Standard Triples of Structured Matrix Polynomials

Al-Ammari, Maha and Tisseur, Francoise

2011

MIMS EPrint: **2011.37**

ISSN 1749-9097

Standard Triples of Structured Matrix Polynomials ☆

Maha Al-Ammari[a,1], Françoise Tisseur[a,2,*]

[a]*School of Mathematics, The University of Manchester, Manchester, M13 9PL, UK*

Abstract

The notion of standard triples plays a central role in the theory of matrix polynomials. We study such triples for matrix polynomials $P(\lambda)$ with structure $\mathcal{S}$, where $\mathcal{S}$ is the Hermitian, symmetric, $\star$-even, $\star$-odd, $\star$-palindromic or $\star$-antipalindromic structure (with $\star = *, T$). We introduce the notion of $\mathcal{S}$-structured standard triple. With the exception of T-(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, we show that $P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, and moreover that every standard triple of a matrix polynomial with structure $\mathcal{S}$ is $\mathcal{S}$-structured. We investigate the important special case of $\mathcal{S}$-structured Jordan triples.

Keywords: standard triple, Jordan triple, structured matrix polynomial, Hermitian matrix polynomial, symmetric matrix polynomial, palindromic matrix polynomial, even matrix polynomial, odd matrix polynomial
2000 MSC: 15A18, 65F15

1. Introduction

Standard and Jordan triples for matrix polynomials were introduced and developed by Gohberg, Lancaster and Rodman (see for example [4], [5], [6]). Jordan triples extend to matrix polynomials of degree m

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \quad A_j \in \mathbb{F}^{n\times n}, \quad \det(A_m) \neq 0, \tag{1}$$

the notion of Jordan pair (X, J) for a single matrix $A \in \mathbb{C}^{n\times n}$, where $X \in \mathbb{C}^{n\times n}$ is nonsingular, J is a Jordan canonical form for A, and $A = XJX^{-1}$. The matrix X in a Jordan triple (X, J, Y) for $P(\lambda)$ is $n \times mn$ and, as for the single matrix case, it contains

☆Version of November 25, 2011
*Corresponding author.
Email addresses: `malammari@ksu.edu.sa` (Maha Al-Ammari), `ftisseur@ma.man.ac.uk` (Françoise Tisseur)
[1]The work of this author was supported by King Saud University, Riyadh, Saudi Arabia.
[2]The work of this author was supported by Engineering and Physical Sciences Research Council grant EP/I005293 and a Fellowship from the Leverhulme Trust.

Table 1: Matrix polynomials $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ with structure $\mathcal{S} \in \mathbb{S}$.

Structure $\mathcal{S}$	**Definition**	**Coefficients property**
Hermitian	$P(\lambda) = P^*(\lambda)$	$A_j = A_j^*$
symmetric	$P(\lambda) = P^T(\lambda)$	$A_j = A_j^T$
$\star$-even	$P(\lambda) = P^\star(-\lambda)$	$A_j = (-1)^j A_j^\star$
$\star$-odd	$P(\lambda) = -P^\star(-\lambda)$	$A_j = (-1)^{j+1} A_j^\star$
$\star$-palindromic	$P(\lambda) = \lambda^m P^\star(\frac{1}{\lambda})$	$A_j = A_{m-j}^\star$
$\star$-antipalindromic	$P(\lambda) = -\lambda^m P^\star(\frac{1}{\lambda})$	$A_j = -A_{m-j}^\star$

the right eigenvectors and generalized eigenvectors of $P(\lambda)$. The matrix $J \in \mathbb{C}^{mn\times mn}$ is in Jordan canonical form, displaying the elementary divisors of $P(\lambda)$, and the matrix $Y \in \mathbb{C}^{mn\times n}$ plays the role of X^{-1} for a single matrix, i.e., the columns of Y^* determine left eigenvectors and generalized eigenvectors of $P(\lambda)$. A Jordan triple is a particular standard triple $(U, \mathcal{T}, V)$ in which the matrix $\mathcal{T}$ is in canonical form. Standard and Jordan triples are defined precisely in section 2.2.

Our objective is to study the standard and Jordan triples of structured matrix polynomials $P(\lambda)$ of the types listed in Table 1, where we use $\star$ to denote the transpose T for real matrices and either the transpose T or the conjugate transpose $*$ for matrices with complex entries. The structure of standard and Jordan triples are well understood for Hermitian matrix polynomials [4], [5] and more recently real symmetric matrix polynomials [2], [11]. With no assumption on the sizes of the Jordan blocks, Gohberg, Lancaster and Rodman [4] show that if (X, J, Y) is a Jordan triple for a Hermitian matrix polynomial then $Y = SX^*$ for some nonsingular $mn \times mn$ matrix S such that $S = S^*$ and $JS = (JS)^*$. We show in section 3 that results of this type also hold for the structures in $\mathbb{S}$, where

$$\begin{aligned}\mathbb{S} = \{&\text{Hermitian, symmetric, } *\text{-even, } *\text{-odd, } T\text{-even, } T\text{-odd,} \\ &*\text{-palindromic, } *\text{-antipalindromic, } T\text{-palindromic, } T\text{-antipalindromic}\}.\end{aligned} \tag{2}$$

For $\mathcal{S} \in \mathbb{S}$, we introduce the notion of $\mathcal{S}$-structured standard triples. With the exception of T-(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, we show that $P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, and that for any $P(\lambda)$ with structure $\mathcal{S}$, all standard triples for $P(\lambda)$ are $\mathcal{S}$-structured. Finally, we study in section 4 the special case of $\mathcal{S}$-structured Jordan triples.

Two important features of this work are (a) a distinction, when necessary, between triples and matrix polynomials defined over the complex ($\mathbb{C}$) or real ($\mathbb{R}$) fields, and (b) a unified presentation of the results, except in section 4, where we provide explicit expressions for the S-matrix of $\mathcal{S}$-structured Jordan triples that are structure-dependent.

2. Preliminaries

The set of all matrix polynomials with coefficient matrices in $\mathbb{F}^{n\times n}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is denoted by $\mathcal{P}(\mathbb{F}^n)$. When the polynomials are structured with structure $\mathcal{S}$, the corresponding set is denoted by $\mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ (see Table 1). Throughout this paper we assume that $P(\lambda)$ has a nonsingular leading coefficient matrix as in (1). Recall that λ is an eigenvalue of $P(\lambda)$ with corresponding right eigenvector $x \neq 0$ and left eigenvector $y \neq 0$ if $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$. We denote by $\Lambda(P)$ the set of eigenvalues of $P(\lambda)$.

2.1. Structured linearizations

Linearizations play a major role in the theory of matrix polynomials. They are $mn \times mn$ linear matrix polynomials $L(\lambda) = \lambda\mathcal{A} + \mathcal{B}$ related to $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ of degree m by

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

for some matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants. For example, the companion form

$$\mathcal{C} = -\begin{bmatrix} A_m^{-1}A_{m-1} & A_m^{-1}A_{m-2} & \dots & A_m^{-1}A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} \tag{3}$$

of $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ defines a linearization $\lambda I - \mathcal{C}$ of $P(\lambda)$.

Some of the results in section 3 and all the results in section 4 rely on the construction of linearizations that preserve the structure of $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. The vector space of pencils

$$\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{F}^m \,\},$$

introduced in [15], provides a rich source of such linearizations. Here $\Lambda = [\,\lambda^{m-1} \quad \dots \quad \lambda \quad 1\,]^T$. It is shown in [7], [12], [14] that for some $v \in \mathbb{F}^m$ satisfying the admissible constraint

(i) $v \in \mathbb{R}^m$ if $\mathcal{S} =$ Hermitian,
(ii) $v = \Sigma_m v$ if $\mathcal{S} \in \{T\text{-even}, T\text{-odd}\}$ or $v = \Sigma_m \bar{v}$ if $\mathcal{S} \in \{*\text{-even}, *\text{-odd}\}$,
(iii) $v = F_m v$ if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ or $v = F_m\bar{v}$ if $\mathcal{S} \in \{*\text{-palindromic}, *\text{-antipalindromic}\}$,

where

$$\Sigma_m = \mathrm{diag}((-1)^{m-1}, \dots, (-1)^0), \qquad F_m = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix},$$

there exists a unique pencil $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} \in \mathbb{L}_1(P)$ with structure $\mathcal{S} \in \mathbb{S}$. This pencil is a linearization of $P(\lambda)$ if the roots of the v-polynomial

$$\mathsf{p}(x; v) = v_1 x^{m-1} + v_2 x^{m-2} + \dots + v_{m-1}x + v_m$$

are not eigenvalues of P [14, Thm. 6.3 & Thm. 6.5]. The vector $v = e_m$, where e_m is the mth column of the $m \times m$ identity matrix, is an admissible vector for $\mathcal{S} \in$ {Hermitian, symmetric, $\star$-even, $\star$-odd} since $e_m \in \mathbb{R}^m$ and $\Sigma_m e_m = e_m$. Also, the roots of $\mathsf{p}(x; e_m)$ are all equal to ∞ and since $\det(A_m) \neq 0$ then $\infty \notin \Lambda(P)$. Hence the structured pencils $\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} \in \mathbb{L}_1(P)$ with vector e_m are linearizations of P. They are given by (see [7] and [14] for the construction)

$$\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \begin{cases} \lambda \mathcal{A}(1) + \mathcal{B}(1) \text{ when } \mathcal{S} \in \{\text{Hermitian, symmetric}\}, \\ \lambda \mathcal{A}(-1) + \mathcal{B}(-1) \text{ when } \mathcal{S} \in \{\star\text{-even, } \star\text{-odd}\}, \end{cases} \tag{4}$$

where

$$\mathcal{A}(\varepsilon) = \begin{bmatrix} 0 & \cdots & 0 & \varepsilon^{m-1} A_m \\ \vdots & & \iddots & \varepsilon^{m-2} A_{m-1} \\ \vdots & \iddots & \iddots & \vdots \\ \varepsilon^0 A_m & \varepsilon^0 A_{m-1} & \cdots & \varepsilon^0 A_1 \end{bmatrix},$$

and

$$\mathcal{B}(\varepsilon) = -\begin{bmatrix} 0 & \cdots & 0 & \varepsilon^{m-1} A_m & 0 \\ \vdots & \iddots & \varepsilon^{m-2} A_m & \varepsilon^{m-2} A_{m-1} & \vdots \\ 0 & \iddots & \iddots & \vdots & \vdots \\ \varepsilon A_m & \varepsilon A_{m-1} & \cdots & \varepsilon A_2 & 0 \\ 0 & \cdots & \cdots & 0 & -A_0 \end{bmatrix}.$$

Note that for $\star$-(anti)palindromic $P(\lambda)$, we have $0 \notin \Lambda(P)$ since $\infty \notin \Lambda(P)$. When $m = 2k+1$, $v = e_{k+1}$ satisfies $v = F_m v = F_m \bar{v}$ and $0, \infty$ are the only roots of the v-polynomial. The corresponding $\star$-(anti)palindromic pencils in $\mathbb{L}_1(P)$ are linearizations. They are given by (see [14] for the construction)

$$\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \begin{cases} \lambda \mathcal{A}^{odd} + (\mathcal{A}^{odd})^\star & \text{when } \mathcal{S} = \star\text{-palindromic with } m = 2k+1, \\ \lambda \mathcal{A}^{odd} - (\mathcal{A}^{odd})^\star & \text{when } \mathcal{S} = \star\text{-antipalindromic with } m = 2k+1, \end{cases} \tag{5}$$

where

$$\mathcal{A}^{odd} = \begin{bmatrix} \mathcal{A}_{11}^{odd} & \mathcal{A}_{12}^{odd} \\ \mathcal{A}_{21}^{odd} & \mathcal{A}_{22}^{odd} \end{bmatrix}, \tag{6}$$

with $\mathcal{A}_{11}^{odd} = (\mathcal{A}_{22}^{odd})^T = 0_{nk \times n(k+1)}$ and

$$\mathcal{A}_{12}^{odd} = \begin{bmatrix} -A_m^\star & 0 & \cdots & 0 \\ -A_{m-1}^\star & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -A_{k+2}^\star & \cdots & -A_{m-1}^\star & -A_m^\star \end{bmatrix}, \quad \mathcal{A}_{21}^{odd} = \begin{bmatrix} A_m & A_{m-1} & \cdots & A_{k+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{m-1} \\ 0 & \cdots & 0 & A_m \end{bmatrix}.$$

For $\star$-(anti)palindromic polynomials of even degree $m = 2k$, a nonzero vector v satisfying $F_m v = v$ when $\star = T$ or $F_m v = \bar{v}$ when $\star = *$ can be taken of the form $v = z e_k + z^\star e_{k+1}$. The corresponding $\star$-(anti)palindromic pencil in $\mathbb{L}_1(P)$ is a linearization of $P(\lambda)$ if $-z/z^\star$ is not an eigenvalue of P and is given by (see [14])

$$\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \begin{cases} \lambda \mathcal{A}_-^{even}(z) + (\mathcal{A}_-^{even}(z))^\star & \text{when } \mathcal{S} = \star\text{-palindromic, } m = 2k, \\ \lambda \mathcal{A}_-^{even}(z) - (\mathcal{A}_-^{even}(z))^\star & \text{when } \mathcal{S} = \star\text{-antipalindromic, } m = 2k, \end{cases} \tag{7}$$

where

$$\mathcal{A}_{-}^{even}(z) = \begin{bmatrix} \mathcal{A}_{11}^{even}(z) & \mathcal{A}_{12}^{even}(z) \\ \mathcal{A}_{21}^{even}(z) & \mathcal{A}_{22}^{even}(z) \end{bmatrix}, \tag{8}$$

with

$$\mathcal{A}_{11}^{even}(z) = z \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ A_m & A_{m-1} & \dots & A_{k+1} \end{bmatrix}, \quad \mathcal{A}_{22}^{even}(z) = z \begin{bmatrix} A_{k+1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m-1} & 0 & \dots & 0 \\ A_m & 0 & \dots & 0 \end{bmatrix},$$

$$\mathcal{A}_{12}^{even}(z) = - \begin{bmatrix} z^\star A_0 & zA_0 & 0 & \cdots & \cdots & 0 \\ z^\star A_1 & z^\star A_0 + zA_1 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ z^\star A_{k-2} & z^\star A_{k-2} + zA_{k-1} & \cdots & z^\star A_1 + zA_2 & z^\star A_0 + zA_1 & zA_0 \\ -zA_k + z^\star A_{k-1} & z^\star A_{k-2} & & \cdots & z^\star A_1 & z^\star A_0 \end{bmatrix},$$

$$\mathcal{A}_{21}^{even}(z) = \begin{bmatrix} z^\star A_m & zA_m + z^\star A_{m-1} & zA_{m-1} + z^\star A_{m-2} & \cdots & \cdots & zA_{k+2} + z^\star A_{k+1} \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & zA_{m-1} + z^\star A_{m-2} \\ \vdots & & & \ddots & z^\star A_m & zA_m + z^\star A_{m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & z^\star A_m \end{bmatrix}.$$

Note that when $\star = *$, we can always pick a $z \in \mathbb{F}$ such that $-z/z^\star \notin \Lambda(P)$. But when $\star = T$, $-z/z^\star = -1$ so if $-1 \in \Lambda(P)$ the corresponding $\star$-(anti)palindromic pencil in $\mathbb{L}_1(P)$ is not a linearization of $P(\lambda)$. In fact it is shown in [14] that some T-(anti)palindromic matrix polynomials of even degree do not have T-(anti)palindromic linearizations. Instead, we allow a linearization with "anti" structure: palindromic becomes antipalindromic and vice versa. For this, let $v = e_{k+1} - e_k$ satisfying $v = -F_m v$. If $P(\lambda)$ is T-palindromic then there is a unique T-antipalindromic pencil in $\mathbb{L}_1(P)$ with vector v. Similarly if $P(\lambda)$ is T-antipalindromic then there is a unique T-palindromic pencil in $\mathbb{L}_1(P)$ with vector v. Such pencils are linearizations of P if $1 \notin \Lambda(P)$ and are given by

$$\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \begin{cases} \lambda \mathcal{A}_{+}^{even} - (\mathcal{A}_{+}^{even})^T & \text{when } \mathcal{S} = T\text{-palindromic with } m = 2k, \\ \lambda \mathcal{A}_{+}^{even} + (\mathcal{A}_{+}^{even})^T & \text{when } \mathcal{S} = T\text{-antipalindromic when } m = 2k, \end{cases} \tag{9}$$

where $\mathcal{A}_{+}^{even}(z)$ has a block structure similar to that of $\mathcal{A}_{-}^{even}(z)$ in (7) with z replaced by -1 and $z^\star$ replaced by 1. In particular, when $m = 2$,

$$\mathcal{A}_{+}^{even} = \begin{bmatrix} -A_2 & -A_1 - A_0 \\ A_2 & -A_2 \end{bmatrix}.$$

The next result, useful later, shows that the linearizations (4)–(9) share a property.

Lemma 2.1 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ with nonsingular leading coefficient. If $\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ is a structured linearization of $P(\lambda)$ as in* (4)*–*(9) *then $\mathcal{C} = -\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{B}_{\mathcal{S}}$, where $\mathcal{C}$ is the companion form of $P(\lambda)$ given in* (3)*.*

Proof. Some easy calculations show that $-\mathcal{A}_{\mathcal{S}}\mathcal{C} = \mathcal{B}_{\mathcal{S}}$. □

Hence, with the exception of T-(anti)palindromic $P(\lambda)$ of even degree with both -1 and 1 as eigenvalues, the companion form of $P(\lambda)$ can be factorized as $\mathcal{C} = -\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{B}_{\mathcal{S}}$, where $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}} = \mathcal{A}_{\mathcal{S}}(\lambda I - \mathcal{C})$ is a structured linearization of $P(\lambda)$.

2.2. Standard triples

Recall that $(U, \mathcal{T})$ is an (m, n)-*standard pair* over $\mathbb{F}$ if $\mathcal{T} \in \mathbb{F}^{mn\times mn}$ and $U \in \mathbb{F}^{n\times mn}$ are such that

$$Q = Q(U, \mathcal{T}) := \begin{bmatrix} U\mathcal{T}^{m-1} \\ \vdots \\ U\mathcal{T} \\ U \end{bmatrix} \tag{10}$$

is nonsingular [11, Def. 2.1]. The triple $(U, \mathcal{T}, V)$ forms an (m, n)-*standard triple* over $\mathbb{F}$ if $(U, \mathcal{T})$ is an (m, n)-standard pair over $\mathbb{F}$ and $V \in \mathbb{F}^{mn\times n}$ is such that $U\mathcal{T}^{m-1}V$ is nonsingular and, if $m \geq 2$,

$$U\mathcal{T}^{j}V = 0, \quad j = 0{:}\, m-2, \tag{11}$$

or equivalently,

$$QV = e_1 \otimes N \tag{12}$$

for some nonsingular $n \times n$ matrix N, where e_1 is the first column of the $m \times m$ identity matrix [11, Def. 2.3]. Note that the definitions of standard pairs and triples make no reference to matrix polynomials.

An (m, n)-standard pair $(U, \mathcal{T})$ over $\mathbb{F}$ is a *standard pair for* $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ if

$$A_m U\mathcal{T}^m + A_{m-1}U\mathcal{T}^{m-1} + \cdots + A_1 U\mathcal{T} + A_0 U = 0 \tag{13}$$

[6, p. 46]. A standard triple $(U, \mathcal{T}, V)$ is a *standard triple for* $P(\lambda)$ if (13) holds and $A_m = (U\mathcal{T}^{m-1}V)^{-1}$ (i.e., $N = A_m^{-1}$ in (12)). Any $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient admits a standard triple. For example, it is easy to check that

$$(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1}) \tag{14}$$

with $\mathcal{C}$ as in (3) is a standard triple for $P(\lambda)$. We refer to (14) as the *primitive standard triple* for $P(\lambda)$.

Let $U_i \in \mathbb{F}^{n\times mn}$, $\mathcal{T}_i \in \mathbb{F}^{mn\times mn}$ and $V_i \in \mathbb{F}^{mn\times n}$, $i = 1, 2$. Then $(U_1, \mathcal{T}_1, V_1)$ is *similar* to $(U_2, \mathcal{T}_2, V_2)$ if there exists a nonsingular $G \in \mathbb{F}^{mn\times mn}$ such that

$$U_2 = U_1 G, \qquad \mathcal{T}_2 = G^{-1}\mathcal{T}_1 G, \qquad V_2 = G^{-1}V_1. \tag{15}$$

It is easy to check that $Q(U_1, \mathcal{T}_1)G = Q(U_2, \mathcal{T}_2)$. Hence G is uniquely defined by $(U_1, \mathcal{T}_1)$, $(U_1, \mathcal{T}_2)$ and is given by

$$G = Q(U_1, \mathcal{T}_1)^{-1}Q(U_2, \mathcal{T}_2). \tag{16}$$

Also, $(U_2, \mathcal{T}_2, V_2)$ defined in (15) is a standard triple if $(U_1, \mathcal{T}_1, V_1)$ is a standard triple [5, Prop. 12.1.3]. Moreover if $(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ then, with $Q = Q(U, \mathcal{T})$ as in (10), we find that

$$(e_m^T \otimes I_n)Q = U, \quad Q^{-1}\mathcal{C}Q = \mathcal{T}, \quad Q^{-1}(e_1 \otimes A_m^{-1}) = V. \tag{17}$$

Table 2: Definition of $u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})$ for some $\mathcal{T} \in \mathbb{F}^{mn\times mn}$ satisfying assumption (b), where α is some scalar in $\mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \Lambda(\mathcal{T})$.

Structure $\mathcal{S}$	$u_{\mathcal{S}}(\mathcal{T})$	$t_{\mathcal{S}}(\mathcal{T})$	$v_{\mathcal{S}}(\mathcal{T})$
Hermitian/symmetric	I	$\mathcal{T}^\star$	I
$\star$-even	$-I$	$-\mathcal{T}^\star$	I
$\star$-odd	I	$-\mathcal{T}^\star$	I
$\star$-palindromic, $m = 2k+1$	$-\mathcal{T}^{\star(k-1)}$	$\mathcal{T}^{-\star}$	$\mathcal{T}^{\star k}$
$\star$-palindromic, $m = 2k$	$-\mathcal{T}^{\star(k-1)}(I+\alpha\mathcal{T}^\star)^{-1}$	$\mathcal{T}^{-\star}$	$(I+\alpha\mathcal{T}^\star)\mathcal{T}^{\star(k-1)}$
$\star$-antipalindromic, $m = 2k+1$	$\mathcal{T}^{\star(k-1)}$	$\mathcal{T}^{-\star}$	$\mathcal{T}^{\star k}$
$\star$-antipalindromic, $m = 2k$	$\mathcal{T}^{\star(k-1)}(I+\alpha\mathcal{T}^\star)^{-1}$	$\mathcal{T}^{-\star}$	$(I+\alpha\mathcal{T}^\star)\mathcal{T}^{\star(k-1)}$

Hence any standard triple $(U, \mathcal{T}, V)$ for $P(\lambda)$ is similar to the primitive standard triple $(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$. Note that because $\mathcal{T}$ is similar to $\mathcal{C}$, $\lambda I - \mathcal{T}$ is a linearization of $P(\lambda)$ and $\Lambda(P) = \Lambda(\mathcal{T})$. The following result [5, Thm. 12.1.4] will be needed.

Lemma 2.2 *Let $U \in \mathbb{F}^{n\times mn}$, $\mathcal{T} \in \mathbb{F}^{mn\times mn}$, $V \in \mathbb{F}^{mn\times n}$ and let $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ be of degree m with nonsingular leading coefficient. Then $(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ if and only if $P(\lambda)^{-1} = U(\lambda I - \mathcal{T})^{-1}V$ for $\lambda \in \mathbb{C} \setminus \Lambda(P)$.*

A Jordan triple (X, J, Y) over $\mathbb{F}$ for $P(\lambda)$ is a standard triple for $P(\lambda)$ for which the matrix J is in Jordan form or real Jordan form if $\mathbb{F} = \mathbb{R}$. By (13) and [6, Prop. 2.1], we have that $\sum_{j=0}^m A_j X J^j = 0$ and $\sum_{j=0}^m J^j Y A_j = 0$. The columns of X and Y^* determine right and left eigenvectors and generalized eigenvectors of $P(\lambda)$. The matrix J is the Jordan form of the companion form $\mathcal{C}$ of $P(\lambda)$.

3. $\mathcal{S}$-structured standard triples

We now consider standard triples in the context of structured matrix polynomials. We start by listing two assumptions used in our analysis. Let $\mathcal{S} \in \mathbb{S}$, $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ have degree m with nonsingular leading coefficient and let $\mathcal{T} \in \mathbb{F}^{mn\times mn}$.

> **Assumption (a)**: if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ and $P(\lambda)$ has degree $m = 2k$ then either $-1 \notin \Lambda(P)$ or $1 \notin \Lambda(P)$.
>
> **Assumption (b)**: if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ and $m = 2k$ then either $-1 \notin \Lambda(\mathcal{T})$ or $1 \notin \Lambda(\mathcal{T})$.

Assumption (a) ensures the existence of a structured linearization. Assumption (b) ensures the existence of $\alpha \in \mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \Lambda(\mathcal{T})$. Also, for $\star$-(anti)palindromic structures, the eigenvalues of $\mathcal{T}$ come in pairs $(\lambda, \lambda^{-\star})$. Hence $0 \notin \Lambda(\mathcal{T})$ since $\infty \notin \Lambda(\mathcal{T})$ and $\mathcal{T}^{-\star}$ is well defined. So for some $\mathcal{T}$ satisfying assumption (b) we define $u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})$ as in Table 2. We note that assumptions (a) and (b) are equivalent when $\lambda I - \mathcal{T}$ is a linearization of $P(\lambda)$.

Before stating our main result in Theorem 3.4, we provide a few lemmas and introduce the notion of $\mathcal{S}$-structured standard triple. The first lemma of this section extends to all structures in $\mathbb{S}$ a result in [6, Thm. 10.1] for Hermitian structure.

Lemma 3.1 *Let $(U, \mathcal{T}, V)$ be an (m,n)-standard triple for $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient and let $\mathcal{S} \in \mathbb{S}$. Assume that $\mathcal{T}$ satisfies assumption (b). Then $P(\lambda)$ has structure $\mathcal{S}$ if and only if $(V^\star u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})U^\star)$ is a standard triple for $P(\lambda)$.*

Proof. ($\Rightarrow$) Assume that $P(\lambda)$ is structured with structure $\mathcal{S}$. Since any standard triple for $P(\lambda)$ is similar to the primitive standard triple $(U_0, \mathcal{C}, V_0) := (e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$ (see comment before Lemma 2.2), it suffices to show that $(U_0, \mathcal{C}, V_0)$ is similar to $(V_0^\star u_{\mathcal{S}}(\mathcal{C}), t_{\mathcal{S}}(\mathcal{C}), v_{\mathcal{S}}(\mathcal{C})U_0^\star)$. Note that under assumption (b), $P(\lambda)$ has a structured linearization $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$, which is one of (4)–(9) and by Lemma 2.1, $\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{B}_{\mathcal{S}} = -\mathcal{C}$. Define

$$G^{-1} := \begin{cases} z^{-\star}\mathcal{A}_{-}^{even}(z) & \text{if } P \text{ is } \star\text{-(anti)palindromic, } m = 2k,\ -z/z^\star \notin \Lambda(P), \\ \mathcal{A}_{\mathcal{S}} & \text{otherwise,} \end{cases} \tag{18}$$

with $\mathcal{A}_{-}^{even}(z)$ as in (8). We aim to show that

$$V_0^\star u_{\mathcal{S}}(\mathcal{C}) = U_0 G, \qquad G^{-1}\mathcal{C}G = t_{\mathcal{S}}(\mathcal{C}), \qquad v_{\mathcal{S}}(\mathcal{C})U_0^\star = G^{-1}V_0, \tag{19}$$

that is, $(U_0, \mathcal{C}, V_0)$ is similar to $(V_0^\star u_{\mathcal{S}}(\mathcal{C}), t_{\mathcal{S}}(\mathcal{C}), v_{\mathcal{S}}(\mathcal{C})U_0^\star)$ for all $\mathcal{S} \in \mathbb{S}$. That (19) holds for $\mathcal{S} \in \{$Hermitian, symmetric, $\star$-even, $\star$-odd$\}$ is easy to check.

For $\mathcal{S} \in \{\star$-palindromic, $\star$-antipalindromic$\}$, the proof that $G^{-1}\mathcal{C}G = \mathcal{C}^{-\star} = t_{\mathcal{S}}(\mathcal{C})$ follows from the definition of G and $\mathcal{C} = \varepsilon\mathcal{A}_{\mathcal{S}}^{-1}\mathcal{A}_{\mathcal{S}}^\star$, where $\varepsilon = \pm 1$ depends on whether $\mathcal{B}_{\mathcal{S}} = \mathcal{A}_{\mathcal{S}}^\star$ or $\mathcal{B}_{\mathcal{S}} = -\mathcal{A}_{\mathcal{S}}^\star$. To prove that the first and third equalities in (19) hold for palindromic structures, we consider three cases.

(i) $m = 2k+1$. In that case, $G^{-1} = \mathcal{A}^{odd}$, with $\mathcal{A}^{odd}$ as in (6). Then

$$G^{-1}V_0 = G^{-1}(e_1 \otimes A_m^{-1}) = e_{k+1} \otimes I = (\mathcal{C}^\star)^k(e_m \otimes I) = v_{\mathcal{S}}(\mathcal{C})U_0^\star,$$

from which it follows that $V_0^\star = (e_m^T \otimes I)\mathcal{C}^k G^\star$ so that, on using $G^{-1}\mathcal{C}G = \mathcal{C}^{-\star}$,

$$\begin{aligned} V_0^\star u_{\mathcal{S}}(\mathcal{C})G^{-1} &= (e_m^T \otimes I)\mathcal{C}^k G^\star(-\mathcal{C}^{\star(k-1)})G^{-1} \\ &= (e_m^T \otimes I)\mathcal{C}^k\mathcal{C}^{(1-k)}(-G^\star G^{-1}) \\ &= (e_m^T \otimes I) = U_0. \end{aligned}$$

(ii) $m = 2k$, $\star = T$ and $-1 \in \Lambda(\mathcal{T})$. In that case, $G^{-1} = \mathcal{A}_{+}^{even}$ with $\mathcal{A}_{+}^{even}$ as in (9). Then

$$v_{\mathcal{S}}(\mathcal{C})U_0^T = (I - \mathcal{C}^T)\mathcal{C}^{T(k-1)}(e_m \otimes I_n) = e_{k+1} \otimes I - e_k \otimes I = G^{-1}(e_1 \otimes I)A_m^{-1} = G^{-1}V_0.$$

From $V_0 = Gv_{\mathcal{S}}(\mathcal{C})U_0^T$ it follows that $V_0^T = U_0\mathcal{C}^{(k-1)}(I - \mathcal{C})G^T$, so that

$$\begin{aligned} V_0^T u_{\mathcal{S}}(\mathcal{C}) &= -U_0\mathcal{C}^{(k-1)}(I - \mathcal{C})G^T\mathcal{C}^{T(k-1)}(I - \mathcal{C}^T)^{-1} \\ &= -U_0\mathcal{C}^{(k-1)}(I - \mathcal{C})\mathcal{C}^{(1-k)}G^T(I - \mathcal{C}^T)^{-1} \\ &= U_0 G(I - \mathcal{C}^T)(I - \mathcal{C}^T)^{-1} = U_0 G, \end{aligned}$$

where we used $\mathcal{C}G^T = G$ and $G^T\mathcal{C}^{T(k-1)}G^{-T} = \mathcal{C}^{-(k-1)}$.

(iii) $m = 2k$, $\star = *, T$ and if $\star = T$ then $-1 \notin \Lambda(\mathcal{T})$. The proof is similar to that in (ii) with $\alpha = z/z^\star$ in the definition of $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$, and $G^{-1} = z^{-\star}\mathcal{A}_{-}^{even}(z)$ with $\mathcal{A}_{-}^{even}(z)$ as in (8).

The case of antipalindromic structures is proved similarly.

($\Leftarrow$) Suppose that $(U, \mathcal{T}, V)$ and $(V^\star u_\mathcal{S}(\mathcal{T}), t_\mathcal{S}(\mathcal{T}), v_\mathcal{S}(\mathcal{T})U^\star)$ are standard triples for $P(\lambda)$. By Lemma 2.2, we have that

$$U(\lambda I - \mathcal{T})^{-1}V = P(\lambda)^{-1} = V^\star u_\mathcal{S}(\mathcal{T})\big(\lambda I - t_\mathcal{S}(\mathcal{T})\big)^{-1} v_\mathcal{S}(\mathcal{T})U^\star. \tag{20}$$

As shown in the proof of [6, Thm. 10.1] for Hermitian structure, (20) implies that

$$\big(P^*(\lambda)\big)^{-1} = \big(P(\bar\lambda)\big)^{-*} = (U(\bar\lambda I - \mathcal{T})^{-1}V)^* = V^*(\lambda I - \mathcal{T}^*)^{-1}U^* = P(\lambda)^{-1}$$

showing that $P(\lambda)$ is Hermitian. This proof extends easily to structures $\mathcal{S} \in \{$symmetric, $\star$-even, $\star$-odd$\}$.

We now concentrate on palindromic structures. Using the left hand side of (20) we find that

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = \lambda^{-m}(U(\lambda^{-\star}I - \mathcal{T})^{-1}V)^\star = \lambda^{1-m}V^\star(I - \lambda\mathcal{T}^\star)^{-1}U^\star.$$

If $\|\lambda\mathcal{T}^\star\| < 1$ for some subordinate matrix norm $\|\cdot\|$ then

$$(I - \lambda\mathcal{T}^\star)^{-1} = I + \lambda\mathcal{T}^\star + \lambda^2\mathcal{T}^{\star 2} + \cdots. \tag{21}$$

Using (21) and the fact that $V^\star\mathcal{T}^{\star j}U^\star = 0$, $j = 0\colon m-2$ (see (11)), we obtain

$$\begin{aligned}\lambda^{-m}(P(\lambda^{-\star}))^{-\star} &= V^\star\mathcal{T}^{\star(m-1)}(I + \lambda\mathcal{T}^\star + \lambda^2\mathcal{T}^{\star 2} + \cdots)U^\star \\ &= V^\star\mathcal{T}^{\star(k-1)}(I - \lambda\mathcal{T}^\star)^{-1}\mathcal{T}^{\star(m-k)}U^\star \\ &= -V^\star\mathcal{T}^{\star(k-1)}(\lambda I - \mathcal{T}^{-\star})^{-1}\mathcal{T}^{\star(m-k-1)}U^\star\end{aligned} \tag{22}$$

for all $|\lambda| < \|\mathcal{T}^\star\|^{-1}$. When $m = 2k+1$, (22) and the right hand side of (20) yield

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = V^\star u_\mathcal{S}(\mathcal{T})(\lambda I - \mathcal{T}^{-\star})^{-1}v_\mathcal{S}(\mathcal{T})U^\star = P(\lambda)^{-1}. \tag{23}$$

Note that $(\lambda I - \mathcal{T}^{-\star})^{-1}$ commutes with $\mathcal{T}^{\star k-1}$, $(I + \alpha\mathcal{T}^\star)$ and $(I + \alpha\mathcal{T}^\star)^{-1}$ so when $m = 2k$, (22) can be rewritten to yield (23). Since $\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = P(\lambda)^{-1}$ holds for many values of λ, $P(\lambda) = \lambda^m P^\star(\lambda^{-1})$ for all λ, that is, $P(\lambda)$ is $\star$-palindromic.

That $P(\lambda) = -\lambda^m P^\star(\lambda^{-1})$ for the $\star$-antipalindromic structure is proved in a similar way. $\square$

Lemma 3.1 naturally leads to the following definition.

Definition 3.2 ($\mathcal{S}$-structured standard triple) *Let $\mathcal{S} \in \mathbb{S}$. An (m, n)-standard triple $(U, \mathcal{T}, V)$ with $\mathcal{T}$ satisfying assumption* (b) *is said to be* $\mathcal{S}$-structured *if it is similar to $(V^\star u_\mathcal{S}(\mathcal{T}), t_\mathcal{S}(\mathcal{T}), v_\mathcal{S}(\mathcal{T})U^\star)$.*

If $(U, \mathcal{T}, V)$ is an $\mathcal{S}$-structured standard triple then there is a nonsingular $S \in \mathbb{F}^{mn\times mn}$ such that

$$US = V^\star u_\mathcal{S}(\mathcal{T}), \quad S^{-1}\mathcal{T}S = t_\mathcal{S}(\mathcal{T}), \quad S^{-1}V = v_\mathcal{S}(\mathcal{T})U^\star. \tag{24}$$

The matrix S is unique and is given by (see (16))

$$S = Q(U, \mathcal{T})^{-1}Q\big(V^\star u_\mathcal{S}(\mathcal{T}), t_\mathcal{S}(\mathcal{T})\big).$$

We refer to S as the the S-matrix of the $\mathcal{S}$-structured standard triple $(U, \mathcal{T}, V)$.

The next lemma shows that any standard triple that is similar to an $\mathcal{S}$-structured standard triple is itself $\mathcal{S}$-structured.

Lemma 3.3 *Let $(U, \mathcal{T}, V)$ be a standard triple similar to $(U_1, \mathcal{T}_1, V_1)$, that is, $(U_1, \mathcal{T}_1, V_1) = (UG, G^{-1}\mathcal{T}G, G^{-1}V)$ for some nonsingular matrix G. Let $\mathcal{S} \in \mathbb{S}$ and assume $\mathcal{T}$ satisfies assumption (b). If $(U, \mathcal{T}, V)$ is $\mathcal{S}$-structured with S-matrix S then $(U_1, \mathcal{T}_1, V_1)$ is $\mathcal{S}$-structured with S-matrix $S_1 = G^{-1}SG^{-\star}$.*

Proof. If $(U_1, \mathcal{T}_1, V_1) = (UG, G^{-1}\mathcal{T}G, G^{-1}V)$ with $(U, \mathcal{T}, V)$ $\mathcal{S}$-structured then

$$\begin{aligned}(V_1^\star G^\star u_{\mathcal{S}}(G\mathcal{T}_1G^{-1}), t_{\mathcal{S}}(G\mathcal{T}_1G^{-1}), v_{\mathcal{S}}(G\mathcal{T}_1G^{-1})G^{-\star}U_1^\star) &= (V^\star u_{\mathcal{S}}(\mathcal{T}), t_{\mathcal{S}}(\mathcal{T}), v_{\mathcal{S}}(\mathcal{T})U^\star) \\ &= (US, S^{-1}\mathcal{T}S, S^{-1}V) \\ &= (U_1G^{-1}S, S^{-1}G\mathcal{T}_1G^{-1}S, S^{-1}GV_1).\end{aligned}$$

Since $u_{\mathcal{S}}(G\mathcal{T}_1G^{-1}) = G^{-\star}u_{\mathcal{S}}(\mathcal{T}_1)G^\star$, $t_{\mathcal{S}}(G\mathcal{T}_1G^{-1}) = G^{-\star}u_{\mathcal{S}}(\mathcal{T}_1)G^\star$, and $v_{\mathcal{S}}(G\mathcal{T}_1G^{-1}) = G^{-\star}v_{\mathcal{S}}(\mathcal{T}_1)G^\star$, it follows that $(U_1, \mathcal{T}_1, V_1)$ is $\mathcal{S}$-structured with S-matrix $G^{-1}SG^{-\star}$. □

We can now state our main result, which is a direct consequence of Lemma 3.1 and Lemma 3.3. It extends a result for Hermitian structure [5, Thm. 12.2.2] to all structures in $\mathbb{S}$.

Theorem 3.4 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient satisfying assumption (a). Then $P(\lambda)$ has structure $\mathcal{S}$ if and only if $P(\lambda)$ admits an $\mathcal{S}$-structured standard triple, in which case every standard triple for $P(\lambda)$ is $\mathcal{S}$-structured.*

The relations in (24) imply certain properties of S, as shown in the next theorem.

Theorem 3.5 *Let $\mathcal{S} \in \mathbb{S}$. An (m, n)-standard triple $(U, \mathcal{T}, V)$ with $\mathcal{T}$ satisfying assumption (b) is $\mathcal{S}$-structured with matrix S if and only if $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ and S satisfies the following properties:*

- *$S = S^\star$, $\mathcal{T}S = (\mathcal{T}S)^\star$ when $\mathcal{S} \in \{$Hermitian, symmetric$\}$,*
- *$S = -S^\star$, $\mathcal{T}S = (\mathcal{T}S)^\star$ when $\mathcal{S} = \star$-even,*
- *$S = S^\star$, $\mathcal{T}S = -(\mathcal{T}S)^\star$ when $\mathcal{S} = \star$-odd,*
- *$\mathcal{T}S^\star = -S$ when $\mathcal{S} = \star$-palindromic and $m = 2k+1$ or $\mathcal{T}S^\star = -\alpha S$ when $\mathcal{S} = \star$-palindromic and $m = 2k$,*
- *$\mathcal{T}S^\star = S$ when $\mathcal{S} = \star$-antipalindromic and $m = 2k+1$ or $\mathcal{T}S^\star = \alpha S$ when $\mathcal{S} = \star$-antipalindromic and $m = 2k$,*

for some $\alpha \in \mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \Lambda(\mathcal{T})$.

Proof. ($\Leftarrow$) Assume that $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ and that S satisfies the properties listed in the theorem. We show that (24) holds. The last equality follows from $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ and the second equality follows from the properties of S. Now from $V = Sv_{\mathcal{S}}(\mathcal{T})U^\star$ we have that $V^\star u_{\mathcal{S}}(\mathcal{T}) = U(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T})$. That $(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T}) = S$ for $\mathcal{S} \in \{$Hermitian, symmetric, $\star$-even, $\star$-odd$\}$ follows from the definition of $u_{\mathcal{S}}, v_{\mathcal{S}}$ and the properties of S. For palindromic structures, $S^{-1}\mathcal{T}S = t_{\mathcal{S}}(\mathcal{T})$ implies that

$$S^\star(\mathcal{T}^\star)^{(k-1)} = \mathcal{T}^{-(k-1)}S^\star. \tag{25}$$

Hence, when $m = 2k + 1$,

$$(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T}) = -\mathcal{T}^k S^\star \mathcal{T}^{\star(k-1)} = -\mathcal{T}^k \mathcal{T}^{-(k-1)} S^\star = -\mathcal{T} S^\star = S,$$

where we used (25) and the assumption that $\mathcal{T} S^\star = -S$. When $m = 2k$,

$$\begin{aligned}(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T}) &= -\mathcal{T}^{(k-1)}(I + \alpha^\star \mathcal{T}) S^\star \mathcal{T}^{\star(k-1)} (I + \alpha \mathcal{T}^\star)^{-1} \\ &= -(I + \alpha^\star \mathcal{T}) S^\star (I + \alpha \mathcal{T}^\star)^{-1} \\ &= (S - S^\star)(I + \alpha \mathcal{T}^\star)^{-1} = S(I + \alpha \mathcal{T}^\star)(I + \alpha \mathcal{T}^\star)^{-1} = S.\end{aligned}$$

In a similar way we can show that $(v_{\mathcal{S}}(\mathcal{T}))^\star S^\star u_{\mathcal{S}}(\mathcal{T}) = S$ for antipalindromic structures. Hence $V^\star u_{\mathcal{S}}(\mathcal{T}) = US$.

($\Rightarrow$) Assume that $(U, \mathcal{T}, V)$ is $\mathcal{S}$-structured with S-matrix S so that (24) holds and hence $V = S v_{\mathcal{S}}(\mathcal{T}) U^\star$. By [11, Thm. 2.4] there exists a unique matrix polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ for which $(U, \mathcal{T}, V)$ is a standard triple. This triple is similar to the primitive triple $(U_0, \mathcal{T}_0, V_0) = (e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$, where $A_m^{-1} = U \mathcal{T}^{m-1} V$. The proof of Lemma 3.1 shows that $(U_0, \mathcal{T}_0, V_0)$ is $\mathcal{S}$-structured with S-matrix $S_0 = G$ defined in (18). It is easy to check that $S_0 = G$ and $\mathcal{T}_0 = \mathcal{C}$ satisfy the properties displayed in the the bullet points of the theorem. By Lemma 3.3, $S = Q^{-1} S_0 Q^{-\star}$ and since $\mathcal{T} = Q^{-1} \mathcal{T}_0 Q$ (see (17)), we have that $\mathcal{T} S = Q^{-1} \mathcal{T}_0 S_0 Q^{-\star}$, $\mathcal{T} S^\star = Q^{-1} \mathcal{T}_0 S_0^\star Q^{-\star}$. This completes the proof since the properties of S_0 and $\mathcal{T}_0 S_0$ are preserved by $\star$-congruences and it is easy to check that $\mathcal{T} S^\star$ is the appropriate multiple of S for the (anti)palindromic structures. □

We point out that Hermitian and symmetric structured standard triples are called *self-adjoint standard triples* in the literature (see for example [5, p. 244]). For (anti)palindromic structures, the matrix $\mathcal{T}$ of an $\mathcal{S}$-structured standard triple $(U, \mathcal{T}, V)$ with S-matrix S is S^{-1}-unitary, that is, $\mathcal{T}^\star S^{-1} \mathcal{T} = S^{-1}$. With additional constraints on $\mathcal{T}$'s structure, Lancaster, Prells and Rodman refer to $(U, \mathcal{T}, V)$ as a *unitary standard triple* [8, Def. 4]. Hence a unitary standard triple is $\mathcal{S}$-structured but the converse is not true in general.

The S-matrix of an $\mathcal{S}$-structured standard triple $(U, \mathcal{T}, V)$ for $P(\lambda)$ can be expressed in terms of $U, \mathcal{T}$ and the matrix coefficients of $P(\lambda)$ as the next result shows.

Proposition 3.6 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ be of degree m with nonsingular leading coefficient and satisfying assumption* (a). *If $(U, \mathcal{T})$ is a standard pair for $P(\lambda)$ then $(U, \mathcal{T}, S v_{\mathcal{S}}(\mathcal{T}) U^\star)$ is an $\mathcal{S}$-structured standard triple for $P(\lambda)$ with S-matrix S given by*

$$S^{-1} = \begin{cases} z^{-\star} Q^\star \mathcal{A}_-^{even}(z) Q & \text{if } P \text{ is } \star\text{-(anti)palindromic, } m = 2k,\ -z/z^\star \notin \Lambda(P), \\ Q^\star \mathcal{A}_{\mathcal{S}} Q & \text{otherwise,} \end{cases}$$

where $Q := Q(U, \mathcal{T})$ is as in (10), *and $\mathcal{A}_{\mathcal{S}}$ and $\mathcal{A}_-^{even}(z)$ are as in* (4)–(9).

Proof. The primitive standard triple $(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1})$ is $\mathcal{S}$-structured with matrix G defined in (18). Since $(U, \mathcal{T})$ is a standard pair of $P(\lambda)$, we easily check that $Q^{-1} \mathcal{C} Q = \mathcal{T}$ and $\left(e_m^T \otimes I_n\right) Q = U$. Define $V = Q^{-1}\left(e_1 \otimes A_m^{-1}\right)$. Then $(U, \mathcal{T}, V)$ is a standard triple for $P(\lambda)$ similar to $\left(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1}\right)$. By Lemma 3.3, $(U, \mathcal{T}, V)$ is $\mathcal{S}$-structured with matrix $S = Q^{-1} G Q^{-\star}$ and $V = S v_{\mathcal{S}}(\mathcal{T}) U^\star$. □

4. $\mathcal{S}$-structured Jordan triples

We now explain how to obtain explicit expressions for the Jordan matrix and S-matrix of $\mathcal{S}$-structured Jordan triples $(X, J, S_J v_{\mathcal{S}}(J) X^\star)$ of $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. We note that the matrix S_J displays the sign characteristic of $P(\lambda)$, whose definition we now give.

Let $(U, \mathcal{T}, S_{\mathcal{T}} v_{\mathcal{S}}(\mathcal{T}) U^\star)$ be a standard triple for $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. The *sign characteristic* of $P(\lambda)$ is defined as the sign characteristic of the pair $(\mathcal{T}, S_{\mathcal{T}}^{-1})$, which is a list of signs, with a sign ($+1$ or -1) attached to each partial multiplicity of

- real eigenvalues of Hermitian or real symmetric matrix polynomials,
- purely imaginary eigenvalues of $*$-even, $*$-odd, real T-even and real T-odd matrix polynomials,
- eigenvalues with unit modulus of $*$-(anti)palindromic and real T-(anti)palindromic matrix polynomials.

These signs can be read off the canonical decomposition of $\lambda S_{\mathcal{T}}^{-1} - S_{\mathcal{T}}^{-1}\mathcal{T}$ via $\star$-congruence (see [5, Sec. 12.4] for Hermitian structure). Note that the definition of the sign characteristic for $P(\lambda)$ is independent of the choice of standard triple. Indeed if $(U_i, \mathcal{T}_i, S_{\mathcal{T}_i} v_{\mathcal{S}}(\mathcal{T}_i) U_i^\star)$, $i = 1, 2$ are $\mathcal{S}$-structured standard triples for $P(\lambda)$, then by Lemma 3.3 there exists a nonsingular G such that $\mathcal{T}_2 = G^{-1}\mathcal{T}_1 G$ and $S_{\mathcal{T}_2} = G^{-1} S_{\mathcal{T}_1} G^{-\star}$. Hence, $\lambda S_{\mathcal{T}_2}^{-1} - S_{\mathcal{T}_2}^{-1}\mathcal{T}_2 = G^\star(\lambda S_{\mathcal{T}_1}^{-1} - S_{\mathcal{T}_1}^{-1}\mathcal{T}_1)G$, that is, the pencils $\lambda S_{\mathcal{T}_i}^{-1} - S_{\mathcal{T}_i}^{-1}\mathcal{T}_i$, $i = 1, 2$ are $\star$-congruent. They share the same canonical decomposition via $\star$-congruence and therefore the same sign characteristic.

We know that the triple $\big((e_m^T \otimes I_n), \mathcal{C}, (e_1 \otimes A_m^{-1})\big)$ is a standard triple for $P(\lambda)$ and by Theorem 3.4, it is $\mathcal{S}$-structured with S-matrix as in Proposition 3.6 with $Q = I_{mn}$. Hence, on using Lemma 2.1, we find that

$$\lambda S_{\mathcal{C}}^{-1} - S_{\mathcal{C}}^{-1}\mathcal{C} = \lambda z^{-\star}\mathcal{A}_{\mathcal{S}} + z^{-\star}\mathcal{B}_{\mathcal{S}},$$

where $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ is a structured linearization of $P(\lambda)$ as in (4)–(9), and $z = 1$ except when $\mathcal{A}_{\mathcal{S}} = \mathcal{A}_{-}^{even}(z)$, in which case $z \in \mathbb{F}$ is chosen such that $-z/z^\star \notin \Lambda(P)$. So what we need is a canonical decomposition of $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ via $\star$-congruence,

$$Z^\star(\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}})Z = \lambda(Z^\star\mathcal{A}_{\mathcal{S}}Z) - (Z^\star\mathcal{A}_{\mathcal{S}}Z)\big(Z^{-1}\mathcal{C}Z\big) = z^\star(\lambda S_J^{-1} - S_J^{-1}J),$$

where $J = Z^{-1}\mathcal{C}Z$ is the Jordan form of $\mathcal{C}$. Fortunately, such decompositions are available in the literature for all the structures in $\mathbb{S}$. We use these canonical decompositions to provide explicit expressions for J and S_J in Appendix A. These expressions show that S_J and J have the same block structure and that we can read the sign characteristic of $P(\lambda)$ from certain diagonal blocks of S_J.

5. Concluding remarks

The results in this paper represent a first step towards the solution of the structured inverse polynomial eigenvalue problem: given a list of admissible elementary divisors for the structure, and possibly, corresponding right eigenvectors and generalized eigenvectors, construct a structured matrix polynomial having these elementary divisors and eigenvectors/generalized eigenvectors. Indeed, using the results in sections 3 and 4 we show in [1] how to construct an $\mathcal{S}$-structured $(2,n)$-Jordan triple (X,J,Y) from a given list of $2n$ prescribed eigenvalues and n linearly independent eigenvectors and generalized eigenvectors, and use the fact that an $\mathcal{S}$-structured $(2,n)$-Jordan triple defines a unique structured quadratic $Q(\lambda)=\lambda^2A_2+\lambda A_1+A_0\in\mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$, where $A_2=(XJSv_{\mathcal{S}}(J)X^\star)^{-1}$,

$$A_1=-A_2XJ^2Sv_{\mathcal{S}}(J)X^\star A_2,\quad A_0=-A_2(XJ^2Sv_{\mathcal{S}}(J)X^\star A_1+XJ^3Sv_{\mathcal{S}}(J)X^\star A_2),$$

and $v_{\mathcal{S}}(\cdot)$ as in Table 2.

Finally, we note that standard triples have been useful to describe structure preserving transformations (SPTs) for matrix polynomials, and in particular quadratic matrix polynomials [3]. We believe that the notion of $\mathcal{S}$-structured standard triples will further our understanding of SPTs for structured matrix polynomials.

Acknowledgement

The authors would like to thank the referee for valuable suggestions, which improved the organization of section 3.

Appendix A. Explicit expressions for J and S_J

Using the canonical decompositions of structured pencils via $\star$-congruences, we provide in this appendix an explicit expression for the Jordan matrix and S-matrix of $\mathcal{S}$-structured Jordan triples $(X,J,S_Jv_{\mathcal{S}}(J)X^\star)$ of $P(\lambda)\in\mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ for each $\mathcal{S}\in\mathbb{S}$. We assume that $P(\lambda)$ is of degree m with nonsingular leading coefficient matrix. To facilitate the description of J and S_J, we introduce the matrices $E_1=F_1=[1]$ and for integers $k>1$

$$E_k=\begin{bmatrix} & & & 1\\ & & -1 & \\ & 1 \,\cdot^{\cdot^{\cdot}} & & \\ (-1)^{k-1} & & & \end{bmatrix}_{k\times k}=(-1)^{k-1}E_k^T,\qquad F_k=\begin{bmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}_{k\times k}.$$

We denote by

$$J_{\ell_k}(\lambda_k)=\begin{bmatrix}\lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1\\ & & & \lambda_k\end{bmatrix}\in\mathbb{C}^{\ell_k\times\ell_k},$$

the Jordan block of size ℓ_k associated with λ_k, and by

$$K_{2m_k}(\lambda_k,\bar\lambda_k)=K_{2m_k}(\Lambda_k)=\begin{bmatrix}\Lambda_k & I_2 & & \\ & \Lambda_k & \ddots & \\ & & \ddots & I_2\\ & & & \Lambda_k\end{bmatrix}\in\mathbb{R}^{2m_k\times 2m_k},\quad \Lambda_k=\begin{bmatrix}\alpha_k & \beta_k\\ -\beta_k & \alpha_k\end{bmatrix},$$

the $2m_k\times 2m_k$ real Jordan block associated with the pair of complex conjugate eigenvalues $(\lambda_k,\bar\lambda_k)$, where $\lambda_k=\alpha_k+i\beta_k$ with $\alpha_k,\beta_k\in\mathbb{R}$, $\beta_k\neq 0$. We use the notation $\bigoplus_{j=1}^r F_j$ to denote the direct sum of the matrices $F_1,\ldots,F_r$.

Note that there are restrictions on the Jordan structure of P. For instance, a regular $n \times n$ matrix polynomial cannot have more than n elementary divisors associated with the same eigenvalue [13]. Also, the elementary divisors have certain pairing, which depends on the structure $\mathcal{S} \in \mathbb{S}$ and the eigenvalue. Hence we describe for each $\mathcal{S} \in \mathbb{S}$ the elementary divisors arising from $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ and then provide an expression for J and S_J.

Appendix A.1. Hermitian structure

Suppose $P(\lambda)$ is Hermitian with

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1\colon r$, and
- s pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \overline{\mu}_j)^{m_j}$, $j = 1\colon s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. It follows from [9, Thm. 6.1] that

$$J = \bigoplus_{j=1}^{r} J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^{s} (J_{m_j}(\bar{\mu}_j) \oplus J_{m_j}(\mu_j)), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j}.$$

Here $\{\varepsilon_1, \dots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the real eigenvalues λ_j, $j = 1\colon r$ of $P(\lambda)$. We easily check that $S_J = S_J^*$ and $JS_J = (JS_J)^*$.

Appendix A.2. Real symmetric structure

Suppose $P(\lambda)$ is real symmetric with

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1\colon r$, and
- s pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \overline{\mu}_j)^{m_j}$, $j = 1\colon s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. On using [9, Thm. 9.2] we find that

$$J = \bigoplus_{j=1}^{r} J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^{s} K_{2m_j}(\mu_j, \bar{\mu}_j), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the real eigenvalues of $P(\lambda)$. Note that $S_J = S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.3. Complex symmetric structure

Suppose $P(\lambda)$ is complex symmetric with q elementary divisors $(\lambda - \lambda_j)^{m_j}$, $\lambda_j \in \mathbb{C}$, $j = 1\colon q$, with m_j such that $\sum_{j=1}^{q} m_j = mn$. Then [19, Prop. 4.3] leads to

$$J = \bigoplus_{j=1}^{q} J_{m_j}(\lambda_j), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{q} F_{m_j},$$

which satisfy $S_J = S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.4. $*$-even structure

Suppose $P(\lambda)$ is $*$-even with

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1{:}\, r$, and
- s pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\overline{\mu}_j)^{m_j}$, $j = 1{:}\, s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. With the change of eigenvalue parameter $\lambda = -i\mu$, the $*$-even linearization of $P(\lambda)$, $\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \mu(-i\mathcal{A}_\mathcal{S}) + \mathcal{B}_\mathcal{S}$ becomes a Hermitian pencil in μ. Using Appendix A.1 we obtain that

$$J = -i\Big(\bigoplus_{j=1}^{r} J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^{s} \big(J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)\big)\Big), \quad S_J = -i\Big(\bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j}\Big).$$

Here $\{\varepsilon_1, \dots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the zero and purely imaginary eigenvalues of $P(\lambda)$. Note that $S_J = -S_J^*$ and $JS_J = (JS_J)^*$.

Appendix A.5. Real T-even structure

Suppose $P(\lambda)$ is real T-even with (see [16])

- t zero elementary divisors λ^{n_j} with n_j even, $j = 1{:}\, t$,
- r pairs of real elementary divisors $(\lambda + \alpha_j)^{p_j}$, $(\lambda - \alpha_j)^{p_j}$ with p_j odd if $\alpha_j = 0$, $j = 1{:}\, r$,
- s pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1{:}\, s$, and
- q quadruples of nonzero and non-purely imaginary elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \overline{\mu}_j)^{m_j}$, $(\lambda - \overline{\mu}_j)^{m_j}$, $j = 1{:}\, q$,

with n_j, p_j, k_j, m_j such that $\sum_{j=1}^{t} n_j + 2\sum_{j=1}^{r} p_j + 2\sum_{j=1}^{s} k_j + 4\sum_{j=1}^{q} m_j = mn$. Using [10, Thm. 16.1], we find that

$$\begin{aligned} J = {} & \bigoplus_{j=1}^{t} J_{n_j}(0) \oplus \bigoplus_{j=1}^{r} \big(J_{p_j}(\alpha_j) \oplus -J_{p_j}(\alpha_j)^T\big) \\ & \oplus \bigoplus_{j=1}^{s} K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^{q} \big(K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T\big), \end{aligned}$$

$$S_J = \bigoplus_{j=1}^{t} \varepsilon_j E_{n_j} \oplus \bigoplus_{j=1}^{r} \begin{bmatrix} 0 & -I_{p_j} \\ I_{p_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^{s} \varepsilon_j (E_{k_j} \otimes E_2^{k_j}) \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & -I_{2m_j} \\ I_{2m_j} & 0 \end{bmatrix},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the purely imaginary eigenvalues and zero eigenvalues of even partial multiplicities (see [18]). We easily check that $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.6. Complex T-even structure

Let $\lambda_j \in \mathbb{C}\setminus\{0\}$ and suppose $P(\lambda)$ is complex T-even with (see [16])

- t zero elementary divisors λ^{m_j} with m_j even, $j = 1{:}\,t$,
- q pairs of elementary divisors $(\lambda-\lambda_j)^{k_j}$, $(\lambda+\lambda_j)^{k_j}$ with k_j odd if $\lambda_j = 0$, $j = 1{:}\,q$,

with m_j, k_j such that $\sum_{j=1}^{r} m_j + 2\sum_{j=1}^{q} k_j = mn$. Then, by [19, Prop. 4.7 (b)], we obtain that

$$J = \bigoplus_{j=1}^{t} J_{m_j}(0) \oplus \bigoplus_{j=1}^{q} (J_{k_j}(\lambda_j) \oplus J_{k_j}(-\lambda_j)), \quad S_J = \bigoplus_{j=1}^{t} \begin{bmatrix} 0 & -F_{\frac{1}{2}m_j} \\ F_{\frac{1}{2}m_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & -F_{k_j} \\ F_{k_j} & 0 \end{bmatrix}.$$

Note that $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

Appendix A.7. ∗-odd structure

Suppose $P(\lambda)$ is ∗-odd with

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1{:}\,r$ and
- s pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\bar{\mu}_j)^{m_j}$, $j = 1{:}\,s$,

with ℓ_j, m_j such that $\sum_{j=1}^{r} \ell_j + 2\sum_{j=1}^{s} m_j = mn$. Note that for the ∗-odd linearization $\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ of $P(\lambda)$ in (4), the pencil $i(\lambda\mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}})$ is ∗-even and the structure for S_J and J follows from Appendix A.4. We find that

$$J = -i\Big(\bigoplus_{j=1}^{r} J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^{s} \big(J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)\big)\Big), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^{r} \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j},$$

which satisfy $S_J = S_J^*$ and $JS_J = -(JS_J)^*$. Here $\{\varepsilon_1, \ldots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the zero and purely imaginary eigenvalues of $P(\lambda)$.

Appendix A.8. Real T-odd structure

Suppose $P(\lambda)$ is real T-odd with (see [16])

- t zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1{:}\,t$,
- r pairs of real elementary divisors $(\lambda+\alpha_j)^{p_j}$, $(\lambda-\alpha_j)^{p_j}$ with p_j even if $\alpha_j = 0$, $j = 1{:}\,r$,
- s pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1{:}\,s$, and
- q quadruples elementary divisors $(\lambda+\mu_j)^{m_j}$, $(\lambda-\mu_j)^{m_j}$, $(\lambda+\bar{\mu}_j)^{m_j}$, $(\lambda-\bar{\mu}_j)^{m_j}$, $j = 1{:}\,q$,

with ℓ_j, p_j, k_j, m_j such that $\sum_{j=1}^{t} \ell_j + 2\sum_{j=1}^{r} p_j + 2\sum_{j=1}^{s} k_j + 4\sum_{j=1}^{q} m_j = mn$. On using [10, Thm. 17.1] we find that

$$\begin{aligned} J = {} & \bigoplus_{j=1}^{t} J_{\ell_j}(0) \oplus \bigoplus_{j=1}^{r} \big(J_{p_j}(\alpha_j) \oplus -J_{p_j}(\alpha_j)^T\big) \\ & \oplus \bigoplus_{j=1}^{s} K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^{q} \big(K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T\big), \end{aligned}$$

$$S_J = S_J^{-1} = \bigoplus_{j=1}^{t} \varepsilon_j E_{\ell_j} \oplus \bigoplus_{j=1}^{r} \begin{bmatrix} 0 & I_{p_j} \\ I_{p_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^{s} \varepsilon_j (E_{k_j} \otimes E_2^{k_j - 1}) \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & I_{2m_j} \\ I_{2m_j} & 0 \end{bmatrix},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the purely imaginary eigenvalues and the zero eigenvalues with odd partial multiplicities. We easily check that $S_J = S_J^T$ and $JS_J = -(JS_J)^T$.

Appendix A.9. Complex T-odd structure

Let $\lambda_j \in \mathbb{C} \setminus \{0\}$ and suppose $P(\lambda)$ is complex T-odd with (see [16])

- s zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1{:}\, s$, and
- q pairs of elementary divisors $(\lambda + \lambda_j)^{k_j}$, $(\lambda - \lambda_j)^{k_j}$ with k_j even if $\lambda_j = 0$, $j = 1{:}\, q$,

with ℓ_j, k_j such that $\sum_{j=1}^{s} \ell_j + 2\sum_{j=1}^{q} k_j = mn$. It follows from [19, Prop. 4.7 (b)] that

$$J = \bigoplus_{j=1}^{s} J_{\ell_j}(0) \oplus \bigoplus_{j=1}^{q} \left(-J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j)\right), \qquad S_J = S_J^{-1} = \bigoplus_{j=1}^{s} E_{\ell_j} \oplus \bigoplus_{j=1}^{q} F_{2k_j}.$$

Clearly, $S_J = S_J^T$ and $JS_J = -(JS_J)^T$.

Notice the difference between the zero elementary divisors associated with T-even and T-odd pencils (see [16, Cor. 4.3]).

Appendix A.10. ∗-(anti)palindromic structure

Suppose $P(\lambda)$ is complex $*$-palindromic with $-1 \notin \Lambda(P)$ and (see [17])

- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\overline{\lambda}_j)^{k_j}$ with $\lambda_j \in \mathbb{C} \setminus \{0\}$, $|\lambda_j| \neq 1$, $j = 1{:}\, q$,
- t elementary divisors $(\lambda - \lambda_j)^{2\ell_j + 1}$ with $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1$, $j = 1{:}\, t$, and
- s elementary divisors $(\lambda - \lambda_j)^{2m_j}$ with $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$, $j = 1{:}\, s$,

with k_j, ℓ_j, m_j such that $2\sum_{j=1}^{q} k_j + \sum_{j=1}^{t}(2\ell_j + 1) + 2\sum_{j=1}^{s} m_j = mn$. Then using either [20, Thm. 5] or [21, Sec. 2.2.2] we find that

$$J = -S_J S_J^{-*}$$

with

$$S_J = \bigoplus_{j=1}^{q} \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{t} \varepsilon_j \begin{bmatrix} 0 & 0 & F_{\ell_j} J_{\ell_j}(-\lambda_j) \\ 0 & (-\lambda_j)^{1/2} & e_1^T \\ F_{\ell_j} & 0 & 0 \end{bmatrix}$$
$$\oplus \bigoplus_{j=1}^{s} \varepsilon_j \begin{bmatrix} 0_{m_j} & F_{m_j} J_{m_j}(-\lambda_j) \\ F_{m_j} & e_1 e_1^T \end{bmatrix}$$

has the above elementary divisors. Here e_1 is the first column of the identity matrix. The scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the eigenvalues of unit modulus of $P(\lambda)$ (see [8]).

For the $*$-antipalindromic structure, $J = S_J S_J^{-*}$ with S_J as above but with $-\lambda_j$ replaced by λ_j.

Appendix A.11. Real T-(anti)palindromic structure

Suppose $P(\lambda)$ is real T-palindromic with $-1 \notin \Lambda(P)$, $\lambda_j \in \mathbb{C}\setminus\{0\}$, and (see [17])

- r pairs of real elementary divisors $(\lambda-\lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with $\lambda_j \in \mathbb{R}$, $|\lambda_j| \neq 1$, $j = 1{:}\,r$,
- q quadruples of nonreal elementary divisors $(\lambda-\lambda_j)^{n_j}$, $(\lambda-\overline{\lambda}_j)^{n_j}$, $(\lambda-1/\lambda_j)^{n_j}$, $(\lambda-1/\overline{\lambda}_j)^{n_j}$ with $|\lambda_j| \neq 1$, $j = 1{:}\,q$,
- s elementary divisors $(\lambda-1)^{2m_j}$, $j = 1{:}\,s$,
- t pairs of elementary divisors $(\lambda-1)^{2\ell_j+1}$, $(\lambda-1)^{2\ell_j+1}$, $j = 1{:}\,t$,
- u pairs of elementary divisors $(\lambda-\lambda_j)^{\ell'_j}$, $(\lambda-\overline{\lambda}_j)^{\ell'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, ℓ'_j odd, $j = 1{:}\,u$, and
- p pairs of elementary divisors $(\lambda-\lambda_j)^{m'_j}$, $(\lambda-\overline{\lambda}_j)^{m'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, m'_j even, $j = 1{:}\,p$.

We have that $2\sum_{j=1}^{r} k_j + 4\sum_{j=1}^{q} n_j + 2\sum_{j=1}^{s} m_j + 2\sum_{j=1}^{t}(2\ell_j+1) + 2\sum_{j=1}^{u}\ell'_j + 2\sum_{j=1}^{p} m'_j = mn$.

Using [21, Thm. 2.8] we find that $J = -S_J S_J^{-T}$ has the above list of elementary divisors, where

$$S_J = \bigoplus_{j=1}^{r}\begin{bmatrix} 0_{k_j} & F_{k_j}J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j}\end{bmatrix} \oplus \bigoplus_{j=1}^{q}\begin{bmatrix} 0_{2n_j} & K_{2n_j}(-\Lambda_j) \\ F_{n_j}\otimes I_2 & 0_{2n_j}\end{bmatrix} \oplus \bigoplus_{j=1}^{s}\begin{bmatrix} 0 & F_{m_j}J_{m_j}(-1) \\ F_{m_j} & 0\end{bmatrix}$$
$$\oplus \bigoplus_{j=1}^{t}\varepsilon_j\begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j}J_{\ell_j}(-1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j}\end{bmatrix} \oplus \bigoplus_{j=1}^{t}\varepsilon_j\begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j}J_{\ell_j}(-1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j}\end{bmatrix}$$
$$\oplus \bigoplus_{j=1}^{u}\varepsilon_j\begin{bmatrix} 0_{\ell'_j-1} & 0 & K_{\ell'_j-1}(-\Lambda_j) \\ 0 & (-\Lambda_j)^{\frac{1}{2}} & e_1^T\otimes I_2 \\ F_{\frac{1}{2}(\ell'_j-1)}\otimes I_2 & 0 & 0_{\ell'_j-1}\end{bmatrix} \oplus \bigoplus_{j=1}^{p}\varepsilon_j\begin{bmatrix} 0_{m'_j} & K_{m'_j}(-\Lambda_j) \\ F_{\frac{1}{2}m'_j}\otimes I_2 & e_1e_1^T\otimes I_2\end{bmatrix}.$$

Here $(-\Lambda_j)^{\frac{1}{2}}$ is the principal square root of $-\Lambda_j$. The scalars ε_j are signs ± 1 and form the sign characteristic associated with the eigenvalues of unit modulus of $P(\lambda)$ except the eigenvalues 1 with even partial multiplicities (see [8]).

For the T-antipalindromic $P(\lambda)$, $J = S_J S_J^{-T}$ where S_J is as above but with $-\lambda_j, -1, -\Lambda_j$ replaced by $\lambda_j, 1, \Lambda_j$, respectively.

Appendix A.12. Complex T-(anti)palindromic structure

Suppose $P(\lambda)$ is complex T-palindromic with $-1 \notin \Lambda(P)$ and (see [17])

- t elementary divisors $(\lambda-1)^{m_j}$ with m_j even, $j = 1{:}\,t$,
- q pairs of elementary divisors $(\lambda-\lambda_j)^{k_j}$, $(\lambda-1/\lambda_j)^{k_j}$ with k_j odd when $\lambda_j = 1$, $j = 1{:}\,q$,

with m_j, k_j such that $\sum_{j=1}^{t} m_j + 2\sum_{j=1}^{q} k_j = mn$. On using either [20, Thm. 1] or [21, Thm. 2.6], we find that with

$$S_J = \bigoplus_{j=1}^{t}\begin{bmatrix} 0_{m_j/2} & F_{m_j/2}J_{m_j/2}(-1) \\ F_{m_j/2} & e_1e_1^T\end{bmatrix} \oplus \bigoplus_{j=1}^{q}\begin{bmatrix} 0_{k_j} & F_{k_j}J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j}\end{bmatrix}$$

the matrix $J = -S_J S_J^{-T}$ has the above elementary divisors.

Now if $P(\lambda)$ is complex T-antipalindromic with $-1 \notin \Lambda(P)$ and (see [17])

- t elementary divisors $(\lambda - 1)^{\ell_j}$ with ℓ_j odd, $j = 1\colon t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\lambda_j)^{k_j}$ with k_j even if $\lambda_j = 1$, $j = 1\colon q$,

with ℓ_j, k_j such that $\sum_{j=1}^{t} \ell_j + 2\sum_{j=1}^{q} k_j = mn$. On using [21, Thm. 2.6], we find that the matrix $J = S_J S_J^{-T}$ with

$$S_J = \bigoplus_{j=1}^{t} \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix}$$

has the above elementary divisors.

Note that J in Appendix A.10–Appendix A.12 is "almost" in Jordan canonical form.

References

[1] M. Al-Ammari. *Analysis of structured polynomial eigenvalue problems.* PhD thesis, The University of Manchester, Manchester, UK, 2011. MIMS EPrint 2011.89, Manchester Institute for Mathematical Sciences, The University of Manchester, UK.

[2] M. T. Chu and S.-F. Xu. Spectral decomposition of real symmetric quadratic λ-matrices and its applications. *Math. Comp.*, 78:293–313, 2009.

[3] S. D. Garvey, P. Lancaster, A. A. Popov, U. Prells, and I. Zaballa. Filters connecting isospectral quadratic systems. *To appear in Linear Algebra Appl.*, 2012.

[4] I. Gohberg, P. Lancaster, and L. Rodman. Spectral analysis of selfadjoint matrix polynomials. *Ann. of Math. (2)*, 112(1):33–71, 1980.

[5] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra and Applications.* Birkhäuser, Basel, Switzerland, 2005.

[6] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2009. Unabridged republication of book first published by Academic Press in 1982.

[7] N. J. Higham, D. S. Mackey, N. Mackey, and F. Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29(1):143–159, 2006.

[8] P. Lancaster, U. Prells, and L. Rodman. Canonical structures for palindromic matrix polynomials. *Oper. Matrices*, 1(4):469–489, 2007.

[9] P. Lancaster and L. Rodman. Canonical forms for Hermitian matrix pairs under strict equivalence and congruence. *SIAM Rev.*, 47(3):407–443, 2005.

[10] P. Lancaster and L. Rodman. Canonical forms for symmetric/skew-symmetric real matrix pairs under strict equivalence and congruence. *Linear Algebra Appl.*, 406:1–76, 2005.

[11] P. Lancaster and I. Zaballa. *A review of canonical forms for selfadjoint matrix polynomials.* To appear in the Gohberg Memorial Volume. Springer, 2011.

[12] D. S. Mackey. *Structured Linearizations for Matrix Polynomials.* PhD thesis, The University of Manchester, Manchester, UK, 2006. MIMS EPrint 2006.68, Manchester Institute for Mathematical Sciences, The University of Manchester, UK.

[13] D. S. Mackey. Quadratic realizability of matrix polynomials. Part I: The unstructured case. Manuscript, 2011.

[14] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[15] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.

[16] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432:867–891, 2010.

[17] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms of palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[18] V. Mehrmann and H. Xu. Perturbation of purely imaginary eigenvalues of Hamiltonian matrices under structured perturbations. *Electron. J. Linear Algebra*, 17:234–257, 2008.

[19] L. Rodman. Comparison of congruences and strict equivalences for real, complex, and quaternionic matrix pencils with symmetries. *Electron. J. Linear Algebra*, 16:248–283, 2007.

[20] C. Schröder. A canonical form for palindromic pencils and palindromic factorizations. Technical Report No. 316, DFG Research Center, MATHEON, Technische Universität Berlin, Berlin, Germany, 2006. 43 pp.

[21] C. Schröder. *Palindromic and Even Eigenvalue Problems — Analysis and Numerical Methods*. PhD thesis, Technischen Universitat Berlin, Germany, 2008.