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On Commuting Graphs for Elements of Order 3 in Symmetric Groups

Athirah Nawawi and Peter Rowley

Abstract
The commuting graph $C(G, X)$, where $G$ is a group and $X$ is a subset of $G$, is the graph with vertex set $X$ and distinct vertices being joined by an edge whenever they commute. Here the diameter of $C(G, X)$ is studied when $G$ is a symmetric group and $X$ a conjugacy class of elements of order 3.

(MSC2000: 05C25 ; keywords: Commuting graph, Symmetric group, Order 3 elements, Diameter)

1 Introduction

Suppose that $G$ is a finite group and $X$ is a subset of $G$. The commuting graph $C(G, X)$ is the graph with $X$ as the vertex set and two distinct elements of $X$ being joined by an edge if they are commuting elements of $G$. This type of graph has been studied for a wide variety of groups $G$ and selection of subsets of $G$. One of the earliest investigations occurred in Brauer and Fowler [8] in which $X = G \setminus \{1\}$. This particular case has recently been the subject of further study by Segev [14], [15] and Segev and Seitz [16]. A great deal of attention has been focussed on the case when $X$ is a conjugacy class of involutions – the so-called commuting involution graphs. Pioneering work on such graphs appeared in Fischer [13] which led to the construction of the three Fischer groups. Recently various properties of other commuting involution graphs have been studied; see, for example, [2], [3], [4], [5], [11] and [12]. When $X$ is a conjugacy class of non-involutions, $C(G, X)$ has to date received less attention. Never-the-less graphs of this type can be of interest – witness the computer-free uniqueness proof of the Lyon’s simple group by Aschbacher and Segev [1] which employed a commuting graph whose vertex set consisted of the 3-central subgroups of order 3. Also see Baumeister and Stein [7], the results obtained there being used to describe the structure of Bruck loops and Bol loops of exponent 2. Further, commuting graphs when $G$ is a symmetric group have been investigated in Bates, Bundy, Perkins and Rowley [6] and Bundy[9]. The former paper concentrates on the structure of discs (around some fixed vertex) and the diameter of the graph while the latter gives a complete answer as to when $C(G, X)$ is a connected graph.

In the present paper we shall determine the diameters of $C(G, X)$ when $G$ is a symmetric group and $X$ is a $G$-conjugacy class of elements of order 3. So for the rest of this paper we assume $G = Sym(\Omega) = Sym(n)$ with $G$ acting upon the set $\Omega = \{1, \ldots, n\}$ in the usual manner. Also let

$$t = (1, 2, 3)(4, 5, 6)(7, 8, 9) \ldots (3r - 2, 3r - 1, 3r).$$
Thus $t$ has order 3 and cycle type $1^{n-3r}3^r$. Set $X = t^G$, the $G$-conjugacy class of $t$, and let $\text{Diam}(C(G,X))$ denote the diameter of the commuting graph $C(G,X)$. Our main results are as follows.

**Theorem 1.1** If $n \geq 8r$, then $\text{Diam}(C(G,X)) = 2$.

**Theorem 1.2** If $6r < n < 8r$, then $\text{Diam}(C(G,X)) = 3$.

Our last theorem only gives a bound on $\text{Diam}(C(G,X))$.

**Theorem 1.3** If $r > 1$ and $n = 6r$, then $\text{Diam}(C(G,X)) \leq 4$.

Consulting Table 1 (or Table 1 of [6]) we see that for $r = 1, n = 7$ or $r = 2, n = 15$ we have that $\text{Diam}(C(G,X)) = 3$ and so Theorem 1.1 is sharp. For $r = 2$ the same table gives $\text{Diam}(C(G,X)) = 4$ when $n = 12$ and 2 when $n = 16$, so Theorems 1.2 and 1.3 are also sharp. We note that for $r = 1$ and $n = 6$, $C(G,X)$ is disconnected which explains the assumption $r > 1$ in Theorem 1.3. All the graphs we consider here are connected – see [9]. For $g \in G$, $\text{supp}(g)$ denotes the set of points of $\Omega$ not fixed by $g$. We use $d(\cdot)$ for the usual distance metric on the graph $C(G,X)$. For $x \in X$, the $i^{th}$ disc, $\Delta_i(x)$, is defined as follows

$$\Delta_i(x) = \{y \mid y \in X \text{ and } d(x,y) = i\}.$$ 

The proofs of Theorems 1.1, 1.2 and 1.3 adopt a similar, somewhat direct, approach. Since $G$ acting by conjugation upon $X$ induces graph automorphisms on $C(G,X)$ and of course is transitive on $X$, it suffices to determine (or bound) $d(t,x)$ for an arbitrary vertex $x$ of $X$. This we do by writing down explicit paths in $C(G,X)$.

## 2 Diameter of $C(G,X)$

We begin by establishing Theorem 1.1.

**Proof of Theorem 1.1**

Let $x \in X$. Set $\Lambda = \text{supp}(t) \cup \text{supp}(x)$ and $s = |\text{supp}(t) \cap \text{supp}(x)|$. Then $|\Lambda| = 6r - s$. If $s \geq r$, then $|\Lambda| \leq 5r$. Hence there exists $y \in X$ with $\text{supp}(t) \cap \text{supp}(y) = \emptyset = \text{supp}(x) \cap \text{supp}(y)$ and so $d(t,x) \leq 2$. Now consider the case $s < r$, and set $e = r - s$. Without loss of generality we may suppose that $\text{supp}(t) \cap \text{supp}(x) \subseteq \{1, 2, 3, \ldots, 3s - 2, 3s - 1, 3s\}$. Put $y_1 = (3s + 1, 3s + 2, 3s + 3) \ldots (3r - 2, 3r - 1, 3r)$ (so $y_1$ is the product of the “last” $r - s = e$ 3-cycles of $t$). Since $|\Omega \setminus \Lambda| = 8r - (6r - s) = 2r + s > 3s$ and $s < r$, we may select $y_2$ with $\text{supp}(y_2) \subseteq \Omega \setminus \Lambda$ and $y_2$ is a product of $s$ pairwise disjoint 3-cycles. So $y = y_1y_2 \in X$, $ty = yt$ and $xy = yx$. Thus $d(t,x) \leq 2$. Clearly $\text{Diam}(C(G,X)) \geq 2$, and so the theorem follows.

Before proving Theorems 1.2 and 1.3 we introduce some notation and certain permutations of $Sym(\Omega)$. These permutations, though elements of order 3, are not in general in $X$. We will assume that $|\Omega| \geq 6r$. For
Let $x \in X$. Denote the product of the $t_i$’s for which $\vartheta_i(t) \cap supp(x) = \emptyset$ by $\tau_0$ and let $\tau_3$ be the product of the $t_i$’s for which $\vartheta_i(t) \subseteq supp(x)$. Also let $\tau_1$ be the product of $r_1$ $t_i$’s where $|\vartheta_i(t) \cap supp(x)| = 1$, $3 \mid r_1$ and $r_1$ is as large as possible. Analogously, $\tau_2$ is the product of $r_2$ $t_i$’s where $|\vartheta_i(t) \cap supp(x)| = 2$, $3 \mid r_2$ and $r_2$ is as large as possible. Setting $\tau_* = t\tau_0^{-1}\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}$ we have $t = \tau_3\tau_0\tau_1\tau_2\tau_3$. Let $r_*$ be the number of $t_i$’s in $\tau_*$, $r_0$ the number of $t_i$’s in $\tau_0$ and $r_3$ the number of $t_i$’s in $\tau_3$. Observe that the maximality of $r_1$ and $r_2$ means $r_* \leq 4$ and that at most two of the $t_i$’s in $\tau_*$ will have $|\vartheta_i(t) \cap supp(x)| = 1$ and at most two will have $|\vartheta_i(t) \cap supp(x)| = 2$. Evidently $r = r_* + r_0 + r_1 + r_2 + r_3$ and, for $i = 0, 1, 2, 3$, $|supp(x) \cap supp(t_i)| = ir_i$. Putting $s_* = |supp(x) \cap supp(\tau_*)|$, we also have

$$|supp(t) \cap supp(x)| = s_* + r_1 + 2r_2 + 3r_3.$$

Set $\Lambda = \Omega \setminus (supp(t) \cup supp(x))$. Since

$$|supp(t) \cup supp(x)| = 3r + 3r - (s_* + r_1 + 2r_2 + 3r_3) = 6r - (s_* + r_1 + 2r_2 + 3r_3)$$

it follows that

$$|\Lambda| = s_* + r_1 + 2r_2 + 3r_3 \text{ if } n = 6r$$

and

$$|\Lambda| \geq 1 + s_* + r_1 + 2r_2 + 3r_3 \text{ if } n > 6r.$$

Since 3 divides $r_1$, we may write

$$\tau_1 = \prod \mu_{i_1i_2i_3}$$

where the product of the $\mu_{i_1i_2i_3} = t_i_1t_i_2t_i_3$ is pairwise disjoint. For each $\mu_{i_1i_2i_3} = t_i_1t_i_2t_i_3 = (3i_1 - 2, 3i_2 - 2, 3i_3 - 2, 3i_1 - 1, 3i_2 - 1, 3i_3 - 1, 3i_2 - 3, 3i_3 - 1)$ we may without loss, suppose that $supp(\mu_{i_1i_2i_3}) \cap supp(x) = \{3i_1 - 2, 3i_2 - 2, 3i_3 - 2\}$. Put

$$\lambda_{i_1i_2i_3} = (3i_1 - 2, 3i_2 - 2, 3i_3 - 2)(3i_1 - 1, 3i_2 - 1, 3i_3 - 1)(3i_1, 3i_2, 3i_3).$$

Then $\lambda_{i_1i_2i_3}$ commutes with $\mu_{i_1i_2i_3}$. Let

$$\rho_1 = \prod \lambda_{i_1i_2i_3}$$

and observe that $\rho_1$ commutes with $t$ and will be a pairwise disjoint product of $r_1$ 3-cycles. Further, $\frac{1}{3}$ of the 3-cycles in $\rho_1$ will have their support contained in $supp(x)$ while the remaining $\frac{2r_1}{3}$ 3-cycles in $\rho_1$ will have their support intersecting $supp(x)$ in the empty set.

Also, as 3 divides $r_2$, we may express

$$\tau_2 = \prod \eta_{i_1i_2i_3}$$
where $\eta_{j_1j_2j_3} = t_{j_1}t_{j_2}t_{j_3}$ with the product being pairwise disjoint. For each $\eta_{j_1j_2j_3}$ we may suppose that $\text{supp}(\eta_{j_1j_2j_3}) \cap \text{supp}(x) = \{3j_1 - 2, 3j_1 - 1, 3j_2 - 2, 3j_2 - 1, 3j_3 - 2, 3j_3 - 1\}$. Define

$$\delta_{j_1j_2j_3} = (3j_1, 3j_2, 3j_3)(3j_1 - 2, 3j_2 - 2, 3j_3 - 2)(3j_1 - 1, 3j_2 - 1, 3j_3 - 1),$$

and let

$$\rho_2 = \prod \delta_{j_1j_2j_3}.$$

Evidently $\rho_2$ commutes with $t$ and $\rho_2$ is a pairwise disjoint product of $r_2$ 3-cycles. Moreover, $\frac{2r_2}{3}$ of the 3-cycles in $\rho_2$ will have their support contained in $\text{supp}(x)$ and the remaining $\frac{r_2}{3}$ have supports intersecting $\text{supp}(x)$ in the empty set.

Let $\sigma_1$ (respectively $\sigma_4$) be the product of the $\frac{2r_2}{3}$ (respectively $\frac{r_2}{3}$) 3-cycles in $\rho_1$ (respectively $\rho_2$) whose support intersects $\text{supp}(x)$ in the empty set. Also let $\sigma_4$ be a pairwise disjoint product of $(\frac{r_2}{3} + \frac{2r_2}{3})$ 3-cycles with $\text{supp}(\sigma_4) \subseteq \Lambda$. Put $\Delta = \Lambda \backslash \text{supp}(\sigma_4)$.

We now summarize the pertinent properties of the permutations just introduced.

**Lemma 2.1**

(i) $\text{supp}(\tau_0\rho_1\rho_2\tau_3) \subseteq \text{supp}(t)$, $\tau_0\rho_1\rho_2\tau_3$ commutes with $t$ and is the product of $r - r_*$ pairwise disjoint 3-cycles.

(ii) $\sigma_1\sigma_2\tau_0\sigma_4$ commutes with $\tau_0\rho_1\rho_2\tau_3$ and is the product of $r - r_*$ pairwise disjoint 3-cycles. Moreover $\text{supp}(\sigma_1\sigma_2\tau_0\sigma_4) \cap \text{supp}(x) = \emptyset$.

(iii) $|\Delta| = s_*$ if $n = 6r$ and $|\Delta| \geq 1 + s_*$ if $n \geq 6r$.

**Proof**

(i) Since $\text{supp}(\rho_1\rho_2) = \text{supp}(\tau_1\tau_2)$, $\tau_0\rho_1\rho_2\tau_3$ is the product of pairwise disjoint 3-cycles, and the number of such 3-cycles is $r - r_*$. Because $\rho_1$ and $\rho_2$ both commute with $t$, $\tau_0\rho_1\rho_2\tau_3$ commutes with $t$.

(ii) Since $\text{supp}(\sigma_4) \subseteq \Delta$ and $\tau_0\rho_1\rho_2\tau_3$ commutes with $\tau_0\rho_1\rho_2\tau_3$, while $\sigma_1\sigma_2\tau_0$ is a product of 3-cycles which appear in $\tau_0\rho_1\rho_2\tau_3$ and therefore $\sigma_1\sigma_2\tau_0\sigma_4$ commutes with $\tau_0\rho_1\rho_2\tau_3$. By construction $\sigma_1 \cap \text{supp}(x) = \emptyset (i = 1, 2)$, $\text{supp}(\tau_0) \cap \text{supp}(x) = \emptyset$ by definition and because we chose $\sigma_4$ so as $\text{supp}(\sigma_4) \subseteq \Lambda$ we get $\text{supp}(\sigma_1\sigma_2\tau_0\sigma_4) \cap \text{supp}(x) = \emptyset$.

(iii) Part (iii) follows from $|\text{supp}(\sigma_4)| = r_1 + 2r_2 + 3r_3$ and $\Delta = \Lambda \backslash \text{supp}(\sigma_4)$.

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2**

Let $y \in X$ be such that $|\text{supp}(y) \cap \vartheta_i(t)| = 1$ for $i = 1, \ldots, r$. Then $C_G(t) \cap C_G(y) = \text{Sym}(\Psi)$ where $\Psi = \Omega \backslash (\text{supp}(t) \cup \text{supp}(y))$. Now $|\text{supp}(t) \cup \text{supp}(y)| = 3r + 3r - r = 5r$ and so $|\Psi| = n - 5r < 8r - 5r = 3r$. Thus $X \cap C_G(t) \cap C_G(y) = \emptyset$ and consequently $d(t, y) \geq 3$. Hence $\text{Diam} (C(G, X)) \geq 3$.

Let $x \in X$. We aim to show that $d(t, x) \leq 3$. On account of $C_G(t)$ having shape $3'\text{Sym}(r) \times \text{Sym}(n - 3r)$ there is no loss in supposing $\tau_* = t_1 \ldots t_*$. 


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where $0 \leq r_* \leq 4$ ($r_*=0$ meaning $\tau_* = 1$). Depending on $\tau_*$ we define two elements $\rho_*$ and $\sigma_*$ which will be the product of $r_*$ pairwise disjoint 3-cycles.

(1) $r_* = 4$

Then we have $\tau_* = t_1t_2t_3t_4 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$, $s_* = 6$ and we may, without loss, assume $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 4, 7, 8, 10, 11\}$. Observe that $|\text{supp}(x) \setminus \text{supp}(t)| \geq 6$ and so we may select $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \text{supp}(x) \setminus \text{supp}(t)$. Also by Lemma 2.1(iii), as $s_* = 6$, $|\Delta| \geq 7$. Thus we may also select $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \Delta$. Define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \alpha_6)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6)$$

and

$$\sigma_* = (2, 3, 5)(6, 9, 12)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6).$$

(2) $r_* = 3$

So $\tau_* = t_1t_2t_3 = (1, 2, 3)(4, 5, 6)(7, 8, 9)$. First we examine the case when $s_* = 4$, and may suppose that $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 4, 7, 8\}$. Here we have $|\text{supp}(x) \setminus \text{supp}(t)| \geq 5$ and $|\Delta| \geq 5$. Choose $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \text{supp}(x) \setminus \text{supp}(t)$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \Delta$, and define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \beta_1)(\beta_2, \beta_3, \beta_4)$$

and

$$\sigma_* = (2, 3, 5)(6, 9, \beta_5)(\beta_2, \beta_3, \beta_4).$$

We move onto the case when $s_* = 5$ and, without loss of generality, assume $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 2, 4, 5, 7\}$. Since $|\text{supp}(x) \setminus \text{supp}(t)| \geq 4$ and $|\Delta| \geq 6$, we may select $\alpha_1, \alpha_2, \alpha_3 \in \text{supp}(x) \setminus \text{supp}(t)$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \Delta$. Then we take

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6)$$

and

$$\sigma_* = (3, 6, 8)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6).$$

(3) $r_* = 2$

So $\tau_* = t_1t_2 = (1, 2, 3)(4, 5, 6)$ with $s_* = 2$, 3 or 4. First we look at the case when $s_* = 2$ or 3. Then we have $|\text{supp}(x) \setminus \text{supp}(t)| \geq 3$, $|\text{supp}(x) \setminus \text{supp}(t)| \geq 3$ and $|\Delta| \geq 3$. Choosing $\alpha_1, \alpha_2, \alpha_3 \in \text{supp}(x) \setminus \text{supp}(t)$, $\beta_1, \beta_2, \beta_3 \in \Delta$ and $\gamma_1, \gamma_2, \gamma_3 \in \text{supp}(t) \setminus \text{supp}(x)$, we let

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)$$

and

$$\sigma_* = (\gamma_1, \gamma_2, \gamma_3)(\beta_1, \beta_2, \beta_3).$$

Now assume that $s_* = 4$, and, without loss, that $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 2, 4, 5\}$. Because $|\text{supp}(x) \setminus \text{supp}(t)| \geq 2$ and $|\Delta| \geq 5$ we may choose $\alpha_1, \alpha_2 \in \text{supp}(x) \setminus \text{supp}(t)$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \Delta$ and then define

$$\rho_* = (\alpha_1, \alpha_2, \beta_1)(\beta_2, \beta_3, \beta_4)$$
and
\[ \sigma_* = (3, 6, \beta_3)(\beta_2, \beta_3, \beta_4). \]

(4) \( r_* = 1 \)

Then \( \tau_* = t_1 = (1, 2, 3) \) and \( s_* = 1 \) or \( 2 \). Suppose \( s_* = 1 \) with \( \text{supp}(\tau_*) \cap \text{supp}(x) = \{1\} \). So \( |\text{supp}(x) \setminus \text{supp}(t)| \geq 2 \leq |\Delta| \). Selecting \( \alpha_1, \alpha_2 \in \text{supp}(x) \setminus \text{supp}(t) \) and \( \beta_1, \beta_2 \in \Delta \), we set
\[ \rho_* = (\alpha_1, \alpha_2, \beta_1) \]
and
\[ \sigma_* = (2, 3, \beta_2). \]
While if \( s_* = 2 \), then \( |\Delta| \geq 3 \) and selecting \( \beta_1, \beta_2, \beta_3 \in \Delta \) we set
\[ \rho_* = \sigma_* = (\beta_1, \beta_2, \beta_3). \]

(5) \( r_* = 0 \)

Here we take \( \rho_* = 1 = \sigma_* \).

Put \( y = \rho_* \tau_0 \rho_1 \rho_2 \tau_3 \). Since \( y \) is the product of \( r_* + r_0 + r_1 + r_2 + r_3 = r \) disjoint 3-cycles, \( y \in X \). Further we have that \( ty = yt \) by Lemma 2.1(i).

Next we consider \( z = \sigma_* \tau_1 \tau_2 \tau_3 \). Each of \( \sigma_* \tau_1, \sigma_2, \tau_0 \) and \( \sigma_4 \) are pairwise disjoint. Recalling that \( \sigma_1, \sigma_2 \) and \( \sigma_4 \) are, respectively, the product of \( \frac{2r_1}{3}, \frac{r_2}{3}, (\frac{r_1}{3} + \frac{2r_2}{3} + r_3) \) disjoint 3-cycles, we see that \( z \in X \). It may be further checked using Lemma 2.1(ii) that \( yz = zy \) and \( xz = zx \), and consequently \( d(t, x) \leq 3 \). This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3**

Let \( x \in X \). Our objective here is to show that \( d(t, x) \leq 4 \) from which it will follow that \( \text{Diam} \left(G, X\right) \leq 4 \). We proceed in a similar fashion to that in the proof of Theorem 1.1 though here, except for some cases, we will define three permutations \( \rho_*, \sigma_*, \xi_* \), each a product of \( r_* \) pairwise disjoint cycles.

(6) \( r_* = 4 \)

So \( \tau_* = t_1 t_2 t_3 t_4 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12) \) with \( s_* = 6 \). Assume, without loss, that \( \text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 4, 7, 8, 10, 11\} \). Since \( |\text{supp}(x) \setminus \text{supp}(t)| \geq 6 \) and so we may choose \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \text{supp}(x) \setminus \text{supp}(t) \). Further, as \( |\Delta| = s_* = 6 \) by Lemma 2.1(iii), we may also choose \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \Delta \). Now define
\[ \rho_* = (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \alpha_6)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6) \]
and
\[ \sigma_* = (2, 3, 5)(6, 9, 12)(\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6). \]

(7) \( r_* = 3 \)
So $\tau_* = t_1 t_2 t_3 = (1, 2, 3)(4, 5, 6)(7, 8, 9)$. If $s_* = 4$ we may suppose without loss that $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 4, 7, 8\}$. Here we have $|\text{supp}(x) \setminus \text{supp}(t)| \geq 5$ and $|\Delta| = s_* = 4$ by Lemma 2.1(iii). Choose $\alpha_1, \alpha_2, \alpha_3 \in \text{supp}(x) \setminus \text{supp}(t)$ and $\beta_1, \beta_2, \beta_3 \in \Delta$, and define

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)(1, 2, 3),$$

$$\sigma_* = (5, 6, 9)(\beta_1, \beta_2, \beta_3)(1, 2, 3)$$

and

$$\xi_* = (5, 6, 9)(\beta_1, \beta_2, \beta_3)(\alpha, \beta, \gamma),$$

where $(\alpha, \beta, \gamma)$ is a 3-cycle of $x$ for which $1 \notin \{\alpha, \beta, \gamma\}$. Note that $\{\alpha, \beta, \gamma\} \cap \text{supp}(\sigma_*) = \emptyset$.

For the case when $s_* = 5$, without loss of generality, we assume $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 4, 5, 7, 8\}$. Since $|\text{supp}(x) \setminus \text{supp}(t)| \geq 4$ and $|\Delta| = s_* = 5$, we may select $\alpha_1, \alpha_2, \alpha_3 \in \text{supp}(x) \setminus \text{supp}(t)$ and $\beta_1, \beta_2, \beta_3 \in \Delta$. Then we take

$$\rho_* = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)(4, 5, 6),$$

$$\sigma_* = (2, 3, 9)(\beta_1, \beta_2, \beta_3)(4, 5, 6)$$

and

$$\xi_* = (2, 3, 9)(\beta_1, \beta_2, \beta_3)(\alpha, \beta, \gamma),$$

where $(\alpha, \beta, \gamma)$ is a 3-cycle of $x$ chosen so as $\{4, 5\} \cap \{\alpha, \beta, \gamma\} = \emptyset$. Since $r \geq r_* = 3$ such a choice is possible.

Before dealing with $r_* = 2$ we analyze a number of small cases.

**(8)** Suppose that $t = (1, 2, 3)(4, 5, 6)$ (so $r = 2$ and $n = 12$).

(i) If $x = (1, 7, 8)(4, 9, 10)$ or $x = (1, 4, 7)(2, 5, 8)$, then $d(t, x) \leq 4$.

(ii) If $x = (1, 4, 7)(8, 9, 10)$, then $d(t, x) \leq 3$.

Assume that $x = (1, 7, 8)(4, 9, 10)$, and let $x_1 = (7, 8, 11)(9, 10, 12), x_2 = (2, 3, 5)(9, 10, 12), x_3 = (2, 3, 5)(1, 7, 8)$. Then $x_1, x_2, x_3 \in X$ and $(t, x_1, x_2, x_3, x)$ is a path in $C(G, X)$ whence $d(t, x) \leq 4$. In the case $x = (1, 4, 7)(2, 5, 8)$ we take $x_1 = (7, 8, 9)(10, 11, 12), x_2 = (1, 3, 6)(10, 11, 12)$ and $x_3 = (2, 5, 8)(10, 11, 12)$. It is easily checked that $(t, x_1, x_2, x_3, x)$ is also a path in $C(G, X)$, so proving part (i). For $x = (1, 4, 7)(8, 9, 10)$ taking $x_1 = (1, 2, 3)(8, 9, 10)$ and $x_2 = (5, 6, 11)(8, 9, 10)$ gives a path $(t, x_1, x_2, x)$ in $C(G, X)$. So (ii) holds and (8) is proved.

**(9)** Suppose $t = (1, 2, 3)(4, 5, 6)(7, 8, 9)$ with $\tau_* = (1, 2, 3)(4, 5, 6)$ (so $r = 3$ and $n = 18$). Let $x \in X$ be such that $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1, 4\}$ and assume 1 and 4 are in different 3-cycles of $x$. Then $d(t, x) \leq 4$.

By assumption $x = (1, *, *) (4, \delta, e)(\alpha, \beta, \gamma)$ with $\{1, 4\} \cap \{\alpha, \beta, \gamma\} = \emptyset$. Because $\tau_* = (1, 2, 3)(4, 5, 6)$ we must have $\text{supp}(t) \cap \text{supp}(x) = \{1, 4\}$ or $\{1, 4, 7, 8, 9\}$. Suppose the former holds and set $x_1 = (1, 2, 3)(\alpha, \beta, \gamma)(7, 8, 9)$ and $x_2 = (4, \delta, e)(\alpha, \beta, \gamma)(7, 8, 9)$. Then $(t, x_1, x_2, x)$ is a path in $C(G, X)$. Hence $d(t, x) \leq 4$. Turning to the latter case we have $|\text{supp}(t) \cup \text{supp}(x)| = 13$.
So we may choose, say, $16, 17, 18 \in \Lambda$ and then take $x_1 = (1, 2, 3)(4, 5, 6)(16, 17, 18)$, $x_2 = (1, 2, 3)(\alpha, \beta, \gamma)(16, 17, 18)$ and $x_3 = (4, \delta, \epsilon)(\alpha, \beta, \gamma)(16, 17, 18)$, giving a path $\langle t, x_1, x_2, x_3, x \rangle$ in $C(G, X)$. Thus $d(t, x) \leq 4$, so proving (9).

(10) $r_\ast = 2$

So we have $\tau_\ast = t_1t_2 = (1, 2, 3)(4, 5, 6)$ with $s_\ast = 2, 3$ or 4. First we consider the case $s_\ast = 2$, and assume $\text{supp}(\tau_\ast) \cap \text{supp}(x) = \{1, 4\}$. For the moment also assume that $r = 2$ (so $t = \tau_\ast$). Then, without loss, $x$ is either $(1, 7, 8)(4, 9, 10)$ or $(1, 4, 7)(8, 9, 10)$ (1 and 4 in different 3-cycles of $x$). By (8)(i) we have $d(t, x) \leq 4$. So, since we are examining to show that $d(t, x) \leq 4$, we may suppose $r \geq 3$. Now consider the possibility that $r = 3$ or $1$ and 4 in three different cycles of $x$. Then, without loss, $x = (1, *, *)(4, \delta, \epsilon)(\alpha, \beta, \gamma)$ in which case $d(t, x) \leq 4$ by (9). Thus, when $r = 4$, we may suppose $1$ and 4 are in the same 3-cycle of $x$. Consequently, as $r \geq 3$, we may find two 3-cycles of $x$, $(\alpha, \beta, \gamma)$ and $(\delta, \epsilon, \lambda)$ such that $\{\alpha, \beta, \gamma, \delta, \epsilon, \lambda\} \cap \{1, 4\} = \emptyset$. Now we define $\rho_\ast, \sigma_\ast$ and $\xi_\ast$ by taking $\rho_\ast = \sigma_\ast = \tau_\ast$ and $\xi_\ast = (\alpha, \beta, \gamma), (\delta, \epsilon, \lambda)$.

Next we look at the case $s_\ast = 3$. Then we have $|\text{supp}(x) \setminus \text{supp}(t)| \geq 3$, $|\text{supp}(t) \setminus \text{supp}(x)| \geq 3$ and $|\Delta| = s_\ast = 3$. Choosing $\alpha_1, \alpha_2, \alpha_3 \in \text{supp}(x) \setminus \text{supp}(t)$, $\beta_1, \beta_2, \beta_3 \in \Delta$ and $\gamma_1, \gamma_2, \gamma_3 \in \text{supp}(t) \setminus \text{supp}(x)$, we let

$$\rho_\ast = (\alpha_1, \alpha_2, \alpha_3)(\beta_1, \beta_2, \beta_3)$$

and

$$\sigma_\ast = (\gamma_1, \gamma_2, \gamma_3)(\beta_1, \beta_2, \beta_3).$$

Finally we come to $s_\ast = 4$. So without loss we have $\text{supp}(\tau_\ast) \cap \text{supp}(x) = \{1, 2, 4, 5\}$. Suppose, for the moment, that for all 3-cycles $(\alpha, \beta, \gamma)$ we have $\{1, 2\} \cap \{\alpha, \beta, \gamma\} \neq \emptyset \neq \{4, 5\} \cap \{\alpha, \beta, \gamma\}$. Then it follows that $r = 2$ and, without loss, $x = (1, 4, 7)(2, 5, 8)$. But then $d(t, x) \leq 4$ by (8)(ii). Thus we may suppose $x$ contains a 3-cycle $(\alpha, \beta, \gamma)$ such that $(\alpha, \beta, \gamma) \cap \{1, 2\} = \emptyset$, and we can now define $\rho_\ast$ and $\sigma_\ast$. Since $|\Delta| = s_\ast = 4$, we have $\beta_1, \beta_2, \beta_3 \in \Delta$. Let $\rho_\ast = (1, 2, 3)(\beta_1, \beta_2, \beta_3)$ and $\sigma_\ast = (\alpha, \beta, \gamma)(\beta_1, \beta_2, \beta_3)$. This completes the case $s_\ast = 4$ and (10).

Yet another special case must be looked at before doing $r_\ast = 1$.

(11) Let $t = (1, 2, 3)(4, 5, 6)$ with $\tau_\ast = (1, 2, 3)$. Suppose $x = (1, *, *)(2, *, *) \in X$ with $\text{supp}(\tau_\ast) \cap \text{supp}(x) = \{1, 2\}$. Then $d(t, x) \leq 3$.

Since $\tau_\ast = (1, 2, 3)$, $\text{supp}(t) \cap \text{supp}(x) = \{1, 2\}$ or $\{1, 2, 4, 5, 6\}$. If $\text{supp}(t) \cap \text{supp}(x) = \{1, 2\}$ and, say $\Omega \setminus (\text{supp}(t) \cap \text{supp}(x)) = \{11, 12\}$, then define $x_1 = (4, 5, 6)(10, 11, 12), x_2 = (4, 5, 6)(\alpha, \beta, \gamma)$ where $(\alpha, \beta, \gamma)$ is a 3-cycle not containing 10. While in the other case with, say $\Omega \setminus (\text{supp}(t) \cap \text{supp}(x)) = \{8, 9, 10, 11, 12\}$ we define $x_1 = (8, 9, 10)(7, 11, 12), x_2 = (8, 9, 10)(\alpha, \beta, \gamma)$ where $(\alpha, \beta, \gamma)$ is a 3-cycle not containing 7. Hence $d(t, x) \leq 3$. 

8
(12) $r_* = 1$

So we have either, without loss, $\text{supp}(\tau_*) \cap \text{supp}(x) = \{1\}$ or \{2, 3\}. In view of (10), as $r > 1$, either $d(t, x) \leq 3$ or we may find a 3-cycle $(\alpha, \beta, \gamma)$ of $x$ for which $\text{supp}(\tau_*) \cap \{\alpha, \beta, \gamma\} = \emptyset$. In the latter case we define $\rho_* = \sigma_* = \tau_*$ and $\xi_* = (\alpha, \beta, \gamma)$.

(13) $r_* = 0$

Just as in (5) we take $\rho_* = 1 = \sigma_*$.  
Now let $y = \rho_* \tau_0 \rho_1 \rho_2 \tau_3$, $z = \sigma_* \tau_0 \sigma_1 \sigma_2 \tau_0 \sigma_4$ and $w = \xi_* \tau_0 \sigma_1 \sigma_2 \sigma_3 \sigma_4$ (where $w$ is only defined if in (6), (7), (10), (12), (13) $\xi_*$ is defined). Then $y, z, w \in X$ with $(t, y, z, w, x)$ is a path in $\mathcal{C}(G, X)$. Consequently $d(t, x) \leq 4$. Since $x$ was an arbitrary vertex, this shows that $\text{Diam} \ (\mathcal{C}(G, X)) \leq 4$ and completes the proof of Theorem 1.3.

We end this paper with a table containing some calculations on diameters and discs using MAGMA[10]. Each entry in the table first gives the size of the relevant $\Delta_i(t)$ for the given $r$ and $n$ with the number in brackets being the number of $C_G(t)$-orbits on $\Delta_i(t)$. A blank entry means that $|\Delta_i(t)| = 0$. 

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9
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</table>

Table 1: Disc sizes and $C_G(t)$-orbits
References


[12] Everett, Alistaire; Rowley, Peter Commuting Involution Graphs for 4-Dimensional Projective Symplectic Groups.


