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2012

MIMS EPrint: 2012.100

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ISSN 1749-9097
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E. I. Khukhro

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(But there is no bound for the exponent of a \( p \)-group with a partition.)
Splitting automorphism approach

Splitting automorphism approach of condition (c) turned out to be most efficient. Recall:

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All groups with a splitting automorphism of order \( p \) form a variety of groups with operators defined by the laws \((*)\).
Analogues of theorems on group of exponent $p$

Analogues of theorems on group of exponent $p$ are natural for finite $p$-groups with a partition (equivalently, for $p$-groups with a splitting automorphism of order $p$).

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For example, EKh-1981: if $P_1$ in condition (c) has derived length $d$, then $P_1$ is nilpotent of $(p, d)$-bounded class.

Plus, based on Kostrikin's theorem for groups of prime exponent, EKh-1986: analogue of the affirmative solution of the Restricted Burnside Problem: the nilpotency class of $P_1$ is bounded in terms of $p$ and the number of generators.

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EKh–Shumyatsky, 1995: if a finite group $G$ of exponent $p$ admits a soluble group of automorphisms $A$ of coprime order such that the fixed-point subgroup $C_G(A)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p, d, |A|)$-bounded class.
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**Theorem 1**

Suppose that a finite $p$-group $P$ with a partition admits a soluble group of automorphisms $A$ of coprime order such that $C_P(A)$ has derived length $d$. Then any maximal subgroup of $P$ containing $H_p(P)$ is nilpotent of $(p, d, |A|)$-bounded class.
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The bound for the nilpotency class of that maximal subgroup can be chosen the same as in EKh–Shumyatsky-95 for groups of exponent $p$. 
Exponent

**Theorem 2**

If a finite $p$-group $P$ with a partition admits a group of automorphisms $A$ that acts faithfully on $P/H_p(P)$, then the exponent of $P$ is bounded in terms of the exponent of $C_P(A)$. 

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Corollary

Suppose that a finite group $G$ admits a Frobenius group of automorphisms $FH$ with cyclic kernel $F = \langle \varphi \rangle$ of prime order $p$ such that $\varphi$ is a splitting automorphism, that is, $x x^\varphi x^{\varphi^2} \cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$. 

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(a) If $C_G(H)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p, d)$-bounded class.

(b) The exponent of $G$ is bounded in terms of $p$ and the exponent of $C_G(H)$.
Proof of Corollary

The group $G$ is nilpotent by Kegel–Thompson–Hughes. $\phi$ is fixed-point-free on $G^p$: for any $g \in C_G(\phi)$ we have

$$1 = g g \phi g \phi^2 \cdots g \phi^{p-1} = g^p.$$

Hence $G^p$ is nilpotent of $p$-bounded class by Higman–Kreknin–Kostrikin.

For (a) it now remains to consider the Sylow $p$-subgroup $G^p$ of $G$. The result follows from Theorem 1 applied to $P = G^p \langle \phi \rangle$ and $A = H$.

By a lemma in EKh–Makarenko–Shumyatsky-2010

$$G^p = \langle C_G^p(H)^f \mid f \in F \rangle.$$

So $G^p$ is generated by elements of orders dividing the exponent of $C_G(H)$.

Plus $p$-bounded nilpotency class of $G^p$ $\Rightarrow$ exponent of $G^p$ is bounded in terms of $p$ and exponent of $C_G(H)$.

So in (b) it remains to consider $G^p$. The result follows from Theorem 2 applied to $P = G^p \langle \phi \rangle$ and $A = H$. 
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Question remains open for the exponent, as well as for the derived length.
Proof of Theorem 1: elimination of automorphisms by nilpotency

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**Theorem 1'**

Suppose that a soluble group $FA$ with normal Sylow $p$-subgroup $F = \langle \varphi \rangle$ of order $p$ and Hall $p'$-subgroup $A$ acts by automorphisms on a finite $p$-group $G$ in such a manner that $\varphi$ is a splitting automorphism, that is, $xx^\varphi x^{\varphi^2} \cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$. If $C_G(A)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p, d, |A|)$-bounded class.
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Free $\text{FA}$-group

The trick of elimination of automorphisms requires passing to a free $\text{FA}$-group $X = \langle x_1, x_2, \ldots \rangle$ of some exponent $p^M$ and some nilpotency class $N$.

There is an $\text{FA}$-homomorphism $\xi : X \to G$ given by $x_i \to g_i$ for any $g_i \in G$ (provided $M, N$ are a large enough.)

Let $C$ be the $\text{FA}$-invariant normal closure of $(C_X(A))^{(d)}$.

Let $S$ be the $\text{FA}$-invariant normal closure of all $xx^\varphi x^\varphi^2 \cdots x^\varphi^{p-1}$.

Clearly, $C, S \leq Ker \xi$ by hypothesis.
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Clearly, $C, S \leq \text{Ker } \xi$ by hypothesis.

Lemma

The subgroups $C$ and $S$ are invariant under any $FA$-endomorphism $\vartheta$ of $X$. 
Trivialization of $F$

Since there is an $FA$-homomorphism $\xi : X \to G$ with $C, S \leq \text{Ker} \, \xi$, it is sufficient (and even necessary) to prove that $\left[x_1, \ldots, x_{c+1}\right] \in CS$, where $c$ is the nilpotency class given by EKh-Shumyatsky theorem when $\varphi = 1$. 
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Let $T = [X, F]F$ (“trivialization of $F$”)
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By EKh-Shumyatsky theorem, $[x_1, \ldots, x_{c+1}] \in CST$,

that is, we need to eliminate $T$. 

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Higman’s lemma

We have
\[ [x_1, \ldots, x_{c+1}] \equiv c_1^{k_1} \cdots c_m^{k_m} \pmod{CS}, \text{ where } c_i \in T. \]
Higman’s lemma

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An analogue of Higman’s lemma gives that we can assume that each \( c_i \) depends on all \( x_1, \ldots, x_{c+1} \), and on \( \varphi \).
Higman’s lemma

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An analogue of Higman’s lemma gives that we can assume that each \( c_i \) depends on all \( x_1, \ldots, x_{c+1} \), and on \( \varphi \).

One can show that we can furthermore assume that each \( c_i \) has the form
\[
[[x_{a^*_i}, \ldots], [x_{a^*_i}, \ldots], \ldots, [x_{a^*_i}, \ldots]] \quad (a_* \in A),
\]
where \( \{i_1, i_2, \ldots, i_{c+1}\} = \{1, 2, \ldots, c+1\} \) and there is at least one \( \varphi \) among “dots” in at least one of the subcommutators \( [x_{i_k}^{a_*}, \ldots] \).
Self-amplification process

\[ [x_1, \ldots, x_{c+1}] \equiv c_1^{k_1} \cdots c_m^{k_m} \pmod{CS} \] (*
Self-amplification process

\[ [x_1, \ldots, x_{c+1}] \equiv c_1^{k_1} \cdots c_m^{k_m} \pmod{CS} \]  

We “iterate”, “self-amplify”: by homomorphisms of the type

\[ x_k \rightarrow [x^{a_k}_1, \ldots], \quad k = 1, \ldots, c + 1 \]

we express each \( c_i = [[x^{a_i}_{i_1}, \ldots], \ldots, [x^{a_i}_{c+1}, \ldots]] \) as the image of the left-hand-side.

Since \( X_\langle \phi \rangle \) is nilpotent (being a finite \( p \)-group!), in the end we get

\[ \equiv 1, \text{ as required}. \]
Self-amplification process

\[ [x_1, \ldots, x_{c+1}] \equiv c_1^{k_1} \cdots c_m^{k_m} \pmod{CS} \]  

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we express each \( c_i = [[x_{i_1}^{a_{i_1}}, \ldots], \ldots, [x_{i_{c+1}}^{a_{i_{c+1}}}, \ldots]] \) as the image of the left-hand-side,

then substitute the result into right-hand side of the original (\ast).
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As a result, the new (*) has the same form but now each new \( c_i \) has at least two occurrences of \( \varphi \).
Self-amplification process

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then substitute the result into right-hand side of the original (\(*\)).

As a result, the new (\(*\)) has the same form but now each new \( c_i \) has at least two occurrences of \( \varphi \).

And so on, at each step we double the number of occurrences of \( \varphi \) in the new \( c_i \).
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And so on, at each step we double the number of occurrences of \( \varphi \) in the new \( c_i \).

Since \( X\langle \varphi \rangle \) is nilpotent (being a finite \( p \)-group!), in the end we get \( \equiv 1 \), as required.
Proof of exponent theorem.

By known results, proof of Theorem 2 reduces to the following result.

**Theorem 2’**

If a finite $p$-group $G$ admits a Frobenius group of automorphisms $FA$ with kernel $F = \langle \varphi \rangle$ of order $p$ and complement $A$ such that $\varphi$ is a splitting automorphism, that is, $xx^\varphi x^{\varphi^2} \cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$, then the exponent of $P$ is bounded in terms of the exponent of $C_P(A)$. 
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Since any $g \in G$ belongs to $\langle g^{FA} \rangle$, we can assume that $G$ is generated by $|FA|$ elements.
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By EKh-86 affirmative solution to an analogue of the Restricted Burnside Problem for groups with a splitting automorphism of prime order $p$, the nilpotency class of $G$ is bounded in terms of $p$ and the number of generators, which is at most $p(p - 1)$. 
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By known results, proof of Theorem 2 reduces to the following result.

**Theorem 2′**

If a finite $p$-group $G$ admits a Frobenius group of automorphisms $FA$ with kernel $F = \langle \varphi \rangle$ of order $p$ and complement $A$ such that $\varphi$ is a splitting automorphism, that is, $xx^\varphi x^{\varphi^2} \cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$, then the exponent of $P$ is bounded in terms of the exponent of $C_P(A)$.

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It remains to obtain a bound for the exponent of $V = G/[G, G]$. 
Abelian case: eigenspaces.

Consider $V = G/[G, G]$ as a $\mathbb{Z}FA$-module, with additive notation. In particular, $v + v\varphi + v\varphi^2 + \cdots + v\varphi^{p-1} = 0$ for all $v \in V$ by hypothesis.
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Extend the ground ring by a primitive $p$th root of unity $\omega$, forming $W = V \otimes \mathbb{Z} [\omega]$. Still have $w + w\varphi + w\varphi^2 + \cdots + w\varphi^{p-1} = 0$ for all $w \in W$. 
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Define analogues of eigenspaces for the “linear transformation” $\varphi$:

$$W_i = \{w \in W \mid w\varphi = \omega^i w\}.$$
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Define analogues of eigenspaces for the “linear transformation” $\varphi$:

$$W_i = \{ w \in W \mid w\varphi = \omega^i w \}.$$ 

Then $W$ is an “almost direct sum” of the $W_i$:

$$pW \subseteq W_0 + W_1 + \cdots + W_{p-1}$$

and

if $w_0 + w_1 + \cdots + w_{p-1} = 0$ for $w_i \in W_i$, then $pw_i = 0$ for all $i$. 
$A$-orbits.

First: since $\varphi = 1$ on $W_0$, for $x \in W_0$ we have $px = x + x\varphi + \cdots + x\varphi^{p-1} = 0$, so that $pW_0 = 0$. 
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Action of \( A \): permutes the \( W_i \) in the same way as it acts on \( \langle \varphi \rangle \).
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Let $A = \langle \alpha \rangle$ and let $\varphi^\alpha = \varphi^r$ for some $1 \leq r \leq p - 1$. Let $|\alpha| = n$; then $r$ is a primitive $n$th root of $1$ in $\mathbb{Z}/p\mathbb{Z}$. 

E. I. Khukhro (Inst. Math., Novosibii; Automorphisms of finite $p$-groups adn

March 2012 18 / 22
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$A$ “almost permutes” the $W_i$:
$W_i\alpha \subseteq W_{ri}$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. Indeed, if $x_i \in W_i$, then
$(x_i\alpha)\varphi = x_i(\alpha\varphi\alpha^{-1}\alpha) = (x_i\varphi^r)\alpha = \omega^{ir}x_i\alpha$. 
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$$(x_i \alpha)\varphi = x_i(\alpha \varphi \alpha^{-1} \alpha) = (x_i \varphi^r)\alpha = \omega^i r x_i \alpha.$$ 

Given $u_k \in W_k$ for $k \neq 0$, to lighten the notation we denote $u_k \alpha^i$ by $u_{r^i k}$; note that $u_{r^i k} \in W_{r^i k}$.
Let $p^e$ be the exponent of $C_G(A)$. Claim: $W_i$ are annihilated by $p^{1+e}$. 
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For any $k \neq 0$ and for any $u_k \in W_k$ we have

$$u_k + u_k \alpha + \cdots + u_k \alpha^{n-1} = u_k + u_{rk} + \cdots + u_{r^{n-1}k} \in CW(A)$$

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Recall that $pW_0 = 0$. As a result,

$$p^{2+e}W \subseteq p^{1+e}(W_0 + W_1 + \cdots + W_{p-1}) = 0,$$

so also $p^{2+e}V = 0$. 
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In multiplicative notation, the exponent of $G/[G, G]$ divides $p^{2+e}$,
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In multiplicative notation, the exponent of $G/[G, G]$ divides $p^{2+e}$, so the exponent of $G$ divides $p^{c(2+e)}$, where $c$ is the nilpotency class of $G$, which is bounded in terms of $p$. 


Remark

If, for some reason, it is known that the derived length \( s \) of the group \( G \) in Theorems 1 or 2, or in the Corollary, is relatively small, then EKh-81 can be used instead to give a possibly better estimate

\[
\frac{(p - 1)^s - 1}{p - 2}
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for the nilpotency class of \( G \) (in Theorems 1' and 2').
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for the nilpotency class of $G$ (in Theorems 1′ and 2′).

A smaller bound for the nilpotency class would also imply a smaller bound for the exponent.
Generalizations

In EKh-91 general nilpotency theorem was proved: if a group $G$ admits a group of operators $\Omega$ such that $G\Omega$ is nilpotent, $G$ satisfies $\Omega$-laws which after $\Omega \to 1$ imply nilpotency of class $c$,
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Similarly, the same arguments as above prove

Theorem 1

Suppose that a soluble group $FA$ with normal Sylow $p$-subgroup $F$ and Hall $p'$-subgroup $A$ acts by automorphisms on a finite $p$-group $G$ in such a manner that for some fixed $\phi_1,...,\phi_p \in F$ we have $x^{\phi_1}x^{\phi_2}...x^{\phi_p}=1$ for all $x \in G$. If $C_G(A)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p,d,|A|)$-bounded class. Furthermore, the bound for the nilpotency class can be chosen to be the same as in the case $G_p=1$ (given by EKh-Shumyatsky-95).
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Generalizations

There is also local nilpotency theorem in EKh-93, which may also have generalizations in the context of additional group of automorphisms...