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Abstract

We give a complete description of the Morita equivalence classes of blocks with elementary abelian defect groups of order 8 and of the derived equivalences between them. A consequence is the verification of Broué’s abelian defect group conjecture for these blocks. It also completes the classification of Morita and derived equivalence classes of 2-blocks of defect at most three defined over a suitable field.

1 Introduction

Throughout let $k$ be an algebraically closed field of prime characteristic $\ell$ and let $\mathcal{O}$ be a discrete valuation ring with residue field $k$ and field of fractions $K$ of characteristic zero. We assume that $K$ is large enough for the groups under consideration. We consider blocks $B$ of $\mathcal{O}G$ with defect group $D$.

We are concerned with the description of the Morita and derived equivalence classes of (module categories for) blocks of finite groups with a given defect group $D$. We briefly review progress on this problem to date. If $D$ is an abelian $p$-group whose automorphism group is a $p$-group, then any block with defect group $D$ must be nilpotent and so Morita equivalent to $\mathcal{O}D$ (see [14] and [22]). There are many other examples of $p$-groups for which it has been proved that every fusion system is nilpotent, but we do not list these here. If $D$ is cyclic, then the Morita equivalence classes can be characterised in terms of Brauer trees, in work going back to Brauer and Dade (see [20]). In a series of papers Erdmann characterises the Morita equivalence classes of tame blocks defined over $k$ except when $D$ is generalised quaternion and $B$ has two simple modules (see [8]), although the problem remains open for blocks defined over $\mathcal{O}$. The (three) Morita equivalence classes of blocks defined over $\mathcal{O}$ with defect group $C_2 \times C_2$ are determined in [19]. When $D = \langle x, y : x^{2^r} = y^{2^s} = [x, y]^2 = [x, [x, y]] = [y, [x, y]] = 1 \rangle$, where $r \geq s \geq 1$ (nonmetacyclic minimal nonabelian 2-group), the Morita equivalence classes of blocks are determined in [24] and [7]. When $D$ is a homocyclic 2-group, the Morita equivalence classes of blocks are determined in [6].

In this paper we use the classification given in [6] to completely determine the Morita and derived equivalence classes of blocks defined over $\mathcal{O}$ with defect group $D \cong C_2 \times C_2 \times C_2$. As a consequence it follows that Broué’s abelian defect group conjecture holds for blocks of defect three. We also note that this completes the classification of Morita equivalence classes of 2-blocks of defect at most three, for
blocks defined over \( k \). Blocks with elementary abelian defect groups of order 8 have already been studied in [10], where it is shown that Alperin’s weight conjecture and the isotypy version of Broué’s abelian defect group conjecture hold for these blocks. The results of [10] are needed here, in particular to achieve Morita equivalences over \( \mathcal{O} \) rather than \( k \).

Before stating the main theorem, we recall the definition of the inertial quotient of \( B \). Let \( b_D \) be a block of \( \mathcal{O}DC_G(D) \) with Brauer correspondent \( B \), and write \( N_G(D, b_D) \) for the stabilizer in \( N_G(D) \) of \( b_D \) under conjugation. Then the inertial quotient of \( B \) is \( E = N_G(D, b_D)/DC_G(D) \), an \( \ell' \)-group unique up to isomorphism.

**Theorem 1.1** Let \( B \) be a block of \( \mathcal{O}G \), where \( G \) is a finite group. If \( B \) has defect group \( D \isomorphic  C_2 \times C_2 \times C_2 \), then \( B \) is Morita equivalent to the principal block of precisely one of the following:

- (i) \( D \);
- (ii) \( D \times C_3 \);
- (iii) \( C_2 \times A_5 \), and the inertial quotient is \( C_3 \);
- (iv) \( D \times C_7 \);
- (v) \( SL_2(8) \), and the inertial quotient is \( C_7 \);
- (vi) \( D \times (C_7 \times C_3) \);
- (vii) \( J_1 \), and the inertial quotient is \( C_7 \times C_3 \);
- (viii) \( 2G_2(3) \isomorphic \text{Aut}(SL_2(8)) \), and the inertial quotient is \( C_7 \times C_3 \);

Blocks are derived equivalent if and only if they have the same inertial quotient.

A block with defect group \( C_2 \times C_2 \times C_2 \) cannot be Morita equivalent to a block with non-isomorphic defect group. This is since Morita equivalence preserves defect and (i) 2-blocks of defect three with abelian defect groups other than \( C_2 \times C_2 \times C_2 \) must be nilpotent (and so Morita equivalent to the group algebra of a defect group), (ii) 2-blocks of defect three with nonabelian defect groups have five irreducible characters (whilst the number is eight for blocks with defect group \( C_2 \times C_2 \times C_2 \)).

**Corollary 1.2** Broué’s abelian defect group conjecture holds for all 2-blocks with defect at most three. That is, let \( B \) be a block of \( \mathcal{O}G \) for a finite group \( G \) with defect group \( D \) of order dividing 8, and let \( b \) be the unique block of \( \mathcal{O}N_G(D) \) with Brauer correspondent \( B \). Then \( B \) and \( b \) have derived equivalent module categories.

**Proof.** If a defect group \( D \) are isomorphic to \( C_2 \), \( C_4 \), \( C_4 \times C_2 \) or \( C_8 \), then the block is nilpotent, in which case the conjecture holds automatically since \( \text{Aut}(D) \) is a 2-group. If \( D \isomorphic C_2 \times C_2 \), then the result follows from [19]. Suppose that \( D \isomorphic C_2 \times C_2 \times C_2 \). By Theorem 1.1 the derived equivalence class of \( B \) is uniquely determined by the number \( l(B) \) of irreducible Brauer characters. Since every block with defect group \( D \) has eight irreducible characters, it is a consequence of Brauer’s second main theorem that \( l(B) = l(b) \) and the result follows. \( \square \)

Note that we do not prove that there are splendid derived equivalences of blocks.

**Corollary 1.3** Let \( B \) be a block with defect group \( D \isomorphic C_2 \times C_2 \times C_2 \). Then \( B \) has Loewy length \( \text{LL}(B) \) equal to 4, 6 or 7.
Proof. By Theorem 1.1 it suffices to consider cases (i)-(viii) in the notation of that theorem. In cases (i), (ii), (iv) and (vi), where $D \triangleleft G$ and $[G : D]$ is odd, we have that $LL(B) = LL(kD) = 4$, by [12, 4.1]. In case (iii) $LL(B) = 6$. In the remaining cases $LL(B) = 7$ by [1] and [18], again using [12, 4.1].

Corollary 1.4 Let $B$ be a 2-block of defect at most 3, then the Cartan invariants of $B$ are at most the order of a defect group.

Of course the above does not hold in generality.

Since we now have a complete list of Cartan matrices (up to ordering of the simple modules), and indeed the decomposition matrices, for 2-blocks of defect at most 3, it would be interesting to look for possible concrete restrictions on Cartan matrices.

2 Quoted results

The following proposition will be used when considering automorphism groups of simple groups. It gathers together two propositions from [10], which in turn gathers results from [5] and [15].

**Proposition 2.1** Let $\ell$ be any prime and let $G$ be a finite group and $N \triangleleft G$ with $[G : N] = w$ a prime not equal to $\ell$. Let $b$ be a $G$-stable $\ell$-block of $O_N$. Then either each block of $OG$ covering $b$ is Morita equivalent to $b$, or there is a unique block of $OG$ covering $b$. In the former case, $B$ and $b$ have isomorphic inertial quotient.

**Proof.** Note that the group $G[b]$ of elements of $G$ acting as inner automorphisms on $b$ is a normal subgroup of $G$ containing $N$. If $G[b] = G$, then each block of $G$ covering $b$ is source algebra equivalent to $b$ by [10, 2.2], and has inertial quotient isomorphic to that of $b$ by [10, 3.4]. If $G[b] = N$, then there is a unique block of $G$ covering $b$ by [10, 2.3].

The following is a distillation of those results in [16] which are relevant here.

**Proposition 2.2 ([16])** Let $G$ be a finite group and $N \triangleleft G$. Let $B$ be a block of $OG$ with defect group $D$ covering a $G$-stable nilpotent block $b$ of $O_N$ with defect group $D \cap N$. Then there is a finite group $L$ and $M \triangleleft L$ such that (i) $M \cong D \cap N$, (ii) $L/M \cong G/N$, (iii) there is a subgroup $D_L$ of $L$ with $D_L \cong D$ and $D_L \cap M \cong D \cap N$, and (iv) there is a a central extension $\tilde{L}$ of $L$ by an $\ell'$-group, and a block $\tilde{B}$ of $O\tilde{L}$ which is Morita equivalent to $B$ and has defect group $\tilde{D} \cong D$.

**Proposition 2.3 ([26])** Let $B$ be an $\ell$-block of $OG$ for a finite group $G$ and let $Z \leq O_\ell(Z(G))$. Let $\bar{B}$ be the unique block of $O(G/Z)$ corresponding to $B$. Then $B$ is nilpotent if and only if $\bar{B}$ is nilpotent.

**Proposition 2.4 ([6])** Let $B$ be a block of $OG$ for a quasisimple group $G$ with elementary abelian defect group $D$ of order 8. Then one of the following occurs:

(i) $G \cong SL_2(8)$ and $B$ is the principal block;
Let $G \cong G_2(q)$, where $q = 3^{2m+1}$ for some $m \in \mathbb{N}$, and $B$ is the principal block;

(ii) $G \cong J_1$ and $B$ is the principal block;

(iii) $G \cong C_2$ and $B$ is the unique non-principal 2-block of defect 3;

(iv) $G$ is of type $D_n(q)$ or $E_7(q)$ for some $q$ of odd prime power order, $O_2(G) = 1$ and $B$ is Morita equivalent to the principal block of $C_2 \times A_5$ or of $C_2 \times A_4$.

(v) $|O_2(G)| = 2$ and $D/O_2(G)$ is a Klein four group;

(vi) $B$ is nilpotent.

Lemma 2.5 Let $B$ be a block of $\mathcal{O}G$ for a finite group $G$ with normal defect group $D \cong C_2 \times C_2 \times C_2$. Then $B$ is Morita equivalent to $\mathcal{O}(D \times E)$, where $E$ has odd order and acts faithfully on $D$.

Proof. This is well known, but may be obtained for instance by applying Proposition 2.2 and noting that the inertial quotient is one of 1, $C_3$, $C_7$ and $C_7 \times C_3$, each having trivial Schur multiplier. \hfill $\Box$

3 Preliminary results

Proposition 3.1 Let $N = G_2(q)$, where $q = 3^{2m+1}$ for some $m \in \mathbb{N} \cup \{0\}$, and $N \leq G \leq \text{Aut}(N)$. Let $b$ be the principal 2-block of $\mathcal{O}N$. Then every block of $\mathcal{O}G$ covering $b$ is source algebra equivalent to $b$. Further, each of these blocks shares a defect group with $b$ and has isomorphic inertial quotient.

Proof. $G/N$ is cyclic of odd order. Let $N = G_0 \leq G_1 \leq \cdots \leq G_n = G$, with each $|G_{i+1}/G_i|$ prime. By [25] $b$ has defect groups of the form $C_2 \times C_2 \times C_2$ and irreducible character degrees occurring with multiplicity either one or two, so that each irreducible character is $G$-stable. Since $[G : N]$ is odd each block of $\mathcal{O}G_i$ covering $b$ shares a defect group with $b$. By [10], every block with defect group $C_2 \times C_2 \times C_2$ (in particular $b$ and every block of $\mathcal{O}G_i$ covering it) has precisely eight irreducible characters, and it follows that for each $i$ there are $[G_{i+1} : N]$ 2-blocks of $\mathcal{O}G_{i+1}$ covering $b$, and amongst these there $[G_{i+1} : G_i]$ blocks of $\mathcal{O}G_{i+1}$ covering each such block of $\mathcal{O}G_i$. It follows from Proposition 2.1 that each block of $\mathcal{O}G_i$ covering $b$ is source algebra equivalent to $b$. That the blocks have isomorphic inertial quotient follows from [10, 3.4]. \hfill $\Box$

Proposition 3.2 Let $G$ be a finite group and $N \triangleleft G$ with $[G : N]$ an odd prime. Let $b$ be a $G$-stable block of $\mathcal{O}N$ with defect group $C_2 \times C_2 \times C_2$ and inertial quotient $C_3$. Suppose that $l(b) = 3$. Let $B$ be a block of $\mathcal{O}G$ covering $b$. Then either $B$ is source algebra equivalent to $b$ or nilpotent. In the former case $B$ has inertial quotient $C_3$ and $[G : N] = 3$.

Proof. By [10] we have $l(B) \leq 7$. Suppose first that $[G : N] \geq 5$. Since we are assuming that $l(b) = 3$, there cannot be a unique block of $\mathcal{O}G$ covering $b$ (since each irreducible Brauer character of $b$ is $G$-stable and so the total number of irreducible Brauer characters in blocks covering $B$ is at least 15), so by Proposition 2.1 $B$ is source algebra equivalent to $b$ and has the same inertial quotient.
Suppose now that $[G : N] = 3$. If every irreducible Brauer character of $b$ is $G$-stable, in which case again by Proposition 2.1 $B$ is source algebra equivalent to $b$ and has the same inertial quotient. If the three irreducible Brauer characters are permuted transitively, then $l(B) = 1$, so that by [10] $B$ is nilpotent. \hfill \Box

The following is a strengthening of a special case of the main result of [11], which is only known to hold for blocks defined over $k$.

**Proposition 3.3** Let $G$ be a finite group and $N < G$ and let $C$ be a $G$-stable block of $\mathcal{O}N$ covered by a block $B$ of $\mathcal{O}G$ with elementary abelian defect group $D$ of order 8. Write $P = N \cap D$ and suppose that $D = P \times Q$ for some $Q$ of order 2 such that $G = N \rtimes Q$. Then $B \cong C \otimes \mathcal{O}Q$. In particular $B$ and the block $C \otimes \mathcal{O}Q$ of $\mathcal{O}(N \times Q)$ are Morita equivalent.

**Proof.** Write $Q = \langle x \rangle$. As noted in [11] $B$ and $C$ share a block idempotent $e$, so that $B$ is a crossed product of $C$ with $Q$ and it suffices to find a graded unit of $Z(B)$ of degree $x$ and order two. We do this by exploiting the existence of a perfect isometry as shown in [10, 5.1], although we must show that this perfect isometry satisfies additional properties. Part of the proof follows that of [10, 5.1], and we take facts from there without explicit further reference. Note however that for convenience we use a different labeling of the irreducible characters.

Denote by $E$ the inertial quotient of $B$, so that $|E| = 1$ or 3. If $|E| = 1$, then $B$ is nilpotent and the result follows from [22]. Hence we may assume that $|E| = 3$ and $E$ acts faithfully on $D$. Write $H = D \rtimes E$. Then $Q \leq Z(H)$ and so $H = (P \rtimes E) \times Q$.

By [17] we have $k(B) = 8$. Label the irreducible characters $\theta_i$ of $H$ so that $\theta_1, \ldots, \theta_4$ have $Q$ in their kernel, $\theta_1(1) = \theta_2(1) = \theta_3(1) = 1$, $\theta_4(1) = 3$ and $\theta_i(g) = \chi_i(g)$ for all $i = 5, \ldots, 8$ and all $g \in P \times E$. We have $\theta_1(x) = -\theta_1(1)$ for $i = 5, \ldots, 8$. Similarly label the irreducible characters $\chi_1, \ldots, \chi_8$ of $B$ so that $\text{Res}_N^G(\chi_i) = \text{Res}_N^G(\chi_{i-4})$ for all $i = 5, \ldots, 8$. Note that $\chi_i(x) = -\chi_{i-4}(x)$ for all $i = 5, \ldots, 8$.

There is a stable equivalence of Morita type between $\mathcal{O}H$ and $B$, leading to an isometry $L^0(H, \mathcal{O}H) \cong L^0(G, B)$ between the groups of generalised characters vanishing on 2-regular elements. $L^0(H, \mathcal{O}H)$ is generated by

$$\{\theta_1 - \theta_5, \theta_2 - \theta_6, \theta_3 - \theta_7, \theta_4 - \theta_8, \theta_1 + \theta_2 + \theta_3 - \theta_4\}.$$

We claim that if $\chi_i - \chi_j \in L^0(G, B)$, then $|i - j| = 4$. For suppose that $\chi_i(g) = \chi_j(g)$ for all $g \in G$ of odd order. Then $\text{Res}_N^G(\chi_i)$ and $\text{Res}_N^G(\chi_2)$ are irreducible characters of $C$ agreeing on 2-regular elements. Noting that $C$ is not nilpotent, and that $C$ has decomposition matrix that of the principal 2-block of $A_4$ or $A_5$, it follows that $\text{Res}_N^G(\chi_i) = \text{Res}_N^G(\chi_2)$ and the claim follows.

Hence the isometry takes elements of the form $\theta_i - \theta_{i-4}$ to elements of the form $\delta_j(\chi_j - \chi_{j-4})$. Now the isometry extends to a perfect isometry $I : Z\text{Irr}(H) \to Z\text{Irr}(B)$, and we have seen that $I(\theta_i)(g) = I(\theta_{i-4})(g)$ for every $i = 5, \ldots, 8$ and every $g \in N$.

Following [3] $I$ induces an $\mathcal{O}$-algebra isomorphism $I^0 : \mathcal{O}(\mathcal{O}H) \to Z(B)$ with $I^0(x) = \frac{1}{|H|} \sum_{g \in G} \mu(g^{-1}, x)g$, where $\mu(g, h) = \sum_{i=1}^8 \theta_i(h)I(\theta_i)(g)$ for $g \in G$ and $h \in H$.

We will show that $I^0(x) = ax$ for some $a \in \mathcal{O}N$, i.e., that $\mu(g, x) = 0$ whenever $g \in N$. Then $I^0(x)$ will be the required graded unit of $Z(B)$ of degree $x$ and order two.
Let \( g \in N \). Then
\[
\mu(g, x) = \sum_{i=1}^{8} \theta_i(x)I(\theta_i)(g) = \sum_{i=5}^{8} \theta_{i-4}(1) (I(\theta_{i-4})(g) - I(\theta_i)(g)) = 0
\]
and we are done. \( \square \)

4 Proof of the main theorem

We prove Theorem 1.1.

**Proof.** Let \( B \) be a block of \( OG \) for a finite group \( G \) with defect group \( D \cong C_2 \times C_2 \times C_2 \) with \([G : Z(G)]\) minimised such that \( B \) is not Morita equivalent to any of (i)-(viii). By minimality and the first Fong reduction \( B \) is quasiprimitive, that is, for every \( N \triangleleft G \) each block of \( ON \) covered by \( B \) is \( G \)-stable. By Proposition 2.2 if \( N \triangleleft G \) and \( B \) covers a nilpotent block of \( ON \), then \( N \leq Z(G)O_2(G) \). In particular \( O_2^*(G) \leq Z(G) \)

Following [2] write \( E(G) \) for the layer of \( G \), that is, the central product of the subnormal quasisimple subgroups of \( G \) (the components). Write \( F(G) \) for the Fitting subgroup, which in our case is \( F(G) = Z(G)O_2(G) \). Write \( F^*(G) = F(G)E(G) \triangleleft G \), the generalised Fitting subgroup, and note that \( C_G(F^*(G)) \leq F^*(G) \). Let \( b \) be the (unique) block of \( OF^*(G) \) covered by \( B \).

Let \( \overline{B} \) be the unique block of \( O(G/O_2(Z(G))) \) corresponding to \( B \). First observe that \(|O_2(Z(G))| \leq 2\), for otherwise \( \overline{B} \) would have defect at most one and so would be nilpotent, which in turn would mean that \( B \) would be nilpotent by Proposition 2.3, a contradiction.

If \(|O_2(G)| > 4\), then \( O_2(G) = D \), a contradiction by Lemma 2.5. Hence \(|O_2(G)| \leq 4\).

Claim. \( O_2(G) \leq Z(G) \) and \(|O_2(G)| \leq 2\).

Suppose that \( O_2(G) \not\leq Z(G) \) (so \(|O_2(G)| = 4\)). If \( O_2(Z(G)) \neq 1 \), then \( O_2(G/O_2(Z(G))) \) has order 2 and so is central in \( G/O_2(Z(G)) \), from which it follows using Proposition 2.3 that \( \overline{B} \), and so \( B \), is nilpotent, again a contradiction. If \( O_2(Z(G)) = 1 \), then \( F^*(G) = O_2(G) \times (Z(G)E(G)) \). Since \(|O_2(G)| = 4\), \( B \) covers a nilpotent block of \( F^*(G) \) and so \( F^*(G) = O_2(G)Z(G) \). But \( C_G(F^*(G)) \leq F^*(G) \) and so \( D \leq C_G(O_2(G)) \leq O_2(G)Z(G) \), a contradiction. Hence \( O_2(G) \leq Z(G) \) (and \(|O_2(G)| \leq 2\) as claimed.

Write \( E(G) = L_1 \cdots L_t \), where each \( L_i \) is a component of \( G \) (arguing as above we have that \( t \geq 1 \)). Now \( B \) covers a block \( b_E \) of \( OE(G) \) with defect group contained in \( D \), and \( b_E \) covers a block \( b_i \) of \( OL_i \). Since \( b_E \) is \( G \)-stable, for each \( i \) either \( L_i \triangleleft G \) or \( L_i \) is in a \( G \)-orbit in which each corresponding \( b_i \) is isomorphic (with equal defect).

Since \( B \) has defect three, it follows that if \( t > 1 \), then \( B \) covers a nilpotent block of a normal subgroup generated by components of \( G \), a contradiction. Hence \( t = 1 \). So \( G \) has a unique component \( L_1 \), and \( G/Z(G) \leq \text{Aut}(L_1Z(G)/Z(G)) \).

Suppose that \( O_2(G) \not\leq [L_1, L_1] \). Then \( F^*(G) = O_2(G) \times Z(G)L_1 \). In this case \( D \leq F^*(G) \), since otherwise \( b \) would be nilpotent. Since \( b \) is \( G \)-stable, this means \([G : F^*(G)] \) odd and so \( O_2(G) \) is in fact a direct factor of \( G \). By [19] it follows that \( B \) is Morita equivalent to one of (ii) or (iii), a contradiction. Hence \( O_2(G) \leq [L_1, L_1] \).
We next show that \( D \leq F^*(G) \). Suppose otherwise. Then since \( D \) is elementary abelian we may write \( D = (D \cap F^*(G)) \times Q \) for some \( Q \) of order 2 (if \( Q \) were to be larger, then \( B \) would cover a nilpotent block of \( OF^*(G) \)). By the Schreier conjecture \( G/F^*(G) \) is solvable. Since \( b \) is \( G \)-stable, \( DF^*(G)/F^*(G) \) is a Sylow 2-subgroup of \( G/F^*(G) \). Hence \( G = H \times Q \) for some \( H \trianglelefteq G \). By Proposition 3.3 \( B \cong b \otimes \mathcal{O}Q \) as \( \mathcal{O} \)-algebras. Now \( b \otimes \mathcal{O}Q \) is a block of \( \mathcal{O}(H \times Q) \) with defect group \( D = (D \cap H) \times Q \).

Since \( b \) is Morita equivalent to the principal block of \( G \) block. Then \( \text{Morita} \equiv \) of the theorem.

By Proposition 3.2 \( G/F \) larger, then \( B \) and we are done in this case.

Since \( b \) is Morita equivalent to one of (ii) or (iii). Hence \( b \)-algebras. Now \( \text{Morita} \equiv \) of decomposition matrices.

To see that the blocks in cases (i)-(viii) represent distinct Morita equivalence classes it suffices to note that they have distinct decomposition matrices.

\[ \square \]

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