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Finite p -groups with a Frobenius group of automorphisms whose kernel is a cyclic p -group

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Abstract

Suppose that a finite p -group G admits a Frobenius group of automorphisms FH with kernel F that is a cyclic p -group and with complement H . It is proved that if the fixed-point subgroup $C_G(H)$ of the complement is nilpotent of class c , then G has a characteristic subgroup of index bounded in terms of c , $|C_G(F)|$, and $|F|$ whose nilpotency class is bounded in terms of c and $|H|$ only. Examples show that the condition of F being cyclic is essential. The proof is based on a Lie ring method and a theorem of the authors and P. Shumyatsky about Lie rings with a metacyclic Frobenius group of automorphisms FH . It is also proved that G has a characteristic subgroup of $(|C_G(F)|, |F|)$ -bounded index whose order and rank are bounded in terms of $|H|$ and the order and rank of $C_G(H)$, respectively, and whose exponent is bounded in terms of the exponent of $C_G(H)$.

Key words. finite p -group, Frobenius group, automorphism, nilpotency class, Lie ring

1 Introduction

It has long been known that results on ‘semisimple’ fixed-point-free automorphisms of nilpotent groups and Lie rings can be applied for studying ‘unipotent’ p -automorphisms of finite p -groups. Alperin [1] was the first to use Higman’s theorem on Lie rings and nilpotent groups with a fixed-point-free automorphism of prime order p in the study of a finite p -group P with an automorphism φ of order p . Namely, Alperin [1] proved that the derived length of P is bounded in terms of the number of fixed points $p^m = |C_P(\varphi)|$. Later the first author [10] improved the argument to obtain a subgroup of P of (p, m) -bounded index and of p -bounded nilpotency class, and the second author [19] noted that this class can be bounded by $h(p)$, where $h(p)$ is Higman’s function bounding the nilpotency class of a Lie ring or a nilpotent group with a fixed-point-free automorphism of order p .

Henceforth we write for brevity, say, “ (a, b, \dots) -bounded” for “bounded above by some function depending only on a, b, \dots ”. Further strong results on p -automorphisms of finite p -groups were obtained by Kiming [17], McKay [23], Shalev [26], Khukhro [11], Medvedev [24, 25], Jaikin-Zapirain [6], Shalev and Zelmanov [27] giving subgroups of bounded index and of bounded derived length or nilpotency class. The proofs of most of these ‘unipotent’ results were also based on the ‘semisimple’ theorems of Higman [4], Kreknin [9], Kreknin and Kostrikin [8] on fixed-point-free automorphisms of Lie rings.

In the present paper ‘unipotent’ theorems are derived from the recent ‘semisimple’ results of the authors and Shumyatsky [16, 21] about groups G (and Lie rings L) admitting a Frobenius group FH of automorphisms with kernel F and complement H . The results concern the connection between the nilpotency class, order, rank, and exponent of G and the corresponding parameters of $C_G(H)$. The more difficult of these results is about the nilpotency class, and its proof is based on the corresponding Lie ring theorem. Namely, it was proved in [16] that if the kernel F is cyclic and acts on a Lie ring L fixed-point-freely, $C_L(F) = 0$, and the fixed-point subring $C_L(H)$ of the complement is nilpotent of class c , then L is nilpotent of $(c, |H|)$ -bounded class (under certain assumptions on the additive group of L , which are satisfied in many important cases, like L being an algebra over a field, or being finite). Note that examples show that the condition of F being cyclic is essential. This Lie ring result also implied a similar result for a finite group G with a Frobenius group FH of automorphisms with cyclic fixed-point-free kernel F such that $C_G(H)$ is nilpotent of class c , with reduction to nilpotent case provided by classification and representation theory arguments. The fixed-point-free action of F alone was known to imply nice properties of the Lie ring (solubility of $|F|$ -bounded derived length by Kreknin’s theorem [9]) and of the group (solubility and well-known bounds for the Fitting height due to Thompson [28], Kurzweil [18], Turull [29], and others — although an analogue of Kreknin’s theorem is still an open problem for groups). But the conclusions of the results in [16] are in a sense much stronger, due to the combination of the hypotheses on fixed points of F and H , either of which on its own is insufficient.

We now state the ‘unipotent’ version of the nilpotency class result in [16].

Theorem 1.1. *Suppose that a finite p -group P admits a Frobenius group FH of automorphisms with cyclic kernel F of order p^k . Let c be the nilpotency class of the fixed-point subgroup $C_P(H)$ of the complement. Then P has a characteristic subgroup of index bounded in terms of c , $|F|$, and $|C_P(F)|$ whose nilpotency class is bounded in terms of c and $|H|$ only.*

The proof is quite similar to the proofs of the aforementioned results of Alperin [1] and Khukhro [10], with the Lie ring theorem in [16] taking over the role of the Higman–Kreknin–Kostrikin theorem. However, first a certain combinatorial corollary of that Lie ring theorem has to be derived (Proposition 2.2). Example 3.5 shows that the condition of the kernel F being cyclic in Theorem 1.1 is essential.

We now state the unipotent versions of the rank, order, and exponent results in [16]. (By the rank we mean the minimum number r such that every subgroup can be generated by r elements.)

Theorem 1.2. *Suppose that a finite p -group P admits a Frobenius group FH of automorphisms with cyclic kernel F of order p^k . Then P has a characteristic subgroup Q of index bounded in terms of $|F|$ and $|C_P(F)|$ such that*

- (a) *the order of Q is at most $|C_P(H)|^{|H|}$;*
- (b) *the rank of Q is at most $r|H|$, where r is the rank of $C_P(H)$;*
- (c) *the exponent of Q is at most p^{2e} , where p^e is the exponent of $C_P(H)$.*

Note that the estimates for the order and rank are best-possible, and for the exponent close to being best-possible (and independent of $|FH|$). The proof is facilitated by a straightforward reduction to powerful p -groups. Then certain versions of the ‘free H -module arguments’ are applied to abelian FH -invariant sections. If a finite group G admits a Frobenius group of automorphisms FH with complement H and with kernel F acting fixed-point-freely, then every elementary abelian FH -invariant section of G is a free kH -module (for various prime fields k). This is exactly what provides a motivation for seeking results bounding various parameters of G in terms of those of $C_P(H)$ and $|H|$. In the ‘semisimple’ situation this fact is a basis of the results on the order and rank in [16]. The exponent result in [16] is more difficult, but in our unipotent situation a simpler argument can be used based on powerful p -groups to produce a much better result, with the estimate for the exponent depending only on the exponent of $C_P(H)$.

It should be mentioned that the ‘semisimple’ results on the order and rank in [16] do not assume the kernel to be cyclic, a ‘unipotent’ analogue of which is unclear at the moment. The results of the present paper can be regarded as generalizations of the results of [16], where the kernel F acts on G fixed-point-freely, to the case of ‘almost fixed-point-free’ kernel. It is natural to expect that similar restrictions, in terms of the complement H and its fixed points $C_G(H)$, should hold for a subgroup of index bounded in terms of $|C_G(F)|$ and other parameters: ‘almost fixed-point-free’ action of F implying that G is ‘almost’ as good as when F acts fixed-point-freely. In the coprime ‘semisimple’ situation such restrictions were recently obtained in [14] for the order and rank of G , and in [15] and [20] for the nilpotency class. For the moment it is unclear how to combine these semisimple and unipotent results in a general setting, without assumptions on the orders of G and FH ; note that the results in [16] for the fixed-point-free kernel were free of such assumptions.

2 Lie ring technique

First we recall some definitions and notation. Products in a Lie ring are called commutators. The Lie subring generated by a subset S is denoted by $\langle S \rangle$ and the ideal by $\text{id}\langle S \rangle$.

Terms of the lower central series of a Lie ring L are defined by induction: $\gamma_1(L) = L$; $\gamma_{i+1}(L) = [\gamma_i(L), L]$. By definition a Lie ring L is nilpotent of class h if $\gamma_{h+1}(L) = 0$.

A simple commutator $[a_1, a_2, \dots, a_s]$ of weight (length) s is by definition the commutator $[...[[a_1, a_2], a_3], \dots, a_s]$.

Let A be an additively written abelian group. A Lie ring L is A -graded if

$$L = \bigoplus_{a \in A} L_a \quad \text{and} \quad [L_a, L_b] \subseteq L_{a+b}, \quad a, b \in A,$$

where the grading components L_a are additive subgroups of L . Elements of the L_a are called *homogeneous* (with respect to this grading), and commutators in homogeneous elements *homogeneous commutators*. An additive subgroup H of L is said to be *homogeneous* if $H = \bigoplus_a (H \cap L_a)$; then we set $H_a = H \cap L_a$. Obviously, any subring or an ideal generated by homogeneous additive subgroups is homogeneous. A homogeneous subring and the quotient ring by a homogeneous ideal can be regarded as A -graded rings with the induced gradings.

Suppose that a Lie ring L admits a Frobenius group of automorphisms FH with cyclic kernel $F = \langle \varphi \rangle$ of order n . Let ω be a primitive n -th root of unity. We extend the ground ring by ω and denote by \tilde{L} the ring $L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. Then φ naturally acts on \tilde{L} and, in particular, $C_{\tilde{L}}(\varphi) = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$.

Definition. We define φ -components L_k for $k = 0, 1, \dots, n-1$ as the ‘eigensubspaces’

$$L_k = \{a \in \tilde{L} \mid a^\varphi = \omega^k a\}.$$

It is well known that $n\tilde{L} \subseteq L_0 + L_1 + \dots + L_{n-1}$ (see, for example, [5, Ch. 10]). This decomposition resembles a $(\mathbb{Z}/n\mathbb{Z})$ -grading because of the inclusions $[L_s, L_t] \subseteq L_{s+t \pmod{n}}$, but the sum of φ -components is not direct in general.

Definition. We refer to commutators in elements of φ -components as being *φ -homogeneous*.

Index Convention. Henceforth a small letter with index i denotes an element of the φ -component L_i , so that the index only indicates the φ -component to which this element belongs: $x_i \in L_i$. To lighten the notation we will not use numbering indices for elements in L_j , so that different elements can be denoted by the same symbol when it only matters to which φ -component these elements belong. For example, x_1 and x_1 can be different elements of L_1 , so that $[x_1, x_1]$ can be a nonzero element of L_2 . These indices will be considered modulo n ; for example, $a_{-i} \in L_{-i} = L_{n-i}$.

Note that under the Index Convention a φ -homogeneous commutator belongs to the φ -component L_s , where s is the sum modulo n of the indices of all the elements occurring in this commutator.

Since the kernel F of the Frobenius group FH is cyclic, the complement H is also cyclic. Let $H = \langle h \rangle$ be of order q and $\varphi^{h^{-1}} = \varphi^r$ for some $1 \leq r \leq n-1$. Then r is a primitive q -th root of unity in the ring $\mathbb{Z}/n\mathbb{Z}$.

The group H permutes the φ -components L_i as follows: $L_i^h = L_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Indeed, if $x_i \in L_i$, then $(x_i^h)^\varphi = x_i^{h\varphi h^{-1}} = (x_i^{\varphi^r})^h = \omega^{ir} x_i^h$, so that $L_i^h \subseteq L_{ir}$; the reverse inclusion is obtained by applying the same argument to h^{-1} .

Notation. In what follows, for a given $u_k \in L_k$ we denote the element $u_k^{h^i}$ by $u_{r^i k}$ under the Index Convention, since $L_k^{h^i} = L_{r^i k}$. We denote the H -orbit of an element x_i by $O(x_i) = \{x_i, x_{ri}, \dots, x_{r^{q-1}i}\}$.

Combinatorial theorem. We are going to prove a combinatorial consequence of the Makarenko–Khukhro–Shumyatsky theorem in [16], which we state in a somewhat different form, in terms of $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie rings with a cyclic group of automorphisms H .

Theorem 2.1 ([16, Theorem 5.5 (b)]). *Let $M = \bigoplus_{i=0}^n M_i$ be a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring with grading components M_i that are additive subgroups satisfying the inclusions $[M_i, M_j] \subseteq M_{i+j \pmod n}$. Suppose M admits a finite cyclic group of automorphisms $H = \langle h \rangle$ of order q such that $M_i^h = M_{r_i}$ for some element $r \in \mathbb{Z}/n\mathbb{Z}$ having multiplicative order q . If $M_0 = 0$ and $C_M(H)$ is nilpotent of class c , then for some functions $u = u(c, q)$ and $f = f(c, q)$ depending only on c and q , the Lie subring $n^u L$ is nilpotent of class $f - 1$, that is, $\gamma_f(n^u L) = n^{uf} \gamma_f(L) = 0$.*

The corresponding theorems in [16] were stated about Lie rings admitting a Frobenius group FH of automorphisms with cyclic kernel $F = \langle \varphi \rangle$ of order n . After extension of the ground ring, the φ -components behave like components of a $(\mathbb{Z}/n\mathbb{Z})$ -grading, as we saw above. In fact, the proofs in [16] only used the ‘grading’ properties of the φ -components, so that Theorem 2.1 was actually proved therein. The following proposition is a combinatorial consequence of this theorem.

Proposition 2.2. *Let $f = f(c, q)$, $u = u(c, q)$ be the functions in Theorem 2.1. Suppose that a Lie ring L admits a Frobenius group of automorphisms FH with cyclic kernel $F = \langle \varphi \rangle$ of order n and with complement H of order q such that the fixed-point subring $C_L(H)$ of the complement is nilpotent of class c . Then for the (c, q) -bounded number $w = (u + 1)f(c, q)$ the n^w -th multiple $n^w[x_{i_1}, x_{i_2}, \dots, x_{i_f}]$ of every simple φ -homogeneous commutator in $\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ of weight f with non-zero indices can be represented as a linear combination of φ -homogeneous commutators of the same weight f in elements of the union of H -orbits $\bigcup_{s=1}^f O(x_{i_s})$ each of which contains a subcommutator with zero sum of indices modulo n .*

Remark 2.3. Similar combinatorial propositions were also proved for Lie algebras in [20] and for Lie rings whose ground ring contains the inverse of n in [15].

Proof. The idea of the proof is application of Theorem 2.1 to a free Lie ring with operators FH . Given arbitrary (not necessarily distinct) non-zero elements $i_1, i_2, \dots, i_f \in \mathbb{Z}/n\mathbb{Z}$, we consider a free Lie ring K over R with qf free generators in the set

$$Y = \left\{ \underbrace{y_{i_1}, y_{ri_1}, \dots, y_{r^{q-1}i_1}}_{O(y_{i_1})}, \underbrace{y_{i_2}, y_{ri_2}, \dots, y_{r^{q-1}i_2}, \dots, y_{i_f}, y_{ri_f}, \dots, y_{r^{q-1}i_f}}_{O(y_{i_2})}, \dots, \underbrace{y_{i_f}, y_{ri_f}, \dots, y_{r^{q-1}i_f}}_{O(y_{i_f})} \right\},$$

where indices are formally assigned and regarded modulo n and the subsets $O(y_{i_s}) = \{y_{i_s}, y_{ri_s}, \dots, y_{r^{q-1}i_s}\}$ are disjoint. Here, as in the Index Convention, we do not use numbering indices, that is, all elements $y_{r^k i_j}$ are by definition different free generators, even if indices coincide. (The Index Convention will come into force in a moment.) For every $i = 0, 1, \dots, n - 1$ we define the additive subgroup K_i generated by all commutators in the generators y_{j_s} in which the sum of indices of all entries is equal to i modulo n . Then $K = K_0 \oplus K_1 \oplus \dots \oplus K_{n-1}$. It is also obvious that $[K_i, K_j] \subseteq K_{i+j \pmod n}$; therefore this is a $(\mathbb{Z}/n\mathbb{Z})$ -grading. The Lie ring K also has the natural \mathbb{N} -grading $K = G_1(Y) \oplus G_2(Y) \oplus \dots$

with respect to the generating set Y , where $G_i(Y)$ is the additive subgroup generated by all commutators of weight i in elements of Y .

We define an action of the Frobenius group FH on K by setting $k_i^\varphi = \omega^i k_i$ for $k_i \in K_i$ and extending this action to K by linearity. An action of H is defined on the generating set Y as a cyclic permutation of elements in each subset $O(y_{i_s})$ by the rule $(y_{r:k_{i_s}})^h = y_{r:k+1_{i_s}}$ for $k = 0, \dots, q-2$ and $(y_{r^{q-1}_{i_s}})^h = y_{i_s}$. Then $O(y_{i_s})$ becomes the H -orbit of the element y_{i_s} . Clearly, H permutes the components K_i by the rule $K_i^h = K_{r_i}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $J = \text{id}\langle K_0 \rangle$ be the ideal generated by the φ -component K_0 . Clearly, the ideal J consists of linear combinations of commutators in elements of Y each of which contains a subcommutator with zero sum of indices modulo n . The ideal J is generated by homogeneous elements with respect to the gradings $K = \bigoplus_i G_i(Y)$ and $K = \bigoplus_{i=0}^{n-1} K_i$ and therefore is homogeneous with respect to both gradings. Note also that the ideal J is obviously FH -invariant.

Let $I = \text{id}\langle \gamma_{c+1}(C_K(H)) \rangle^F$ be the smallest F -invariant ideal containing the subring $\gamma_{c+1}(C_K(H))$. The ideal I is obviously homogeneous with respect to the grading $K = \bigoplus_i G_i(Y)$ and is FH -invariant. The fact that the ideal I is F -invariant, implies that $nI \subseteq I_0 \oplus \dots \oplus I_{n-1}$, where $I_k = I \cap K_k$ for $k = 0, 1, \dots, n-1$. Indeed, for $z \in I$, for every $i = 0, \dots, n-1$ we have $z_i := \sum_{s=0}^{n-1} \omega^{-is} z^{\varphi^s} \in K_i$ and $nz = \sum_{j=0}^{n-1} z_j$. We denote $\hat{I} = I_0 \oplus \dots \oplus I_{n-1}$. This is an ideal of K , which is homogeneous with respect to both gradings $K = \bigoplus_i G_i(Y)$ and $K = \bigoplus_{i=0}^{n-1} K_i$. It is also FH -invariant, since I is FH -invariant and the components K_i are permuted by FH .

Consider the quotient Lie ring $N = K/(J + \hat{I})$. Since the ideals J and \hat{I} are homogeneous with respect to the gradings $K = \bigoplus_i G_i(Y)$ and $K = \bigoplus_{i=0}^{n-1} K_i$, the quotient ring N has the corresponding induced gradings. We use indices to denote the components N_i of the $(\mathbb{Z}/n\mathbb{Z})$ -grading induced by $K = \bigoplus_{i=0}^{n-1} K_i$. Note that $N_0 = 0$ by the construction of J .

The group H permutes the grading components of $N = N_1 \oplus \dots \oplus N_{n-1}$ with regular orbits of length q . Therefore elements of $C_N(H)$ have the form $a + a^h + \dots + a^{h^{q-1}}$. Hence $C_N(H)$ is contained in the image of $C_K(H)$ in $N = K/(J + \hat{I})$ and therefore $\gamma_{c+1}(C_N(H))$ is contained in the image of the ideal I by its construction. Then $n\gamma_{c+1}(C_N(H)) = 0$, since $nI \subseteq \hat{I}$.

The group H also permutes the $(\mathbb{Z}/n\mathbb{Z})$ -grading components of $M := nN = \bigoplus_{i=0}^{n-1} M_i$, where $M_i = nN_i$, with regular orbits of length q . Therefore, $C_M(H) = nC_N(H)$ and $\gamma_{c+1}(C_M(H)) = \gamma_{c+1}(nC_N(H)) = n^{c+1}\gamma_{c+1}(C_N(H)) = 0$.

Since $N_0 = 0$, we also have $M_0 = 0$.

By Theorem 2.1 for some (c, q) -bounded function $u = u(c, q)$ the Lie ring $n^u M$ is nilpotent of (c, q) -bounded class $f - 1 = f(c, q) - 1$. Consequently,

$$n^{(u+1)f}[y_{i_1}, y_{i_2}, \dots, y_{i_f}] = [n^{u+1}y_{i_1}, n^{u+1}y_{i_2}, \dots, n^{u+1}y_{i_f}] \in J + \hat{I}.$$

Since both ideals J and \hat{I} are homogeneous with respect to the grading $K = \bigoplus_i G_i(Y)$, this means that the left-hand side is equal modulo the ideal \hat{I} to a linear combination of commutators of the same weight f in elements of Y each of which contains a subcommutator with zero sum of indices modulo n .

Now suppose that L is an arbitrary Lie ring satisfying the hypothesis of Proposition 2.2, and let $\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. Let $x_{i_1}, x_{i_2}, \dots, x_{i_f}$ be arbitrary φ -homogeneous elements of \tilde{L} .

We define the homomorphism δ from the free Lie ring K into \tilde{L} extending the mapping

$$y_{r^k i_s} \rightarrow x_{i_s}^{h^k} \quad \text{for } s = 1, \dots, f \quad \text{and } k = 0, 1, \dots, q-1.$$

It is easy to see that δ commutes with the action of FH on K and \tilde{L} . Therefore $\delta(O(y_{i_s})) = O(x_{i_s})$ and $\delta(I) = 0$, since $\gamma_{c+1}(C_{\tilde{L}}(H)) = 0$ and $\delta(C_K(H)) \subseteq C_{\tilde{L}}(H)$. We now apply δ to the representation of $n^{(u+1)f}[y_{i_1}, y_{i_2}, \dots, y_{i_f}]$ constructed above. Since $\delta(\hat{I}) \subseteq \delta(I) = 0$, as the image we obtain a required representation of $n^{(u+1)f}[x_{i_1}, x_{i_2}, \dots, x_{i_f}]$ as a linear combination of commutators of weight f in elements of the set $\delta(Y) = \bigcup_{s=1}^f O(x_{i_s})$ each of which has a subcommutator with zero sum of indices modulo n . \square

3 Nilpotency class

We begin with two lemmas that are well-known in folklore. Induced automorphisms of invariant subgroups and sections are denoted by the same letters. Fixed-point subgroups are denoted as centralizers in the natural semidirect products.

Lemma 3.1 (see, e. g., [12, Theorem 1.5.1]). *If α is an automorphism of a finite group G and N is an α -invariant subgroup of G , then $|C_{G/N}(\alpha)| \leq |C_G(\alpha)|$.* \square

Lemma 3.2 (see, e. g., [12, Corollary 1.7.4]). *If φ is an automorphism of order p^k of a finite abelian p -group A and $|C_A(\varphi)| = p^s$, then the rank of A is at most sp^k .*

The following lemma is a well-known consequence of the theory of powerful p -groups [22].

Lemma 3.3 (see, e. g., [13, Corollary 11.21]). *If a finite p -group P has rank r and exponent p^e , then $|P|$ is (p, r, e) -bounded.*

Proof of Theorem 1.1. Recall that P is a finite p -group admitting a Frobenius group FH of automorphisms with cyclic kernel $F = \langle \varphi \rangle$ of order p^k and complement H of order q . Let $p^m = |C_P(F)|$ and let $C_P(H)$ be nilpotent of class c . We need to find a characteristic subgroup of (p, k, m, c) -bounded index and of (c, q) -bounded nilpotency class.

Consider the associated Lie ring $L(P) = \bigoplus_i \gamma_i(P)/\gamma_{i+1}(P)$, where γ_i denote terms of the lower central series (see, e. g., § 3.2 in [12]). Extend the ground ring by a p^k -th primitive root of unity ω setting $L = L(P) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ and regarding $L(P)$ as $L(P) \otimes 1$. The group FH naturally acts on L . We define the φ -components as in § 2 (with $n = p^k$); recall that $p^k L \subseteq L_0 + L_1 + \dots + L_{p^k-1}$. Since any φ -homogeneous commutator with zero sum of indices modulo p^k belongs to L_0 , by Proposition 2.2 we obtain

$$p^{k(f+w)} \gamma_f(L) = p^{kw} \gamma_f(p^k L) \subseteq p^{kw} \gamma_f(L_0 + L_1 + \dots + L_{p^k-1}) \subseteq \text{id} \langle L_0 \rangle$$

for the functions $f = f(c, q)$, $w = w(c, q)$ in that proposition. Since $L_0 = C_{L(P)}(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ and $p^m C_{L(P)}(\varphi) = 0$ by Lemma 3.1 and the Lagrange theorem, we obtain

$$p^{k(f+w)+m} \gamma_f(L) \subseteq p^m \text{id} \langle L_0 \rangle = 0.$$

In particular, $p^{k(f+w)+m}\gamma_f(L(P)) = 0$. In terms of the group P this means that the factors $\gamma_i(P)/\gamma_{i+1}(P)$ have exponent dividing $p^{k(f+w)+m}$ for all $i \geq f$.

By Lemmas 3.1 and 3.2, the rank of every factor $\gamma_i(P)/\gamma_{i+1}(P)$ is at most mp^k . Together with the bound for the exponent, this gives a bound for the order, which we state as a lemma.

Lemma 3.4. *Suppose that P is a finite p -group admitting a Frobenius group FH of automorphisms with cyclic kernel $F = \langle \varphi \rangle$ of order p^k and complement H of order q . Let $p^m = |C_P(F)|$ and let $C_P(H)$ be nilpotent of class c . Then $|\gamma_i(P)/\gamma_{i+1}(P)| \leq p^{(kf+kw+m)mp^k}$ for all $i \geq f$, where $f = f(c, q)$ and $w = w(c, q)$ are the functions in Proposition 2.2.*

Lemma 3.4 can be applied to any FH -invariant subgroup Q of P . In particular, we choose $Q = \gamma_{U+1}(P\langle \varphi \rangle)$, where $U = (kf + kw + m)mp^k$. Clearly, $Q \leq P$, so that $|C_Q(\varphi)| \leq p^m$. By Lemma 3.4, $|\gamma_i(Q)/\gamma_{i+1}(Q)| \leq p^U$ for all $i \geq f$. On the other hand, by the well-known theorem of P. Hall [3, Theorem 2.56] we have $|\gamma_i(Q)/\gamma_{i+1}(Q)| \geq p^{U+1}$ if $\gamma_{i+1}(Q) \neq 1$. To avoid a contradiction we must conclude that $\gamma_{f+1}(Q) = 1$. Thus, Q is nilpotent of (c, q) -bounded class $\leq f$.

The automorphism φ acts trivially on the factors of the lower central series of $P\langle \varphi \rangle$. Since $|C_{P\langle \varphi \rangle}(\varphi)| = p^{m+k}$, by Lemma 3.1 the orders of all these factors are at most p^{m+k} . Since the quotient $P\langle \varphi \rangle/Q$ is nilpotent of class U by construction, its order is at most $p^{(m+k)U} = p^{(m+k)(kf+kw+m)mp^k}$, which is a (p, k, m, c) -bounded number. Thus, Q has (p, k, m, c) -bounded index in P and (c, q) -bounded nilpotency class. The subgroup Q contains a characteristic subgroup P^{p^e} for some (p, k, m, c) -bounded number e . Since the rank of P is (p, k, m, c) -bounded, the index of P^{p^e} in P is also (p, k, m, c) -bounded by Lemma 3.3. \square

We now produce an example showing that the condition of the kernel being cyclic in Theorem 1.1 is essential.

Example 3.5. Let L be a Lie ring whose additive group is the direct sum of three copies of \mathbb{Z}_2 , the group of 2-adic integers, with generators e_1, e_2, e_3 as a \mathbb{Z}_2 -module, and let the structure constants of L be $[e_1, e_2] = 4e_3$, $[e_2, e_3] = 4e_1$, $[e_3, e_1] = 4e_2$. A Frobenius group FH of order 12 acts on L as follows: $F = \{1, f_1, f_2, f_3\}$, where $f_i(e_i) = e_i$ and $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1 \pmod{3}}$. Since L is a powerful Lie \mathbb{Z}_2 -algebra, by [2, Theorem 9.8] the Baker–Campbell–Hausdorff formula defines the structure of a uniformly powerful pro-2-group P on the same set L . For any positive integer n , the quotient of P by $P^{2^n} = 2^n L$ is a finite 2-group T . The induced action of FH on T is such that $|C_T(F)| = 8$ and $C_T(H)$ is cyclic, while the derived length of T is about $\log_4 n$.

4 Order, rank, and exponent

Suppose that a finite abelian group V admits a Frobenius group of automorphisms FH with cyclic kernel $F = \langle \varphi \rangle$ of order n . We can extend the ground ring by a primitive n -th root of unity ω forming $W = V \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ and define the natural action of the group

FH on W . As a \mathbb{Z} -module (abelian group), $\mathbb{Z}[\omega] = \bigoplus_{i=0}^{E(n)-1} \omega^i \mathbb{Z}$, where $E(n)$ is the Euler function. Hence,

$$W = \bigoplus_{i=0}^{E(n)-1} V \otimes \omega^i \mathbb{Z}, \quad (1)$$

so that $|W| = |V|^{E(n)}$. Similarly, $C_W(\varphi) = \bigoplus_{i=0}^{E(n)-1} C_V(\varphi) \otimes \omega^i \mathbb{Z}$, so that $|C_W(\varphi)| = |C_V(\varphi)|^{E(n)}$.

As in § 2 for \tilde{L} , we define φ -components W_k for $k = 0, 1, \dots, n-1$ as the ‘eigensubspaces’

$$W_k = \{a \in W \mid a^\varphi = \omega^k a\}.$$

Recall that W is an ‘almost direct sum’ of the W_i : namely,

$$nW \subseteq W_0 + W_1 + \dots + W_{n-1} \quad (2)$$

and

$$\text{if } w_0 + w_1 + \dots + w_{n-1} = 0 \text{ for } w_i \in W_i, \text{ then } nw_i = 0 \text{ for all } i. \quad (3)$$

As in § 2 we refer to elements of φ -components as being φ -homogeneous, and apply the Index Convention using lower indices of small Latin letters to only indicate the φ -component containing this element.

As before, since the kernel F of the Frobenius group FH is cyclic, the complement H is also cyclic, $H = \langle h \rangle$, say, of order q , and $\varphi^{h^{-1}} = \varphi^r$ for some $1 \leq r \leq n-1$, which is a primitive q -th root of unity in $\mathbb{Z}/n\mathbb{Z}$. The group H permutes the φ -components W_i by the rule $W_i^h = W_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. For $u_k \in W_k$ we denote $u_k^{h^i}$ by $u_{r^i k}$ under the Index Convention.

From now on we assume in addition that V is an abelian FH -invariant section of the p -group P in Theorem 1.2. Recall that $|\varphi| = n = p^k$ and $|C_P(\varphi)| = p^m$.

Lemma 4.1. *There is a characteristic subgroup U of V such that $|U|$ is (p, k, m) -bounded and*

- (a) $|V/U| \leq |C_V(H)|^{|H|}$;
- (b) *the rank of V/U is at most $r|H|$, where r is the rank of $C_P(H)$;*
- (c) *the exponent of V/U is at most p^e , where p^e is the exponent of $C_P(H)$.*

Proof. The group H acts on the set of φ -components W_i with one single-element orbit $\{W_0\}$ and $(p^k - 1)/q$ regular orbits. We choose one element in every regular H -orbit and let $Y = \sum_{j=1}^{(p^k-1)/q} W_{i_j}$ be the sum of these chosen φ -components. The mapping $\vartheta : y \rightarrow y + y^h + \dots + y^{h^{q-1}}$ is a homomorphism of the abelian group Y into $C_W(H)$. We claim that $p^k \text{Ker } \vartheta = 0$. Indeed, if $y \in \text{Ker } \vartheta$ is written as $y = \sum_{j=1}^{(p^k-1)/q} y_{i_j}$ for $y_{i_j} \in W_{i_j}$, then $\vartheta(y)$ is equal to y plus a linear combination of elements of φ -components $W_{r^l i_j}$ with all the indices $r^l i_j$ being different from the indices $i_1, \dots, i_{(p^k-1)/q}$. Therefore the equation $\vartheta(y) = 0$ implies $p^k y_{i_j} = 0$ by (3), so that $p^k y = 0$. Clearly, $|Y/\text{Ker } \vartheta| \leq |C_W(H)|$, the rank of $Y/\text{Ker } \vartheta$ is at most the rank of $C_W(H)$, and the exponent of $Y/\text{Ker } \vartheta$ is at most the exponent of $C_W(H)$.

Let p^f be the maximum of p^k and the exponent of W_0 , which is a (p, k, m) -bounded number. Then $\Omega_f(W) \geq W_0 + \text{Ker } \vartheta$ (where we use the standard notation Ω_i for the subgroup generated by all elements of order dividing p^i). Since

$$p^k W \leq W_0 + W_1 + \cdots + W_{p^k-1} = W_0 + Y + Y^h + \cdots + Y^{h^{q-1}},$$

we obtain the following.

Lemma 4.2. *The image of $p^k W$ in $W/\Omega_f(W)$ is contained in the image of $Y + Y^h + \cdots + Y^{h^{q-1}}$ in $W/\Omega_f(W)$, and the image of Y is a homomorphic image of $Y/\text{Ker } \vartheta$.*

We claim that $U = \Omega_{f+k}(V)$ is the required characteristic subgroup. The rank of the abelian group V is at most mp^k by Lemmas 3.1 and 3.2. Hence $\Omega_{f+k}(V)$ being of bounded exponent has (p, k, m) -bounded order. We now verify that parts (a), (b), (c) are satisfied.

(a) In the abelian p -group W the order of the image of $p^k W$ in $W/\Omega_f(W)$ is equal to $|W/\Omega_{f+k}(W)|$. Therefore Lemma 4.2 implies

$$|W/\Omega_{f+k}| \leq |Y/\text{Ker } \vartheta|^{|H|} \leq |C_W(H)|^{|H|}. \quad (4)$$

Clearly, $\Omega_{f+k}(W) = \Omega_{f+k}(V) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ and therefore $|\Omega_{f+k}(W)| = |\Omega_{f+k}(V)|^{E(p^k)}$. Since $|W| = |V|^{E(p^k)}$ and $|C_W(\varphi)| = |C_V(\varphi)|^{E(p^k)}$, taking the $E(p^k)$ -th root of both sides of (4) gives $|V/\Omega_{f+k}(V)| \leq |C_V(H)|^{|H|}$.

(b) Similarly, the rank of the image of $p^k W$ in $W/\Omega_f(W)$ is equal to the rank of W/Ω_{f+k} . By Lemma 4.2 we obtain that the rank of $W/\Omega_{f+k}(W)$ is at most $|H|$ times the rank of $C_W(H)$. Since the ranks are multiplied by $E(p^k)$ when passing from V to W , we obtain that the rank of $V/\Omega_{f+k}(V)$ is at most $|H|$ times the rank of $C_V(H)$, which in turn does not exceed r , the rank of $C_P(H)$, because $C_P(H)$ covers $C_V(H)$ since the action of H is coprime.

(c) Finally, the exponent of the image of $p^k W$ in $W/\Omega_f(W)$ is equal to the exponent of W/Ω_{f+k} . By Lemma 4.2 we obtain that the exponent of $W/\Omega_{f+k}(W)$ is at most that of $C_W(H)$, so that the exponent of $V/\Omega_{f+k}(V)$ is at most that of $C_V(H)$, which is at most p^e , the exponent of $C_P(H)$, since the action is coprime. \square

Proof of Theorem 1.2. Recall that P is a finite p -group admitting Frobenius group FH of automorphisms with cyclic kernel F of order p^k with $p^m = |C_P(F)|$ fixed points of the kernel. Let $p^s = |C_P(H)|$, let r be the rank of $C_P(H)$, and p^e the exponent of $C_P(H)$. We need to find a characteristic subgroup Q of (p, k, m) -bounded index with required bounds for the order, rank, and exponent. We can of course find such a subgroup separately for each of these parameters and then take the intersection.

By Lemmas 3.1 and 3.2, the rank of P is at most mp^k . Hence P has a characteristic powerful subgroup of (p, k, m) -bounded index by [22, Theorem 1.14]. Therefore we can assume P to be powerful from the outset.

By [11] (see also [13, Theorem 12.15]), the group P has a characteristic subgroup P_1 of (p, k, m) -bounded index that is soluble of p^k -bounded derived length at most $2K(p^k)$ (where K is Kreknin's function bounding the derived length of a Lie ring with a fixed-point-free automorphism of order p^k). Let V be any of the factors of the derived series

of P_1 . By Lemma 4.1 we have $|V| \leq p^g |C_V(H)|^{|H|}$ for some (p, k, m) -bounded number $g = g(p, k, m)$. Then

$$|P_1| = \prod_V |V| \leq p^{2gK(p^k)} \prod_V |C_V(H)|^{|H|} = p^{2gK(p^k)} |C_{P_1}(H)|^{|H|},$$

since the action of H is coprime. Since the rank of the powerful p -group P is at most mp^k , by taking a sufficiently large but (p, k, m) -bounded power $P^{f(p, k, m)}$ we obtain a characteristic subgroup of order at most $|C_P(H)|^{|H|}$, which has (p, k, m) -bounded index by Lemma 3.3.

The powerful p -group P has a series

$$P > P^{p^{k_1}} > P^{p^{k_2}} > \dots > 1 \quad (5)$$

with uniformly powerful factors of strictly decreasing ranks. For every factor S of this series having exponent, say, p^t , its subgroup $V = Sp^{[(t+1)/2]}$ is abelian. By Lemma 4.1 the subgroup V has a characteristic subgroup U of (p, k, m) -bounded order such that the rank of V/U is at most $r|H|$. Therefore the rank of S can be higher than $r|H|$ only if the exponent of S is (p, k, m) -bounded. Since the rank of P is at most mp^k , all the factors in (5) of rank higher than $r|H|$ combine in a quotient $P/P^{p^{k_u}}$ of (p, k, m) -bounded order; then $P^{p^{k_u}}$ is the required characteristic subgroup of (p, k, m) -bounded index and of rank at most $r|H|$.

Let p^v be the exponent of P . Since in the powerful group P the series $P > P^p \geq P^{p^2} \geq P^{p^3} \geq \dots$ is central, the subgroup $P^{p^{[(v+1)/2]}}$ is abelian. By Lemma 4.1 the exponent of $P^{p^{[(v+1)/2]}}$ is at most p^{e+f} for some (p, k, m) -bounded number f . Hence the exponent of P is at most p^{2e+g} for some (p, k, m) -bounded number $g = g(p, k, m)$. Since the rank of P is at most mp^k , the characteristic subgroup P^{p^g} has (p, k, m) -bounded index and exponent at most p^{2e} . \square

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