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Finite groups admitting a Frobenius group of automorphisms with fixed-point-free kernel

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Abstract

Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. There are good reasons to expect many properties and parameters of G to be close to the same properties and parameters of $C_G(H)$ (possibly, also depending on $|H|$). We discuss several recent results in this direction. The properties and parameters in question include the order, rank, Fitting height, nilpotency class, and the exponent.

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1 Introduction

Recall that a finite Frobenius group FH with kernel F and complement H is a semidirect product of a normal subgroup F and a subgroup H such that every element of H acts fixed-point-freely on F, that is, $C_F(h) = 1$ for all $1 \neq h \in H$. By the celebrated theorem of Thompson, F is nilpotent, and by the Higman–Kreknin–Kostrikin theorem its nilpotency class is at most $h(p)$, where p is the least prime dividing $|H|$. The structure of H is also quite restricted: all abelian subgroups of H are cyclic, and its composition factors are known, and so on.

Frobenius groups often occur in finite groups; they naturally induce groups of automorphisms by conjugation.

Suppose that a finite group G admits a Frobenius group of automorphisms $FH \leqslant \operatorname{Aut} G$ with kernel F and complement H such that $C_G(F) = 1$. The condition $C_G(F) = 1$ alone already implies strong restrictions on the structure of G. Since F is nilpotent being a Frobenius kernel, G is soluble by a theorem of Belyaev and Hartley [2, Theorem 0.11] based on the classification of finite simple groups. The Fitting height of G is bounded in terms of the number of primes whose product gives $|F|$ by a special case of Dade's theorem [3]. In the case of coprime orders of G and F, this result is in the earlier paper of Thompson [15]. The bounds in the latter paper were improved by various authors, including the linear bounds by Kurzweil [11] and Turull [16].

The "additional" action of the Frobenius complement H suggests another approach to the study of G. By Clifford's theorem, every FH-invariant abelian section V of G is a direct sum of $|H|$ subgroups freely permuted by H, so that $C_V(H)$ is the "diagonal", and V is "$|H|$ times $C_V(H)$". Therefore it is natural to expect that many properties or parameters of G should be close to the corresponding properties or parameters of $C_G(H)$, possibly also depending on H. We discuss several recent results of the author, N. Yu. Makarenko, and P. Shumyatsky in this direction. The properties and parameters in question include the order, rank, Fitting height, nilpotency class, and the exponent. Many of the results rely on Lie ring methods, and analogous results are proved for Lie rings admitting Frobenius groups of automorphisms with fixed-point-free kernel. There still remain many open problems in this area, which are discussed at appropriate places of the talk.

2 Applications of Clifford's theorem.

We consider a finite group G admitting a Frobenius group $FH \leqslant \operatorname{Aut} G$ with complement H and kernel F such that $C_G(F) = 1$. Why the properties of G are expected to be close to the properties $C_G(H)$? Consider the simplest case, where $G = V$ is a vector space in additive notation (in particular, $C_V(F) = 0$). An application of Clifford's theorem gives the following lemma.

Lemma 2.1 *If a Frobenius group of linear transformations FH with kernel F acts on a vector space V over a field k in such a manner that $C_V(F) = 0$, then V is a free kH-module.*

By definition, a free kH-module is a direct sum of several copies of the group algebra kH; equivalently, $V = \bigoplus_{h\in H} W_h$, where $W_h g = W_{hg}$ for $g, h \in H$.

The fixed points of H in a free kH-module

$$V = \bigoplus_{h\in H} W_h, \qquad W_h g = W_{hg},$$

are obviously the diagonal elements:

$$C_V(H) = \left\{ \sum_{h \in H} wh \mid w \in W_1 \right\}.$$

Hence, $\dim V = |H| \cdot \dim C_V(H)$. Roughly speaking, "$V = |H|$ times $C_V(H)$". The idea is that something like this must also hold in general, for any group G admitting a Frobenius group $FH \leqslant \operatorname{Aut} G$ with kernel F such that $C_G(F) = 1$.

We now discuss covering fixed points in quotient groups. Let $A \leqslant \operatorname{Aut} G$, and let N be a normal A-invariant subgroup; consider the induced action of A on G/N. It is well known that if $(|G|, |A|) = 1$, then $C_G(A)N/N = C_{G/N}(A)$; but this is not true in general.

Lemma 2.2 *If a nilpotent group $F \leqslant \operatorname{Aut} G$ acts fixed-point freely: $C_G(F) = 1$, then $C_{G/N}(F) = 1$ for any normal F-invariant N.*

This is useful for us because we do not assume $|G|$ to be coprime to $|FH|$.

Less obvious is the behaviour of fixed points "over" free modules.

Lemma 2.3 *Let $A \leqslant \operatorname{Aut} G$, and let M be an A-invariant elementary abelian p-subgroup that is a free $\mathbb{F}_p A$-module. Then $C_{G/M}(A) = C_G(A)M/M$.*

Proof. Free module $\Rightarrow$ first cohomology group is trivial $H^1(A, M) = 0$ $\Leftrightarrow$ all complements of M in MA are conjugate. If $cM \in C_{G/M}(A)$, then A^c is another complement of M in MA. We have $A^c = A^{am} = A^m$ for $a \in A$, $m \in M \Rightarrow A^{cm^{-1}} = A \Rightarrow [A, cm^{-1}] \in A \cap \langle c \rangle M = 1$, so that $cm^{-1} \in C_G(A) \cap cM$. □

(It is also possible to compute a fixed point directly, without using cohomology.)

Theorem 2.4 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ with complement H and kernel F such that $C_G(F) = 1$. If N is an FH-invariant normal subgroup such that $C_N(F) = 1$, then $C_{G/N}(H) = C_G(H)N/N$.*

Proof. Easy induction: consider an unrefinable normal FH-invariant series

$$N = N_1 > N_2 > \cdots > N_l > N_{l+1} = 1.$$

The hypothesis is inherited by G/N_i by Lemma 2.2. Each factor N_i/N_{i+1} is a free $\mathbb{F}_p H$-module by Lemma 2.1. Then Lemma 2.3 is applied to N_i/N_{i+1} and G/N_{i+1}. □

Theorem 2.5 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ with complement H and kernel F such that $C_G(F) = 1$. Then $|G| = |C_G(H)|^{|H|}$.*

Proof. This follows immediately from Theorem 2.4, Lemma 2.1, and the description of fixed points for H in free H-modules. □

Recall that the rank is the least r such that every subgroup can be generated by r elements. The proof of the following theorem is only a little more complicated than that of Theorem 2.5.

Theorem 2.6 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ with complement H and kernel F such that $C_G(F) = 1$. Then the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$.*

Recall that the Fitting series starts with the Fitting subgroup $F_1(G) := F(G)$, the largest normal nilpotent subgroup, and then by induction $F_{i+1}(G)$ is the full inverse image of $F(G/F_i(G))$ in G. The least l such that $F_l(G) = G$ is called the Fitting height of a soluble group G. The following theorem is proved in [8].

Theorem 2.7 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ with complement H and kernel F such that $C_G(F) = 1$. Then*

(a) *$F_i(C_G(H)) = F_i(G) \cap C_G(H)$ for all i;*

(b) *the Fitting height of G is equal to the Fitting height of $C_G(H)$.*

Note: $|G|$ is not assumed to be coprime with $|FH|$.

Clearly, (b) immediately follows from (a). In part (a) the main case is proving that $F(C_G(H)) = F(G) \cap C_G(H)$. Proof is using Clifford's theorem.

Problem 2.8 *Prove similar results for other "radicals" $R(G)$ instead of $F(G)$.*

Corollary 2.9 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ with complement H and kernel F such that $C_G(F) = 1$. If $C_G(H)$ is nilpotent, then G is nilpotent.*

(This was actually proved earlier than Theorem 2.7.)

Theorem 2.7 largely reduces further study to the case of nilpotent groups.

We state one more corollary in the spirit of Problem 2.8. Recall that $O_\pi(G)$ is the maximal normal π-subgroup; then by induction $O_{\pi_1,\dots,\pi_k}(G)$ is the full inverse image of $O_{\pi_k}\big(G/O_{\pi_1,\dots,\pi_{k-1}}(G)\big)$ in G.

The π-length of a π-soluble group G is the minimum number of symbols π in $O_{\pi',\pi,\pi',\pi\dots,\pi,\pi'}(G) = G$, where π' is the complementary set of primes of π.

Corollary 2.10 *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then*

(a) $O_\pi(C_G(H)) = O_\pi(G) \cap C_G(H)$*;*

(b) *the π-length of G is equal to the π-length of $C_G(H)$;*

(c) $O_{\pi_1,\pi_2,\dots,\pi_k}(C_G(H)) = O_{\pi_1,\pi_2,\dots,\pi_k}(G) \cap C_G(H)$*.*

Proof. Parts (b) and (c) follow by easy induction from (a), since $C_G(H)$ covers $C_{G/N}(H)$ by Theorem 2.4. Obviously, we only need to prove $O_\pi(C_G(H)) \leqslant O_\pi(G) \cap C_G(H)$. We argue by contradiction: suppose that

$$O_\pi(C_G(H)) \nleqslant O_\pi(G) \cap C_G(H);$$

let $\bar{G} = G/O_\pi(G)$. Then there is $q \in \pi$ such that $O_q(C_{\bar{G}}(H)) \neq 1$. We have $O_q(C_{\bar{G}}(H)) \leqslant F(C_{\bar{G}}(H)) = F(\bar{G}) \cap C_{\bar{G}}(H)$ by Theorem 2.7(a). Hence, $O_q(\bar{G}) \neq 1$, a contradiction with $O_\pi(\bar{G}) = 1$. $\square$

3 Bounding nilpotency class

The following theorem was proved in [9].

Theorem 3.1 *Suppose that a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel F and complement H such that $C_G(F) = 1$. If $C_G(H)$ is nilpotent of class c, then G is nilpotent of $(c, |H|)$-bounded class.*

Recall that G is soluble by [2] (mod CFSG), and nilpotent by Corollary 2.9. Thus, it is all about bounding the nilpotency class. Earlier Makarenko and Shumyatsky [12] proved this in the special case where GF is also Frobenius (so that GFH is "2-Frobenius"), which answered Mazurov's question 17.72(a) in Kourovka Notebook [17]. (Earlier still, this was proved in [6] for 2-Frobenius groups GFH with abelian $C_G(H)$.)

Problem 3.2 *Is the dependence on $|H|$ essential?*

So far there are only examples in [1] with class of G greater than that of $C_G(H)$.

The proof of Theorem 3.1 is based on an analogous theorem in [9] for Lie rings.

Theorem 3.3 *Let L be a finite Lie ring admitting a Frobenius group of automorphisms FH with cyclic kernel F and complement H such that $C_G(F) = 0$. If $C_L(H)$ is nilpotent of class c, then L is nilpotent of $(c, |H|)$-bounded class.*

Theorem 3.1 easily follows from Theorem 3.3: the associated Lie ring $L(G) = \bigoplus_i \gamma_i/\gamma_{i+1}$, where γ_i are terms of the lower central series of G, has exactly the same nilpotency class as G. We have

$$C_{L(G)}(H) = \bigoplus_i C_{\gamma_i/\gamma_{i+1}}(H) = \bigoplus_i C_{\gamma_i}(H)\gamma_{i+1}/\gamma_{i+1}$$

by Theorem 2.4. Hence $C_{L(G)}(H)$ is also nilpotent of class c, and $C_{L(G)}(F) = 0$. By Theorem 3.3, $L(G)$ is nilpotent of $(c, |H|)$-bounded class, and therefore so is G.

The metacyclicity of FH is essential, as shown by the following examples.

Example 3.4 The simple 3-dimensional Lie algebra L of characteristic $\neq 2$ with basis e_1, e_2, e_3 and structure constants $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$ admits the Frobenius group of automorphisms FH with non-cyclic F of order 4 and H of order 3: $F = \{1, f_1, f_2, f_3\}$, where $f_i(e_i) = e_i$, $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1\,(\mathrm{mod}\,3)}$.
Then $C_L(F) = 0$, while $C_L(H)$ is one-dimensional (hence abelian).

We can also make L nilpotent of unbounded derived length.

Example 3.5 Let the additive group of L be the direct sum of three copies of $\mathbb{Z}/p^m\mathbb{Z}$ for a prime $p \neq 2$ with generators e_1, e_2, e_3; let the structure constants be $[e_1, e_2] = pe_3$, $[e_2, e_3] = pe_1$, $[e_3, e_1] = pe_2$. Consider "the same" non-metacyclic Frobenius group of automorphisms FH: $F = \{1, f_1, f_2, f_3\}$, where $f_i(e_i) = e_i$ and $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1\,(\mathrm{mod}\,3)}$.
Then $C_L(F) = 0$ and $C_L(H) = \langle e_1 + e_2 + e_3 \rangle$.
It is easy to see that L is nilpotent of class m, and its derived length is $\approx \log m$.

And we can produce similar nilpotent groups.

Example 3.6 The nilpotent Lie ring in the preceding example can be turned into a nilpotent group if $p > m$. Then the Lazard correspondence can be applied based on the "truncated" Baker–Campbell–Hausdorff formula. Then L becomes a finite p-group P of the same derived length admitting the same group of automorphisms FH with $C_P(F) = 1$ and with cyclic $C_P(H)$.

A few words about the proof of Theorem 3.3 on Lie rings. Recall that we have a Frobenius group $FH \leqslant \operatorname{Aut} L$ with cyclic kernel F such that $C_L(F) = 0$, and $C_L(H)$ is nilpotent of class c.

Let $|F| = n$. Extend the ground ring by a primitive nth root of unity ω. Define the eigenspaces for $F = \langle \varphi \rangle$ as $L_i = \{x \in L \mid x^\varphi = \omega^i x\}$. Roughly speaking,

$$L = L_1 \oplus \cdots \oplus L_{n-1} \qquad \text{and} \qquad [L_i, L_j] \subseteq L_{i+j\,(\mathrm{mod\,n})},$$

which is a $(\mathbb{Z}/n\mathbb{Z})$-grading. Plus we have the condition $L_0 = C_L(F) = 0$. Then by Kreknin's theorem L is soluble of n-bounded derived length. But we need nilpotency, and of class bounded in terms of $C_L(H)$ and $|H|$.

Consider the simplest case of abelian $C_L(H)$. The group $H = \langle h \rangle$ permutes the components L_i "freely": ${L_i}^h = L_{ri}$, where r is determined from $\varphi^{h^{-1}} = \varphi^r$.

For $u_k \in L_k$, denote $u_k^{h^i} = u_{r^i k} \in L_{r^i k}$.

The sum over an H-orbit belongs to $C_L(H)$, which is abelian. Therefore,

$$[x_k + x_{rk} + \cdots + x_{r^{|H|-1}k},\ x_l + x_{rl} + \cdots + x_{r^{|H|-1}l}] \;=\; 0.$$

Expand brackets. If $[x_k, x_l] \neq 0$, then there must be other terms in the same component L_{k+l} for cancellation to happen. Therefore, $k + l = kr^i + lr^j$, so that $l = -\dfrac{r^i - 1}{r^j - 1}k$.

Hence, for a given k there are at most $|H|^2$ values of l such that $[L_k, L_l] \neq 0$.

The following theorem was proved in [6].

Theorem 3.7 *Let $L = \bigoplus_{i=0}^{n-1} L_i$ be a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring such that $L_0 = 0$ and for some m every grading component L_k may not commute with at most m components: $|\{i \mid [L_k, L_i] \neq 0\}| \leqslant m$.*

(a) *Then L is soluble of m-bounded derived length.*

(b) *If in addition n is a prime, then L is nilpotent of m-bounded class.*

This works for the case of abelian $C_L(H)$ in Theorem 3.3. The proof of Theorem 3.7 uses versions of Kreknin's theorem (due to Shalev and EIKh), when there are only few non-zero grading components.

In the general case in [9], when $C_L(H)$ is not abelian but nilpotent of class c, a more complicated technical "selective nilpotency" condition arises, from which the required result is derived by rather difficult arguments.

The following recent result [7] generalizes the condition of F being fixed-point-free to being generated by a splitting automorphism.

Theorem 3.8 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ with kernel $F = \langle\varphi\rangle$ of prime order p such that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$. If $C_G(H)$ is soluble of derived length d, then G is nilpotent of (p,d)-bounded class.*

Examples show that one cannot get rid of the dependence of the nilpotency class on p. It is easy to produce examples of finite p-groups of unbounded nilpotency class admitting a Frobenius group of automorphisms with cyclic kernel of order p^2 generated by a splitting automorphism and with complement of order 2 having abelian fixed-point subgroup. At the same time it remains unclear if bounds for the derived length can be obtained in this situation when the kernel is generated by a splitting automorphism of composite order.

The proof of Theorem 3.8 is based on the theorem of the author and Shumyatsky in [10] for groups of exponent p (when $\varphi = 1$) and the method of "elimination of automorphisms by nilpotency", which was developed earlier by the author, in particular, for studying finite p-groups with a splitting automorphism of order p; see [4] and [5, Ch. 6].

4 Bounding the exponent

The following theorem was proved in [9].

Theorem 4.1 *Suppose that a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel F and complement H such that $C_G(F) = 1$. Then the exponent of G is bounded in terms of $|FH|$ and the exponent of $C_G(H)$.*

Problem 4.2 *Is the hypothesis "F is cyclic" essential?*

The difficulty is that without "cyclic" we do not get a bound for the derived length by Kreknin's theorem or its generalizations. As we saw in examples, there is no bound for the derived length for non-cyclic F.

Problem 4.3 *Does the exponent of G really depend on $|F|$?*

Problem 4.4 *Does the exponent of G really depend on $|F|$ at least for F cyclic?*

Problem 4.5 *Does the exponent of G really depend on $|F|$ at least for GFH being 2-Frobenius?*

This is Mazurov's problem 17.72(b) in Kourovka Notebook [17].

Problem 4.6 *Does the exponent of G really depend on $|H|$?*

So far there is only an example in [1] where the exponent of G is greater than that of $C_G(H)$.

The proof of Theorem 4.1 is easily reduced to the case where G is a finite p-group.

Then the Lie algebra is used, based on the Jennings–Zassenhaus filtration $D_i = D_i(G) = \prod_{jp^k \geqslant i} \gamma_j(G)^{p^k}$. The Lie algebra $DL(G) = \bigoplus D_i/D_{i+1}$ is defined via group commutators. The subalgebra $L_p(G) = \langle D_1/D_2 \rangle$ is generated by D_1/D_2.

Lazard proved that if $x \in G$ is of order p^t, then its image $\bar{x}$ in the appropriate factor as an element of $DL(G)$ is ad-nilpotent of index p^t. A theorem, also going back to Lazard, says that if X is a d-generator finite p-group such that the Lie algebra $L_p(X)$ is nilpotent of class c, then X contains a powerful characteristic subgroup of (p, c, d)-bounded index. (Recall that a finite p-group G is powerful if $G^p \geqslant [G, G]$ for $p \neq 2$, or $G^4 \geqslant [G, G]$ for $p = 2$.)

The Lie algebra $L_p(G)$ is soluble by Kreknin theorem. Plus all factors are generated by ad-nilpotent elements, since $G = \langle C_G(H)^F \rangle$ by Lemma 2.1 and by Theorem 2.4. Together this implies the nilpotency of $L_p(G)$ of bounded class.

Therefore G can be assumed to be powerful, which are easy to handle: if a powerful p-group is generated by elements of given order e, then the group is of exponent e. It remains to use again $G = \langle C_G(H)^F \rangle$.

A first step for exponent with non-metacyclic FH was recently made by Shumyatsky [13], who proved that if a finite group G admits a Frobenius group $FH \leqslant \operatorname{Aut} G$ of order $|FH| = 12$ with kernel F such that $C_G(F) = 1$, then the exponent of G is bounded in terms of the exponent of $C_G(H)$ (and "12"). Shumyatsky [14] also combined the metacyclic exponent and nilpotency results by proving that if a finite group G admits a metacyclic Frobenius group $FH \leqslant \operatorname{Aut} G$ with kernel F such that $C_G(F) = 1$ and $C_G(H)$ satisfies a positive law of degree k, then G satisfies a positive law of degree bounded in terms of k and $|FH|$.

Recall that a positive law is a law of the form $v = w$, where group words v, w involve only positive powers of variables. A positive law of degree k for a finite group implies that it is an extension of a nilpotent group of k-bounded class by a group of k-bounded exponent. Conversely, every such an extension satisfies a positive law of bounded degree.

We also mention the generalization in [7], where the condition of F being fixed-point-free is relaxed to being generated by a splitting automorphism.

Theorem 4.7 *Suppose that a finite group G admits a Frobenius group $FH \leqslant \mathrm{Aut}\, G$ with kernel $F = \langle\varphi\rangle$ of prime order p such that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$. Then the exponent of G is bounded in terms of p and the exponent of $C_G(H)$.*

Problem 4.8 *Can a similar bound for the exponent be obtained in the case where F is generated by a splitting automorphism of composite order?*

5 Some other open problems

We mention here some open questions in addition to the ones in the above sections. Throughout, G is a finite group admitting a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$.

Problem 5.1 *Is G supersoluble if $C_G(H)$ is supersoluble?*

Problem 5.2 *If $C_G(H)$ satisfies an n-Engel law, does G satisfy an $f(n)$-Engel law?*

Problem 5.3 *If FH is metacyclic, is the derived length of G bounded in terms of $|H|$ and the derived length of $C_G(H)$? If so, furthermore, is the dependence on $|H|$ essential?*

Problem 5.4 *The same question in the case when GFH is a 2-Frobenius group.*

For the derived length, it is unclear how to use reduction to Lie rings, since the associated Lie ring may have smaller derived length than G.

Problem 5.5 *The same question for a Lie ring L admitting a metacyclic Frobenius group of automorphisms FH with kernel F and complement H such that $C_L(F) = 0$.*

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