Local Fusion Graphs for Symmetric Groups

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Abstract

For a group $G$, a set of odd positive integers and $X$ a set of involutions of $G$ we define a graph $\mathcal{F}_\pi(G, X)$. This graph, called a $\pi$-local fusion graph, has vertex set $X$ with $x, y \in X$ joined by an edge provided $x \neq y$ and the order of $xy$ is in $\pi$. In this paper we investigate $\mathcal{F}_\pi(G, X)$ when $G$ is a finite symmetric group for various choices of $X$ and $\pi$.

1 Introduction

There is a long and rich history of conjuring up various types of important combinatorial structures from a group. For example Cayley graphs, constructed from a group together with a generating set for that group, have had a considerable presence in such areas as geometric group theory and the study of expander graphs [22]. While groups with a $BN$-pair (such as groups of Lie-type) via their parabolic subgroups give rise to buildings (see Chapter 15 of [15]). For a group $G$ and $X$ a subset of $G$ we have the commuting graph, $\mathcal{C}(G, X)$, whose vertices are the elements of $X$ with two distinct elements of $X$ adjacent whenever they commute (for recent work on commuting graphs see [7], [8], [9], [10], [11], [12], [13], [18] and [19]). Such graphs have had an important impact in the study of finite simple groups, the commuting graphs associated with the Fischer groups [20], which featured in their construction, being a prime example. Variations on this theme have also played a role – see for example the so called root 4-subgroups of the Held group, on page 230 of [2]. For yet another variety of graph consult [14].

We now discuss a recent combinatorial structure of this genre. Suppose that $G$ is a group, $\pi$ is a set of positive integers and $X$ is a subset of $G$. The graph $\mathcal{C}_\pi(G, X)$ is defined to be the graph with $X$ as its vertex set and for $x, y \in X$ and $y$ are adjacent if $x \neq y$ and the order of $xy$ is in $\pi$. We observe, as $xy$ and $yx$ are conjugate elements of $G$, that the graph $\mathcal{C}_\pi(G, X)$ is undirected. Further, we observe that $\mathcal{C}_\{2\}(G, X)$ when $X$ is a set of involutions in $G$ is exactly the commuting involution graph $\mathcal{C}(G, X)$. When the orders of the elements in $X$ are coprime to all the integers in $\pi$, we shall call $\mathcal{C}_\pi(G, X)$ a $\pi$-coprimality graph (or just coprimality graph if $\pi$ is understood).

An important type of coprimality graph arises when $X$ is a set of involutions. For $\pi$ a set of odd positive integers, we write $\mathcal{F}_\pi(G, X)$ instead of $\mathcal{C}_\pi(G, X)$, and refer to $\mathcal{F}_\pi(G, X)$ as the $\pi$-local fusion graph for $X$. In the case when $\pi$ consists of all odd positive integers, we just write $\mathcal{F}(G, X)$ instead of $\mathcal{F}_\pi(G, X)$, and call
$\mathcal{F}(G, X)$ the local fusion graph for $X$. The name ‘local fusion’ comes from the fact that if $x = x_0, x_1, x_2, \ldots, x_m = y$ is a path in the graph $\mathcal{F}(G, X)$, then $g = g_1g_2\ldots g_m$ conjugates $x$ to $y$ where each $g_i$, $1 \leq i \leq m$, is an element of the dihedral group $\langle g_{i-1}, g_i \rangle$. In [17] $\{3\}$-local fusion graphs, $\mathcal{F}_{\{3\}}(G, X)$ are investigated for $X$ a $G$-conjugacy class of involutions. There issues, such as connectedness and what kind of triangles the graph contains, are examined. Further, the case when $G \cong PSL_2(q)$ ($q$ a prime power) is analysed in detail, the work in [17] being prompted by a tower of graphs associated with a subgroup chain $Alt(5) \leq PSL_2(11) \leq M_{11} \leq M_{12}$. Each of the graphs in this tower may be viewed as being a restricted type of $\{3\}$-local fusion graph.

The famous Baer-Suzuki Theorem (see (39.6) in [1] or Theorem 3.8.2 in [21]), when $X$ is a $G$-conjugacy class of involutions, may be rephrased using the local fusion graph in the following way. The graph $\mathcal{F}(G, X)$ is totally disconnected if and only if $\langle X \rangle$ is a $2$-subgroup of $G$. For suppose $\mathcal{F}(G, X)$ is totally disconnected, and let $x, y \in X$, with $x \neq y$. Assume that the order of $xy$ is $2^km$, where $m$ is odd. If $m > 1$, then $(xy)^{2^km} = x(yx\cdots xy) = xx^g$ has odd order $m$ and $x \neq x^g$. Hence $x$ and $x^g$ are adjacent in $\mathcal{F}(G, X)$, a contradiction. Therefore $xy$ has order $2^k$. Since, as is well known, $\langle x, y \rangle$ is a dihedral group of order twice that of $xy$, $\langle x, y \rangle$ is a $2$-group, and so $\langle X \rangle$ is a $2$-group by the Baer-Suzuki Theorem.

The aim of the present paper is to begin the investigation of $\pi$-local fusion graphs for finite symmetric groups.

**Theorem 1.1.** Suppose that $G = \text{Sym}(n)$ with $n \geq 5$ and $X$ is a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected with $\text{Diam}(\mathcal{F}(G, X)) = 2$.

For $n = 2$, $\mathcal{F}(G, X)$ consists of just one vertex and for $n = 3$, $\mathcal{F}(G, X)$ is the complete graph on 3 vertices. While for $n = 4$ and $X$ the conjugacy class of $(1, 2)(3, 4)$ in $\text{Sym}(4)$, $\mathcal{F}(G, X)$ consists of three vertices with no edges – if $X$ is the conjugacy class of transpositions in $\text{Sym}(4)$, then $\mathcal{F}(G, X)$ is connected of diameter 2. There are $\pi$-local fusion graphs where we do encounter larger diameters. For example with $G = \text{Sym}(9)$ and $X$ the $G$-conjugacy class of $(1, 2)(3, 4)(5, 6)$ we have $\text{Diam}(\mathcal{F}_{\{3\}}(G, X)) = \text{Diam}(\mathcal{F}_{\{5\}}(G, X)) = \text{Diam}(\mathcal{F}_{\{7\}}(G, X)) = 3$. This all prompts the question as to whether there are groups in which the diameter of local fusion graphs can be arbitrarily large - the answer is yes, and we direct the reader to [3]. For further work on coprimality graphs and symmetric groups see [5], and for more recent developments on local fusion graphs see [4] and [6].

The question of connectivity for $\pi$-local fusion graphs is the subject of our second theorem.

**Theorem 1.2.** Suppose that $G = \text{Sym}(n)$, $X$ is a $G$-conjugacy class of involutions and $\pi$ is a set of odd positive integers. Then $\mathcal{F}_{\pi}(G, X)$ is either totally disconnected or connected.

This paper is arranged as follows. Section 2 is mostly concerned with the notion of an ‘$x$-graph’ which, for $G \cong \text{Sym}(n)$, encodes the $C_G(t)$-orbits on the
conjugacy class of $t$ where $t$ is an involution. Then in Section 3 the $x$-graphs are put to work in establishing the diameter of local fusion graphs thereby proving Theorem 1.1. The proof of Theorem 1.2, which also employs $x$-graphs, is to be found in Section 4. Our group theoretic notation is standard as given, for example, in [1].

2 Background Results

Throughout this paper $t$ will denote a fixed involution of $X$, a conjugacy class of $\text{Sym}(n)$. We will sometimes denote $\text{Sym}(m) \ (m \in \mathbb{N})$ by $\text{Sym}(\Omega)$ where $\Omega$ is an $m$-element set upon which the permutations act. For $g \in \text{Sym}(\Omega)$, the support of $g$, $\text{supp}(g)$, is $\Omega \setminus \text{fix}(g)$, where $\text{fix}(g) = \{ \alpha \in \Omega \mid \alpha^g = \alpha \}$. We use $d(\ ,\ )$ to denote the standard graph theoretic distance on $\mathcal{F}_x(G, X)$.

The proofs of Theorems 1.1 and 1.2 feature another graph $G_x$ referred to as the $x$-graph. Assuming that $G = \text{Sym}(n)$ acts upon $\Omega = \{1, 2, \ldots, n\}$ and that $t = (1,2)(3,4)\ldots(2b-1,2b)$, we set

$$V = \{\{1,2\}, \{3,4\}, \ldots, \{2b-1,2b\}, \{2b+1\}, \ldots, \{n\}\}.$$ 

Thus the elements of $V$ are just the orbits of $\langle t \rangle$ upon $\Omega$. For each $x \in X$, we define the $x$-graph $G_x$ to be the graph with $V$ as vertex set, and $v_1, v_2 \in V$ are joined by an edge whenever there exist $\alpha \in v_1$ and $\beta \in v_2$ with $\alpha \neq \beta$ for which $\{\alpha, \beta\}$ is a $\langle x \rangle$-orbit. Additionally the vertices of $G_x$ corresponding to 2-cycles of $t$ will be coloured black (●) and the other vertices white (○). Therefore $G_x$ has $b$ black vertices and $n - 2b$ white vertices. Note that the edges in $G_x$ are in one-to-one correspondence with the 2-cycles of $x$. So the number of edges in $G_x$ is the same as the number of black vertices. As an example, taking $n = 16$, $t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ and $x = (1,3)(2,4)(5,6)(9,11)(12,13)(14,15)$, $G_x$ looks like

\[
\begin{array}{cccccccc}
\bullet & \bullet & & & & & & \\
(1,2) & (3,4) & (5,6) & (7,8) & (9,10) & (11,12) & (13,14) & (15,16) \\
\end{array}
\]

Lemma 2.1. For $x \in X$, the possible connected components of $G_x$ are

(i) ● ● ● ● ● ● ● ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ......
(ii) If \( x, y \in X \), then \( x \) and \( y \) are in the same \( C_G(t) \)-orbit if and only if \( G_x \) and \( G_y \) are isomorphic graphs (where isomorphisms preserve the colour of vertices).

(iii) Let \( C_1, C_2, \ldots, C_\ell \) be the connected components of \( G_x \). Assume that \( x_i \) and \( t_i \) are the corresponding parts of \( x \) and \( t \), and let \( b_i, w_i \) and \( c_i \), respectively, the number of black vertices, white vertices and cycles in \( C_i \). Then

(a) the order of \( tx \) is the least common multiple of the orders of \( t_i x_i \), \( i = 1, \ldots, \ell \); and

(b) for \( i = 1, \ldots, \ell \), the order of \( t_i x_i \) is \( (2b_i + w_i)/(c_i + 1) \).

Proof. See Lemma 2.1 and Proposition 2.2 of \([7]\). □

Suppose for \( x \in X \) the connected components of \( G_x \) are \( C_1, C_2, \ldots, C_\ell \), and for each such component let \( x_i \) and \( t_i \) be the corresponding parts of \( x \) and \( t \). Observe that for \( i \neq j \) both \( t_i \) and \( x_i \) commute with both \( t_j \) and \( x_j \). So in the above example, \( \ell = 6 \) with \( t_1 = (1, 2)(3, 4) \), \( t_2 = (5, 6) \), \( t_3 = (7, 8) \), \( t_4 = (9, 10)(11, 12)(13) \), \( t_5 = (14)(15) \), \( t_6 = (16) \), and \( x_1 = (1, 3)(2, 4) \), \( x_2 = (5, 6) \), \( x_3 = (7)(8) \), \( x_4 = (9, 11)(12, 13)(10) \), \( x_5 = (14, 15) \), \( x_6 = (16) \).

We remark that, as \( G_x \) has \( b \) edges, the number of connected components of type \( \bullet \cdots \bullet \) and of type \( \circ \cdots \circ \) must be equal (including \( \circ \circ \circ \) in the latter type). This is an important observation for part of the proof of Theorem 1.1. Consider the following simple situation: \( t = (1, 2)(3, 4)(5, 6)(7)(8, 9)(10) \) and \( x = (1)(2, 3)(4, 5)(6)(7, 8)(9, 10) \), with \( n = 10 \). Then \( G_x \) is

\[
\bullet \cdots \bullet \quad \circ \cdots \circ
\]

with \( t_1 = (1, 2)(3, 4)(5, 6) \), \( t_2 = (7)(8, 9)(10) \), \( x_1 = (1)(2, 3)(4, 5)(6) \) and \( x_2 = (7, 8)(9, 10) \) being the parts of \( t \) and \( x \) corresponding to the two connected components. In our proof of Theorem 1.1 we argue by induction on \( n \), and seek to exploit the symmetric subgroups \( \text{Sym}(\Lambda) \), where \( \Lambda \) is the support of a connected part of \( t \). But as we see in this small example, \( t_1 \) and \( x_1 \) are not conjugate in \( \text{Sym}\{1, 2, 3, 4, 5, 6\} \), nor are \( t_2 \) and \( x_2 \) in \( \text{Sym}\{7, 8, 9, 10\} \), and so our inductive strategy will fail. However, this obstacle may be overcome by pairing up connected components \( \bullet \cdots \bullet \) and \( \circ \cdots \circ \) of \( G_x \) and applying induction to \( \text{Sym}(\Lambda) \) where \( \Lambda \) is the union of the support of \( t \) on these two connected components. This kind of issue does not arise with any of the other types of connected components of \( G_x \). While on the subject of potential pitfalls in the proof of Theorem 1.1 we mention the connected components \( \bullet \bullet \cdot \) of \( G_x \). Let \( t_i \) and \( x_i \) be the parts of \( t \) and \( x \) corresponding to this connected component, and set \( \Lambda = \text{supp}(t_i) \). Then \( \text{Sym}(\Lambda) \cong \text{Sym}(4) \) with \( t_i \) and \( x_i \) having cycle type \( 2^2 \) in \( \text{Sym}(\Lambda) \), and there is no path between \( t_i \) and \( x_i \) in the \( \text{Sym}(\Lambda) \) local fusion graph. To deal with such connected components of
we are forced to bring all of $G_x$ into play - this turns out to be a substantial part of the proof of Theorem 1.1.

Suppose $x, y \in X$. We shall use $G'_y$ to denote the $x$-graph where $y$ plays the role of $t$ - so the vertices of $G'_y$ are the orbits of $\langle y \rangle$ on $\Omega$ with vertices $w_1, w_2$ joined if there exists $\alpha$ in $w_1$ and $\beta$ in $w_2$ with $\alpha \neq \beta$ and $\{\alpha, \beta\}$ an $\langle x \rangle$-orbit. So $G'_y$ is just $G_x$. For more on $x$-graphs, see Section 2.1 of [7].

3 The Diameter of $\mathcal{F}(G, X)$

In this section we prove Theorem 1.1. So we have $G = \text{Sym}(n)$ with $n \geq 5$, $X$ a $G$-conjugacy class of involutions and $t$ a fixed element of $X$. For $n \leq 16$, Magma [16] makes relatively short work of checking that $\mathcal{F}(G, X)$ is connected and has diameter 2. So we may assume $n > 16$.

We proceed by induction on $n$. Let $x \in X$. We aim to show that $d(t, x) \leq 2$. Since there are plainly $x \in X$ for which $d(t, x) > 1$, this would prove that $\text{Diam}(\mathcal{F}(G, X)) = 2$. Suppose for the moment that $G_x$ contains no connected components of type $\circ$. If $G_x$ is not connected and not of type $\bullet - \circ$, then, by induction, $d(t, x) \leq 2$. Thus, using Lemma 2.1, we may assume $G_x$ is one of $\bullet - \circ$ or $\circ - \bullet$ (allowing $\circ - \circ$ as a possibility in the latter component). In (3.1), (3.2) and (3.3) we deal with each of these possibilities in turn.

(3.1) If $G_x$ is $\bullet - \circ$, then $d(t, x) \leq 2$.

Assume, without loss of generality, that $t = (1, 2)(3, 4), \ldots, (2m-1, 2m)$. So $G_x$ has $m$ black vertices. If $m$ is odd, then $tx$ has odd order by Lemma 2.2(iii)(b), and so $d(t, x) \leq 1$. While if $m$ is even, we assume that $G_x$ is labelled like so

and that

$$x = (1, 2m)(2, 3)(4, 5) \ldots (2m - 4, 2m - 3)(2m - 2, 2m - 1).$$

Note that we have $m \geq 4$. We select

$$y = (1, 2)(3, 5)(4, 2m)(6, 2m - 1)(7, 8)(9, 10) \ldots (2m - 3, 2m - 2).$$

Then $y \in X$ and $ty = (3, 2m, 6)(4, 5, 2m - 1)$, and hence $d(t, y) \leq 1$. Now $G'_y$ is seen to be
Since the two connected components of $G^y_x$ have sizes $3$ and $m - 3$, both of which are odd, Lemma 2.2(iii) implies that $yx$ has odd order. Therefore $d(x, y) \leq 1$ and so (3.1) holds.

(3.2) If $G^x_x$ is $\circ \bullet \cdots \bullet$, then $d(t, x) = 1$.

Since $G^x_x$ is a connected component with one white vertex, (3.2) follows from Lemma 2.2(iii).

(3.3) If $G^x_x$ is $\bullet \cdots \circ \bullet \cdots \bullet \circ$, then $d(t, x) \leq 2$.

Without loss we may label $G^x_x$ as follows

\[ t = (1, 2)(3, 4)(5, 6) \cdots (2r - 1, 2r)(2r + 1, 2r + 2)(2r + 3, 2r + 3) \cdots (2m - 2, 2m - 1, 2m). \]

We may assume that

\[ x = (2, 3)(4, 5) \cdots (2r - 2, 2r - 1)(2r + 1, 2r + 2) \cdots (2m - 1, 2m). \]

Set $t_0 = (1, 2)t$ and $x_0 = x(2m - 1, 2m)$. Then $t_0$ and $x_0$ are $H$-conjugate, where $H = \text{Sym}(\Omega \setminus \{1, 2m\})$. Observing that $G^t_{x_0}$ (thinking of $t_0$, $x_0$ as involutions in $H$) has two connected components of type $\circ \bullet \cdots \bullet$ we deduce from Lemma 2.2(iii) that $t_0x_0$ has odd order. Let $y = (1, 2m)t_0$. Then $y \in X$ and

\[ ty = (1, 2)t_0(1, 2m)t_0 = (1, 2)(1, 2m) = (1, 2, 2m), \]

whence $d(t, y) \leq 1$. Also, as $t_0$ and $x_0$ fix $1$ and $2m$,

\[ yx = (1, 2m)t_0x_0(2m - 1, 2m) = t_0x_0(1, 2m)(2m - 1, 2m) = t_0x_0(1, 2m - 1, 2m). \]

Now $t_0x_0 \in H$ is a product of two disjoint odd cycles of lengths, say, $m_1, m_2$. If $2m - 1$ is in say the latter cycle of $t_0x_0$, then $tx$ is a disjoint product of an $m_1$-cycle and an $(m_2 + 2)$-cycle. Thus $yx$ has odd order and so $d(x, y) \leq 1$. Therefore $d(t, x) \leq 2$, which proves (3.3).

Taken together (3.1), (3.2) and (3.3) prove Theorem 1.1 when $G^x_x$ contains no connected components of type $\bullet \bullet$. It therefore remains to analyse $G^x_x$ when
it has connected components of type \( \bullet \rightarrow \bullet \). If there are an even number of \( \bullet \rightarrow \bullet \) connected components, then, as the local fusion graphs for \( \text{Sym}(8) \) have diameter two, by pairing them up and using induction we obtain our result. Thus we may assume \( \mathcal{G}_x \) contains exactly one \( \bullet \rightarrow \bullet \) connected component. Let \( \mathcal{H}_x \) denote the union of all the other connected components of \( \mathcal{G}_x \). Also we may assume \( t = (1, 2)(3, 4)t_0, x = (1, 3)(2, 4)x_0 \) where \( t_0 \) and \( x_0 \) are involutions in \( H = \text{Sym}(\Omega \setminus \{1, 2, 3, 4\}) \).

Let \( C_x \) be a subgraph of \( \mathcal{H}_x \), where \( C_x \) is one of \( \circ, \bullet, \circ \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet, \circ \rightarrow \bullet \rightarrow \circ \rightarrow \cdots \circ \rightarrow \cdot \).

Then \( t_0 = t_1t_2, x_0 = x_1x_2 \) where \( t_1, x_1 \) are the parts in \( C_x \) and \( t_2, x_2 \) the parts in \( \mathcal{H}_x \setminus C_x \). Then \( t_2 \) and \( x_2 \) are conjugate involutions in some symmetric subgroup of \( G \) and the \( x_2 \)-graph (with respect to \( t_2 \)) is \( \mathcal{H}_x \setminus C_x \). Since \( \mathcal{H}_x \) contains no subgraph \( \bullet \rightarrow \bullet \rightarrow \) we can find \( y_2 \) in this conjugacy class such that \( t_2y_2 \) and \( y_2x_2 \) have odd order. Since \( y_2 \) commutes with both \( (1, 2)(3, 4)t_1 \) and \( (1, 3)(2, 4)x_1 \), without loss we may assume that \( \mathcal{H}_x = C_x \). We now work through the possibilities for \( \mathcal{H}_x \) making repeated use of Lemma 2.2(iii) to show \( d(t, x) \leq 2 \). The first three possibilities listed above do not need attention as \( n \geq 16 \).

If \( \mathcal{H}_x \) is

\[
\begin{array}{cccc}
\{6, 7\} & \{7, 8\} & \{2m-1, 2m\} & \{2m+1\} \\
\{5, 6\} & \{6, 7\} & \{2m-1, 2m\} & \{2m+1\} \\
\{4, 5\} & \{5, 6\} & \{2m-1, 2m\} & \{2m+1\} \\
\{3, 4\} & \{4, 5\} & \{2m-1, 2m\} & \{2m+1\} \\
\{2, 3\} & \{3, 4\} & \{2m-1, 2m\} & \{2m+1\} \\
\{1, 2\} & \{2, 3\} & \{2m-1, 2m\} & \{2m+1\} \\
\{0, 1\} & \{1, 2\} & \{2m-1, 2m\} & \{2m+1\} \\
\{0\} & \{1\} & \{2m-1, 2m\} & \{2m+1\} \\
\end{array}
\]

then

\[
t = (1, 2)(3, 4)(5, 6)(7, 8) \cdots (2m + 1, 2m) \]

and, without loss of generality,

\[
x = (1, 3)(2, 4)(5)(6, 7)(8, 9) \cdots (2m + 1, 2m + 1).
\]

In the case when \( m \) is odd, we select

\[
y = (1, 5)(2, 3)(4, 2m)(6, 7)(8, 9) \cdots (2m - 2, 2m - 1)(2m + 1),
\]

and then \( \mathcal{G}_y \) is

\[
\begin{array}{cccc}
\{3m-1, 3m\} & \{3m-1, 3m\} & \{3m-1, 3m\} & \{3m-1, 3m\} \\
\{3m-2, 3m-1\} & \{3m-2, 3m-1\} & \{3m-2, 3m-1\} & \{3m-2, 3m-1\} \\
\{3m-3, 3m-2\} & \{3m-3, 3m-2\} & \{3m-3, 3m-2\} & \{3m-3, 3m-2\} \\
\{3m-4, 3m-3\} & \{3m-4, 3m-3\} & \{3m-4, 3m-3\} & \{3m-4, 3m-3\} \\
\{3m-5, 3m-4\} & \{3m-5, 3m-4\} & \{3m-5, 3m-4\} & \{3m-5, 3m-4\} \\
\{3m-6, 3m-5\} & \{3m-6, 3m-5\} & \{3m-6, 3m-5\} & \{3m-6, 3m-5\} \\
\{3m-7, 3m-6\} & \{3m-7, 3m-6\} & \{3m-7, 3m-6\} & \{3m-7, 3m-6\} \\
\{3m-8, 3m-7\} & \{3m-8, 3m-7\} & \{3m-8, 3m-7\} & \{3m-8, 3m-7\} \\
\{3m-9, 3m-8\} & \{3m-9, 3m-8\} & \{3m-9, 3m-8\} & \{3m-9, 3m-8\} \\
\{3m-10, 3m-9\} & \{3m-10, 3m-9\} & \{3m-10, 3m-9\} & \{3m-10, 3m-9\} \\
\end{array}
\]

while \( \mathcal{G}_x^y \) is

\[
\begin{array}{cccc}
\{3, 4\} & \{4, 2m\} & \{2m-1, 2m\} & \{2m+1\} \\
\{2, 3\} & \{3, 4\} & \{2m-1, 2m\} & \{2m+1\} \\
\{1, 2\} & \{2, 3\} & \{2m-1, 2m\} & \{2m+1\} \\
\{0, 1\} & \{1, 2\} & \{2m-1, 2m\} & \{2m+1\} \\
\{0\} & \{1\} & \{2m-1, 2m\} & \{2m+1\} \\
\end{array}
\]
So $G_y$ consists of a cycle of $m$ black vertices and one white vertex while $G_y^x$ has one connected component with three black vertices and one white vertex with each of the other components being a cycle with one black vertex. Consequently $ty$ and $yx$ both have odd order by Lemma 2.2(iii), whence $d(t, x) \leq 2$. If $m$ were to be even, instead we choose

$$y = (1, 2m - 1)(2, 6)(3, 4)(5, 2m)(7, 8)(9, 10)\ldots(2m - 3, 2m - 2)$$

which gives $G_y$ as

![Graph $G_y$](image)

and $G_y^x$ as

![Graph $G_y^x$](image)

Here the cycle of black vertices in $G_y^x$ has $m - 1$ black vertices whence using Lemma 2.2(iii) again we deduce that $d(t, x) \leq 2$, and this settles the case when $\mathcal{H}_x$ is.

Now we examine the case when $\mathcal{H}_x$ is

![Graph $\mathcal{H}_x$](image)

So

$$t = (1, 2)(3, 4)(5, 6)\ldots(2r - 1, 2r)(2r + 1)(2r + 2, 2r + 3)\ldots(2m - 2, 2m - 1)(2m)$$

and

$$x = (1, 3)(2, 4)(5, 7)(8, 9)\ldots(2r - 2, 2r - 1)(2r)(2r + 1, 2r + 2)\ldots(2m - 1, 2m).$$

Choosing

$$y = (1, 2m)(2, 3)(4, 5)\ldots(2r - 2, 2r - 1)(2r + 1, 2r + 2)\ldots(2m - 3, 2m - 2),$$

we observe that $G_y$ is

![Graph $G_y$](image)

and $G_y^x$ is

![Graph $G_y^x$](image)
Yet again Lemma 2.2(iii) shows that $d(t, y) \leq 1 \geq d(y, x)$ so dealing with this possibility for $H_x$.

We now consider our final case which is when $H_x$ is

Thus

$$t = (1, 2)(3, 4)(5, 6)(7, 8) \ldots (2m - 1, 2m)$$

and, without loss,

$$x = (1, 3)(2, 4)(6, 7)(8, 9) \ldots (2m, 5).$$

When $m$ is even we select

$$y = (1, 5)(2, 2m)(3, 4)(6, 2m - 1)(7, 8)(9, 10) \ldots (2m - 3, 2m - 2)$$

and as a result $G_y$ is

Before dealing with $m$ odd we recall that we are assuming $n = 2m \geq 16$. So $2m - 4 > 10$ and therefore the choice we now make gives us an element of $X$. Take

$$y = (1, 2m - 4)(2, 2m)(3, 4)(5, 7)(6, 9)(8, 2m - 1)(10, 11)(12, 13) \ldots (2m - 6, 2m - 5)(2m - 3, 2m - 2).$$

Hence $G_y$ is
and $G_x^y$ is

Use of Lemma 2.2(iii) shows that whether $m$ is even or odd we have $d(t, x) \leq 2$.

Having successfully dealt with all the possibilities for $H_x$, the proof of Theorem 1.1 is complete.

4 Connectedness of $F_\pi(G, X)$

As promised here we prove Theorem 1.2 which we restate.

Theorem 4.1. Suppose that $G = Sym(n)$, $X$ is a $G$-conjugacy class of involutions and $\pi$ is a set of odd positive integers. Then $F_\pi(G, X)$ is either totally disconnected or connected.

Proof. We argue by induction on $n$, with $n = 1$ clearly holding. Assume that $F_\pi(G, X)$ is not totally disconnected, and let $t \in X$ be such that $Y$, the connected component of $t$ in $F_\pi(G, X)$, has $|Y| > 1$. Put $K = Stab_G(Y)$. If $K = G$, then $Y = X$ and hence $F_\pi(G, X)$ is connected. So we now suppose $K \neq G$, and argue for a contradiction.

Let $x \in Y$ with $d(t, x) = 1$. Then $z = tx$ has order in the set $\pi$, and we have

\begin{equation}
\langle C_G(t), C_G(x) \rangle \leq K, \quad \text{and}
\end{equation}

\begin{equation}
\text{supp}(t) \cup \text{supp}(x) = \Omega.
\end{equation}

If (4.2) is false, then $t$ and $x$ both fix some $\alpha \in \Omega$. So $t, x \in G_{\alpha} \cong Sym(n - 1)$. Since $t$ and $x$ are $G_{\alpha}$-conjugate and the order of $tx$ is in $\pi$, by induction $F_\pi(G_{\alpha}, X \cap G_{\alpha})$ is connected. Therefore $G_{\alpha} \leq K$, and so, as $K \neq G$ and $G_{\alpha}$ is a maximal subgroup of $G$, $K = G_{\alpha}$. If $t$ fixes a further element of $\Omega$, say $\beta$, then, by (4.1), $(\alpha, \beta) \in C_G(t) \leq K$, contrary to $K = G_{\alpha}$. So $t$ (and hence also $x$) fixes only $\alpha$. Thus $G_x$ has only one white node (namely $\{\alpha\}$) with the remaining connected components being either $\bullet$ or $\circ$.

Without loss we assume $\alpha = n$.

Suppose that $G_x$ has $\circ$ as a component. So, without loss of generality,

$$t = (1, 2)(3, 4) \ldots (n - 2, n - 1) = (1, 2)t_1$$

and $x = (1, 2)x_1$, where $x_1 \in Sym(\{3, 4, \ldots, n - 1\})$. Since $K \neq G$, we must have $n > 3$. Thus $t_1, x_1 \in H = Sym(\{3, 4, \ldots, n - 1\})$ with $t_1$ and $x_1$ being $H$-conjugate involutions and the order of $t_1x_1$, being the same as that of $tx$, lies
in $\pi$. Using induction again we infer that $F_\pi(H, t_1^H)$ is connected. Hence, in $F_\pi(H, t_1^H)$ there is a path from $t_1$ to

$$s_1 = (3, 4)(5, 6) \ldots (n - 4, n - 3)(n - 1, n),$$

say $t_1 = y_0, y_1, \ldots, y_m = s_1$ ($y_i \in t_1^H$). Consequently

$$t = (1, 2)t_1 = (1, 2)y_0, (1, 2)y_1, \ldots, (1, 2)y_m = (1, 2)s_1$$

is a path in $F_\pi(G, X)$ from $t_1$ to

$$(1, 2)(3, 4)(5, 6) \ldots (n - 4, n - 3)(n - 1, n).$$

But then $(n - 1, n) \in K$, whereas $K = G_{\alpha}$. This rules out $\bullet$ as being a connected component of $G_x$.

Let $t = t_1t_2 \cdots t_k$ and $x = x_1x_2 \cdots x_k$, where

$$
t_1 = (1, 2) \ldots (\ell_1 - 1, \ell_1),
$$

$$
t_2 = (\ell_1 + 1, \ell_1 + 2) \ldots (\ell_1 + \ell_2 - 1, \ell_1 + \ell_2),
$$

$$
\vdots
$$

and

$$
x_1 = (2, 3)(4, 5) \ldots (\ell_1 - 2, \ell_1 - 1)(1, \ell_1),
$$

$$
x_2 = (\ell_1 + 2, \ell_1 + 3) \ldots (\ell_1 + \ell_2 - 2, \ell_1 + \ell_2 - 1)(\ell_1 + 1, \ell_1 + \ell_2),
$$

$$
\vdots
$$

So the elements $x_1, x_2, \ldots$ correspond to the connected components of $G_x$. By Lemma 2.2(iii)(b) $t_1x_1$ has order $m = \ell_1/2$. Now the order of $z = tx$ is the least common multiple of the orders of $t_1x_1, t_2x_2, \ldots, t_kx_k$, whence $m$ must be odd. Put

$$w = (n, 1, 3, 5, \ldots, m - 2, m - 1, m - 3, \ldots, 6, 4, 2).$$

Then $w$ is a cycle of length $m$, and so of order $m$. Further (by design) $w^t_1 = w^{-1}$ and hence

$$y_1 = t_1w = (1, n)(2, 3)(4, 5) \ldots (m - 3, m - 2)(m, m + 1)(m + 2, m + 3) \ldots (\ell_1 - 1, \ell_1)$$

is conjugate to $t_1$. Also, of course, $t_1y_1 = w$ has order $m$. So $y = y_1t_2 \cdots t_k \in X$ and the order of $ty$ is the same as that of $tx$. Therefore $y \in Y$ and hence $(1, n) \in K$. This contradicts the earlier deduction that $K = G_{\alpha}$, and with this we have proven (4.2).

(4.3) $K$ acts transitively and primitively on $\Omega$.

Since $C_G(t)$ and $C_G(x)$ have shape $2^k \text{Sym}(2k) \times \text{Sym}(n - 2k)$, where $k =$
$|\text{supp}(t)|/2$, and $t$ and $x$ do not commute, (4.1) and (4.2) imply that $K$ is transitive on $\Omega$. Suppose $K$ does not act primitively on $\Omega$. Then we may choose a nontrivial block $\Lambda$ for $K$ with $\alpha \in \Lambda \cap \text{supp}(t)$. If $\Lambda \not\subseteq \text{supp}(t)$, then the action of $C_G(t)$ on $\Omega$ results in $\Lambda = \Omega$. Thus $\Lambda \subseteq \text{supp}(t)$. Again, using the action of $C_G(t)$ on $\Omega$ we deduce that either $\Lambda = \text{supp}(t)$ or $\Lambda = \{\alpha, \beta\}$ where $\beta = \alpha^t$. Since $t$ and $x$ do not commute, we may further assume that $\alpha \in \text{supp}(x)$ is such that $\alpha^x \notin \{\alpha, \beta\}$. So $\alpha \in \text{supp}(x)$ and a similar argument yields that either $\Lambda = \text{supp}(x)$ or $\Lambda = \{\alpha, \alpha^x\}$. In view of (4.2) this then implies that $\Lambda = \Omega$, contrary to $\Lambda$ being a nontrivial block. Thus (4.3) holds.

Plainly $C_G(t)$, and hence $K$, contains transpositions. Thus Jordan’s theorem [23] and (4.3) force $K = G$. With this contradiction the proof of the theorem is complete. \square

References


