\mathcal{T}}^{-1}\mathcal{T}$ via $\star$-congruence (see [5, Sec. 12.4] for Hermitian structure). Note that the definition of the sign characteristic for $P(\lambda)$ is independent of the choice of standard triple. Indeed if $(U_i, \mathcal{T}_i, S_{\mathcal{T}_i} v_{\mathcal{S}}(\mathcal{T}_i)U_i^\star)$, $i = 1, 2$ are $\mathcal{S}$-structured standard triples for $P(\lambda)$, then by Lemma 3.3 there exists a nonsingular G such that $\mathcal{T}_2 = G^{-1}\mathcal{T}_1 G$ and $S_{\mathcal{T}_2} = G^{-1}S_{\mathcal{T}_1}G^{-\star}$. Hence, $\lambda S_{\mathcal{T}_2}^{-1} - S_{\mathcal{T}_2}^{-1}\mathcal{T}_2 = G^\star(\lambda S_{\mathcal{T}_1}^{-1} - S_{\mathcal{T}_1}^{-1}\mathcal{T}_1)G$, that is, the pencils $\lambda S_{\mathcal{T}_i}^{-1} - S_{\mathcal{T}_i}^{-1}\mathcal{T}_i$, $i = 1, 2$ are $\star$-congruent. They share the same canonical decomposition via $\star$-congruence and therefore the same sign characteristic.

We know that the triple $\big((e_m^T \otimes I_n), \mathcal{C}, (e_1 \otimes A_m^{-1})\big)$ is a standard triple for $P(\lambda)$ and by Theorem 3.4, it is $\mathcal{S}$-structured with S-matrix as in Proposition 3.6 with $Q = I_{mn}$. Hence, on using Lemma 2.1, we find that

$$\lambda S_{\mathcal{C}}^{-1} - S_{\mathcal{C}}^{-1}\mathcal{C} = \lambda z^{-\star}\mathcal{A}_{\mathcal{S}} + z^{-\star}\mathcal{B}_{\mathcal{S}},$$

where $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ is a structured linearization of $P(\lambda)$ as in (4)–(9), and $z = 1$ except when $\mathcal{A}_{\mathcal{S}} = \mathcal{A}_{-}^{even}(z)$, in which case $z \in \mathbb{F}$ is chosen such that $-z/z^\star \notin \Lambda(P)$. So what we need is a canonical decomposition of $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ via $\star$-congruence,

$$Z^\star(\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}})Z = \lambda(Z^\star\mathcal{A}_{\mathcal{S}}Z) - (Z^\star\mathcal{A}_{\mathcal{S}}Z)\big(Z^{-1}\mathcal{C}Z\big) = z^\star(\lambda S_J^{-1} - S_J^{-1}J),$$

where $J = Z^{-1}\mathcal{C}Z$ is the Jordan form of $\mathcal{C}$. Fortunately, such decompositions are available in the literature for all the structures in $\mathbb{S}$. We use these canonical decompositions to provide explicit expressions for J and S_J in Appendix A. These expressions show that S_J and J have the same block structure and that we can read the sign characteristic of $P(\lambda)$ from certain diagonal blocks of S_J.

5. Concluding remarks

The results in this paper represent a first step towards the solution of the structured inverse polynomial eigenvalue problem: given a list of admissible elementary divisors for the structure, and possibly, corresponding right eigenvectors and generalized eigenvectors, construct a structured matrix polynomial having these elementary divisors and eigenvectors/generalized eigenvectors. Indeed, using the results in sections 3 and 4 we show in [1] how to construct an $\mathcal{S}$-structured $(2, n)$-Jordan triple (X, J, Y) from a given list of $2n$ prescribed eigenvalues and n linearly independent eigenvectors and generalized eigenvectors, and use the fact that an $\mathcal{S}$-structured $(2, n)$-Jordan triple defines a unique structured quadratic $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0 \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$, where $A_2 = (XJSv_{\mathcal{S}}(J)X^{\star})^{-1}$,

$$A_1 = -A_2 X J^2 S v_{\mathcal{S}}(J) X^{\star} A_2, \quad A_0 = -A_2(XJ^2 S v_{\mathcal{S}}(J) X^{\star} A_1 + XJ^3 S v_{\mathcal{S}}(J) X^{\star} A_2),$$

and $v_{\mathcal{S}}(\cdot)$ as in Table 2.

Finally, we note that standard triples have been useful to describe structure preserving transformations (SPTs) for matrix polynomials, and in particular quadratic matrix polynomials [3]. We believe that the notion of $\mathcal{S}$-structured standard triples will further our understanding of SPTs for structured matrix polynomials.

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Appendix A. Explicit expressions for J and S_J

Using the canonical decompositions of structured pencils via $\star$-congruences, we provide in this appendix an explicit expression for the Jordan matrix and S-matrix of $\mathcal{S}$-structured Jordan triples $(X, J, S_J v_{\mathcal{S}}(J) X^{\star})$ of $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ for each $\mathcal{S} \in \mathbb{S}$. We assume that $P(\lambda)$ is of degree m with nonsingular leading coefficient matrix. To facilitate the description of J and S_J, we introduce the matrices $E_1 = F_1 = [1]$ and for integers $k > 1$

$$E_k = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & 1 \;\cdot^{\cdot^{\cdot}} & & \\ (-1)^{k-1} & & & \end{bmatrix}_{k\times k} = (-1)^{k-1} E_k^T, \qquad F_k = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}_{k\times k}.$$

We denote by

$$J_{\ell_k}(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{C}^{\ell_k \times \ell_k},$$

the Jordan block of size ℓ_k associated with λ_k, and by

$$K_{2m_k}(\lambda_k, \bar{\lambda}_k) = K_{2m_k}(\Lambda_k) = \begin{bmatrix} \Lambda_k & I_2 & & \\ & \Lambda_k & \ddots & \\ & & \ddots & I_2 \\ & & & \Lambda_k \end{bmatrix} \in \mathbb{R}^{2m_k \times 2m_k}, \quad \Lambda_k = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix},$$

the $2m_k \times 2m_k$ real Jordan block associated with the pair of complex conjugate eigenvalues $(\lambda_k, \bar{\lambda}_k)$, where $\lambda_k = \alpha_k + i\beta_k$ with $\alpha_k, \beta_k \in \mathbb{R}$, $\beta_k \neq 0$. We use the notation $\bigoplus_{j=1}^r F_j$ to denote the direct sum of the matrices $F_1, \ldots, F_r$.

Note that there are restrictions on the Jordan structure of P. For instance, a regular $n \times n$ matrix polynomial cannot have more than n elementary divisors associated with the same eigenvalue [13]. Also, the elementary divisors have certain pairing, which depends on the structure $\mathcal{S} \in \mathbb{S}$ and the eigenvalue. Hence we describe for each $\mathcal{S} \in \mathbb{S}$ the elementary divisors arising from $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ and then provide an expression for J and S_J.

Appendix A.1. Hermitian structure

Suppose $P(\lambda)$ is Hermitian with

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1\colon r$, and
- s pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \overline{\mu}_j)^{m_j}$, $j = 1\colon s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. It follows from [9, Thm. 6.1] that

$$J = \bigoplus_{j=1}^{r} J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^{s} (J_{m_j}(\bar{\mu}_j) \oplus J_{m_j}(\mu_j)), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j}.$$

Here $\{\varepsilon_1, \ldots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the real eigenvalues λ_j, $j = 1\colon r$ of $P(\lambda)$. We easily check that $S_J = S_J^*$ and $JS_J = (JS_J)^*$.

Appendix A.2. Real symmetric structure

Suppose $P(\lambda)$ is real symmetric with

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1\colon r$, and
- s pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \overline{\mu}_j)^{m_j}$, $j = 1\colon s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. On using [9, Thm. 9.2] we find that

$$J = \bigoplus_{j=1}^{r} J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^{s} K_{2m_j}(\mu_j, \bar{\mu}_j), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the real eigenvalues of $P(\lambda)$. Note that $S_J = S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.3. Complex symmetric structure

Suppose $P(\lambda)$ is complex symmetric with q elementary divisors $(\lambda - \lambda_j)^{m_j}$, $\lambda_j \in \mathbb{C}$, $j = 1\colon q$, with m_j such that $\sum_{j=1}^{q} m_j = mn$. Then [19, Prop. 4.3] leads to

$$J = \bigoplus_{j=1}^{q} J_{m_j}(\lambda_j), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{q} F_{m_j},$$

which satisfy $S_J = S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.4. $*$-even structure

Suppose $P(\lambda)$ is $*$-even with

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1\colon r$, and
- s pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\overline{\mu}_j)^{m_j}$, $j = 1\colon s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. With the change of eigenvalue parameter $\lambda = -i\mu$, the $*$-even linearization of $P(\lambda)$, $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \mu(-i\mathcal{A}_{\mathcal{S}}) + \mathcal{B}_{\mathcal{S}}$ becomes a Hermitian pencil in μ. Using Appendix A.1 we obtain that

$$J = -i\Big(\bigoplus_{j=1}^{r} J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^{s} \big(J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)\big)\Big), \quad S_J = -i\Big(\bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j}\Big).$$

Here $\{\varepsilon_1, \ldots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the zero and purely imaginary eigenvalues of $P(\lambda)$. Note that $S_J = -S_J^*$ and $JS_J = (JS_J)^*$.

Appendix A.5. Real T-even structure

Suppose $P(\lambda)$ is real T-even with (see [16])

- t zero elementary divisors λ^{n_j} with n_j even, $j = 1\colon t$,
- r pairs of real elementary divisors $(\lambda + \alpha_j)^{p_j}$, $(\lambda - \alpha_j)^{p_j}$ with p_j odd if $\alpha_j = 0$, $j = 1\colon r$,
- s pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1\colon s$, and
- q quadruples of nonzero and non-purely imaginary elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \overline{\mu}_j)^{m_j}$, $(\lambda - \overline{\mu}_j)^{m_j}$, $j = 1\colon q$,

with n_j, p_j, k_j, m_j such that $\sum_{j=1}^{t} n_j + 2\sum_{j=1}^{r} p_j + 2\sum_{j=1}^{s} k_j + 4\sum_{j=1}^{q} m_j = mn$. Using [10, Thm. 16.1], we find that

$$\begin{aligned} J = &\bigoplus_{j=1}^{t} J_{n_j}(0) \oplus \bigoplus_{j=1}^{r} \big(J_{p_j}(\alpha_j) \oplus -J_{p_j}(\alpha_j)^T\big) \\ &\oplus \bigoplus_{j=1}^{s} K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^{q} \big(K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T\big), \end{aligned}$$

$$S_J = \bigoplus_{j=1}^{t} \varepsilon_j E_{n_j} \oplus \bigoplus_{j=1}^{r} \begin{bmatrix} 0 & -I_{p_j} \\ I_{p_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^{s} \varepsilon_j (E_{k_j} \otimes E_2^{k_j}) \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & -I_{2m_j} \\ I_{2m_j} & 0 \end{bmatrix},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the purely imaginary eigenvalues and zero eigenvalues of even partial multiplicities (see [18]). We easily check that $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.6. Complex T-even structure

Let $\lambda_j \in \mathbb{C} \setminus \{0\}$ and suppose $P(\lambda)$ is complex T-even with (see [16])

- t zero elementary divisors λ^{m_j} with m_j even, $j = 1{:}\, t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda + \lambda_j)^{k_j}$ with k_j odd if $\lambda_j = 0$, $j = 1{:}\, q$,

with m_j, k_j such that $\sum_{j=1}^{r} m_j + 2\sum_{j=1}^{q} k_j = mn$. Then, by [19, Prop. 4.7 (b)], we obtain that

$$J = \bigoplus_{j=1}^{t} J_{m_j}(0) \oplus \bigoplus_{j=1}^{q} (J_{k_j}(\lambda_j) \oplus J_{k_j}(-\lambda_j)), \quad S_J = \bigoplus_{j=1}^{t} \begin{bmatrix} 0 & -F_{\frac{1}{2}m_j} \\ F_{\frac{1}{2}m_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & -F_{k_j} \\ F_{k_j} & 0 \end{bmatrix}.$$

Note that $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.7. ∗-odd structure

Suppose $P(\lambda)$ is ∗-odd with

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1{:}\, r$ and
- s pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\bar{\mu}_j)^{m_j}$, $j = 1{:}\, s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. Note that for the ∗-odd linearization $\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S}$ of $P(\lambda)$ in (4), the pencil $i(\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S})$ is ∗-even and the structure for S_J and J follows from Appendix A.4. We find that

$$J = -i\Big(\bigoplus_{j=1}^{r} J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^{s} \big(J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)\big)\Big), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j},$$

which satisfy $S_J = S_J^*$ and $JS_J = -(JS_J)^*$. Here $\{\varepsilon_1, \ldots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the zero and purely imaginary eigenvalues of $P(\lambda)$.

Appendix A.8. Real T-odd structure

Suppose $P(\lambda)$ is real T-odd with (see [16])

- t zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1{:}\, t$,
- r pairs of real elementary divisors $(\lambda + \alpha_j)^{p_j}$, $(\lambda - \alpha_j)^{p_j}$ with p_j even if $\alpha_j = 0$, $j = 1{:}\, r$,
- s pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1{:}\, s$, and
- q quadruples elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \bar{\mu}_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$, $j = 1{:}\, q$,

with ℓ_j, p_j, k_j, m_j such that $\sum_{j=1}^{t} \ell_j + 2\sum_{j=1}^{r} p_j + 2\sum_{j=1}^{s} k_j + 4\sum_{j=1}^{q} m_j = mn$. On using [10, Thm. 17.1] we find that

$$\begin{aligned} J = {} & \bigoplus_{j=1}^{t} J_{\ell_j}(0) \oplus \bigoplus_{j=1}^{r} \big(J_{p_j}(\alpha_j) \oplus -J_{p_j}(\alpha_j)^T\big) \\ & \oplus \bigoplus_{j=1}^{s} K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^{q} \big(K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T\big), \end{aligned}$$

$$S_J = S_J^{-1} = \bigoplus_{j=1}^{t} \varepsilon_j E_{\ell_j} \oplus \bigoplus_{j=1}^{r} \begin{bmatrix} 0 & I_{p_j} \\ I_{p_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^{s} \varepsilon_j (E_{k_j} \otimes E_2^{k_j-1}) \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & I_{2m_j} \\ I_{2m_j} & 0 \end{bmatrix},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the purely imaginary eigenvalues and the zero eigenvalues with odd partial multiplicities. We easily check that $S_J = S_J^T$ and $JS_J = -(JS_J)^T$.

Appendix A.9. Complex T-odd structure

Let $\lambda_j \in \mathbb{C} \setminus \{0\}$ and suppose $P(\lambda)$ is complex T-odd with (see [16])

- s zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1\colon s$, and
- q pairs of elementary divisors $(\lambda + \lambda_j)^{k_j}$, $(\lambda - \lambda_j)^{k_j}$ with k_j even if $\lambda_j = 0$, $j = 1\colon q$,

with ℓ_j, k_j such that $\sum_{j=1}^{s} \ell_j + 2\sum_{j=1}^{q} k_j = mn$. It follows from [19, Prop. 4.7 (b)] that

$$J = \bigoplus_{j=1}^{s} J_{\ell_j}(0) \oplus \bigoplus_{j=1}^{q} \left(-J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j)\right), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{s} E_{\ell_j} \oplus \bigoplus_{j=1}^{q} F_{2k_j}.$$

Clearly, $S_J = S_J^T$ and $JS_J = -(JS_J)^T$.

Notice the difference between the zero elementary divisors associated with T-even and T-odd pencils (see [16, Cor. 4.3]).

Appendix A.10. ∗-(anti)palindromic structure

Suppose $P(\lambda)$ is complex $*$-palindromic with $-1 \notin \Lambda(P)$ and (see [17])

- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\overline{\lambda}_j)^{k_j}$ with $\lambda_j \in \mathbb{C} \setminus \{0\}$, $|\lambda_j| \neq 1$, $j = 1\colon q$,
- t elementary divisors $(\lambda - \lambda_j)^{2\ell_j+1}$ with $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1$, $j = 1\colon t$, and
- s elementary divisors $(\lambda - \lambda_j)^{2m_j}$ with $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$, $j = 1\colon s$,

with k_j, ℓ_j, m_j such that $2\sum_{j=1}^{q} k_j + \sum_{j=1}^{t}(2\ell_j + 1) + 2\sum_{j=1}^{s} m_j = mn$. Then using either [20, Thm. 5] or [21, Sec. 2.2.2] we find that

$$J = -S_J S_J^{-*}$$

with

$$S_J = \bigoplus_{j=1}^{q} \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{t} \varepsilon_j \begin{bmatrix} 0 & 0 & F_{\ell_j} J_{\ell_j}(-\lambda_j) \\ 0 & (-\lambda_j)^{1/2} & e_1^T \\ F_{\ell_j} & 0 & 0 \end{bmatrix}$$
$$\oplus \bigoplus_{j=1}^{s} \varepsilon_j \begin{bmatrix} 0_{m_j} & F_{m_j} J_{m_j}(-\lambda_j) \\ F_{m_j} & e_1 e_1^T \end{bmatrix}$$

has the above elementary divisors. Here e_1 is the first column of the identity matrix. The scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the eigenvalues of unit modulus of $P(\lambda)$ (see [8]).

For the $*$-antipalindromic structure, $J = S_J S_J^{-*}$ with S_J as above but with $-\lambda_j$ replaced by λ_j.

Appendix A.11. Real T-(anti)palindromic structure

Suppose $P(\lambda)$ is real T-palindromic with $-1 \notin \Lambda(P)$, $\lambda_j \in \mathbb{C} \setminus \{0\}$, and (see [17])

- r pairs of real elementary divisors $(\lambda-\lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with $\lambda_j \in \mathbb{R}$, $|\lambda_j| \neq 1$, $j = 1\colon r$,
- q quadruples of nonreal elementary divisors $(\lambda-\lambda_j)^{n_j}$, $(\lambda-\overline{\lambda}_j)^{n_j}$, $(\lambda-1/\lambda_j)^{n_j}$, $(\lambda-1/\overline{\lambda}_j)^{n_j}$ with $|\lambda_j| \neq 1$, $j = 1\colon q$,
- s elementary divisors $(\lambda-1)^{2m_j}$, $j = 1\colon s$,
- t pairs of elementary divisors $(\lambda-1)^{2\ell_j+1}$, $(\lambda-1)^{2\ell_j+1}$, $j = 1\colon t$,
- u pairs of elementary divisors $(\lambda-\lambda_j)^{\ell'_j}$, $(\lambda-\overline{\lambda}_j)^{\ell'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, ℓ'_j odd, $j = 1\colon u$, and
- p pairs of elementary divisors $(\lambda-\lambda_j)^{m'_j}$, $(\lambda-\overline{\lambda}_j)^{m'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, m'_j even, $j = 1\colon p$.

We have that $2\sum_{j=1}^{r} k_j + 4\sum_{j=1}^{q} n_j + 2\sum_{j=1}^{s} m_j + 2\sum_{j=1}^{t}(2\ell_j+1) + 2\sum_{j=1}^{u}\ell'_j + 2\sum_{j=1}^{p} m'_j = mn$.

Using [21, Thm. 2.8] we find that $J = -S_J S_J^{-T}$ has the above list of elementary divisors, where

$$
\begin{aligned}
S_J = {} & \bigoplus_{j=1}^{r} \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0_{2n_j} & K_{2n_j}(-\Lambda_j) \\ F_{n_j} \otimes I_2 & 0_{2n_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{s} \begin{bmatrix} 0 & F_{m_j} J_{m_j}(-1) \\ F_{m_j} & 0 \end{bmatrix} \\
& \oplus \bigoplus_{j=1}^{t} \varepsilon_j \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(-1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{t} \varepsilon_j \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(-1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \\
& \oplus \bigoplus_{j=1}^{u} \varepsilon_j \begin{bmatrix} 0_{\ell'_j-1} & 0 & K_{\ell'_j-1}(-\Lambda_j) \\ 0 & (-\Lambda_j)^{\frac{1}{2}} & e_1^T \otimes I_2 \\ F_{\frac{1}{2}(\ell'_j-1)} \otimes I_2 & 0 & 0_{\ell'_j-1} \end{bmatrix} \oplus \bigoplus_{j=1}^{p} \varepsilon_j \begin{bmatrix} 0_{m'_j} & K_{m'_j}(-\Lambda_j) \\ F_{\frac{1}{2}m'_j} \otimes I_2 & e_1 e_1^T \otimes I_2 \end{bmatrix}.
\end{aligned}
$$

Here $(-\Lambda_j)^{\frac{1}{2}}$ is the principal square root of $-\Lambda_j$. The scalars ε_j are signs ± 1 and form the sign characteristic associated with the eigenvalues of unit modulus of $P(\lambda)$ except the eigenvalues 1 with even partial multiplicities (see [8]).

For the T-antipalindromic $P(\lambda)$, $J = S_J S_J^{-T}$ where S_J is as above but with $-\lambda_j, -1, -\Lambda_j$ replaced by $\lambda_j, 1, \Lambda_j$, respectively.

Appendix A.12. Complex T-(anti)palindromic structure

Suppose $P(\lambda)$ is complex T-palindromic with $-1 \notin \Lambda(P)$ and (see [17])

- t elementary divisors $(\lambda-1)^{m_j}$ with m_j even, $j = 1\colon t$,
- q pairs of elementary divisors $(\lambda-\lambda_j)^{k_j}$, $(\lambda-1/\lambda_j)^{k_j}$ with k_j odd when $\lambda_j = 1$, $j = 1\colon q$,

with m_j, k_j such that $\sum_{j=1}^{t} m_j + 2\sum_{j=1}^{q} k_j = mn$. On using either [20, Thm. 1] or [21, Thm. 2.6], we find that with

$$
S_J = \bigoplus_{j=1}^{t} \begin{bmatrix} 0_{m_j/2} & F_{m_j/2} J_{m_j/2}(-1) \\ F_{m_j/2} & e_1 e_1^T \end{bmatrix} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix}
$$

the matrix $J = -S_J S_J^{-T}$ has the above elementary divisors.

Now if $P(\lambda)$ is complex T-antipalindromic with $-1 \notin \Lambda(P)$ and (see [17])

- t elementary divisors $(\lambda - 1)^{\ell_j}$ with ℓ_j odd, $j = 1\colon t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\lambda_j)^{k_j}$ with k_j even if $\lambda_j = 1$, $j = 1\colon q$,

with ℓ_j, k_j such that $\sum_{j=1}^{t} \ell_j + 2\sum_{j=1}^{q} k_j = mn$. On using [21, Thm. 2.6], we find that the matrix $J = S_J S_J^{-T}$ with

$$S_J = \bigoplus_{j=1}^{t} \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix}$$

has the above elementary divisors.

Note that J in Appendix A.10–Appendix A.12 is "almost" in Jordan canonical form.

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