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# Completions of the Goldschmidt $G_3$-amalgam and Alternating Groups

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**Abstract**

Here we show that the alternating group of degree $n$ is a completion of the Goldschmidt $G_3$-amalgam if and only if $n \notin \{1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 16, 17, 19, 23\}$

## 1 Introduction

In [3] Goldschmidt classified the primitive rank 2 amalgams of index $(3, 3)$. These amalgams are now referred to as Goldschmidt amalgams. We recall that a rank 2 amalgam consists of groups $P_1, P_2$ and $B$ together with group monomorphisms $\phi_1, \phi_2$ such that

$$\phi_1 : B \longrightarrow P_1 \text{ and } \phi_2 : B \longrightarrow P_2.$$

Suppressing the $\phi_i$, we refer to this amalgam as $\mathcal{A}(P_1, P_2, B)$. To say $\mathcal{A}(P_1, P_2, B)$ is primitive means that if $K \leqslant B$ and $\phi_i(K) \trianglelefteq P_i$ for $i = 1, 2$, then $K = 1$. If $P_1$ and $P_2$ are finite groups, then $\mathcal{A}(P_1, P_2, B)$ is a finite amalgam whose index is $([P_1 : \phi_1(B)], [P_2 : \phi_2(B)])$. A group $G$ is said to be a completion of $\mathcal{A}(P_1, P_2, B)$ if there exist group homomorphisms $\psi_i : P_i \longrightarrow G$ $i = 1, 2$, for which $\psi_1\phi_1 = \psi_2\phi_2 : B \longrightarrow G$ and $G = \langle im\psi_1, im\psi_2 \rangle$. Should the $\psi_i, i = 1, 2$, be monomorphisms we then say this is a faithful completion. Here we shall only consider faithful completions, and so from now on will identify $P_i$ with $\psi_i(P_i)$ and $B$ with $\psi_i\phi_i(B)$. Hence we have $P_i \leqslant G, i = 1, 2$ with $P_1 \cap P_2 \geqslant B$ and $\langle P_1, P_2 \rangle = G$ (though in all the completions we deal with we have $P_1 \cap P_2 = B$). The amalgams arrayed in [3] consist of fifteen amalgams, among which is the Goldschmidt $G_3$-amalgam. This amalgam has $P_1 \cong Sym(4) \cong P_2$ and $B \cong Dih(8)$. By $Sym(n)$ we mean the symmetric group of degree $n$ and $Dih(8)$ the dihedral group of order 8. We also use $Alt(n)$ to denote the alternating group of degree $n$. It is completions of the $G_3$-amalgam we study here. Putting $\mathcal{E} = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 16, 17, 19, 23\}$ we state our main result.

**Theorem 1.1.** *For $n \in \mathbb{N}$, $Alt(n)$ is a completion of the Goldschmidt $G_3$-amalgam if and only if $n \notin \mathcal{E}$.*

Earlier work on the Goldschmidt $G_3$-amalgam for alternating groups [7] verified that for $n \leqslant 24$, $Alt(n)$ is a completion precisely when $n \in \{6, 10, 13, 14, 15, 18, 20, 21, 22, 24\}$. In this paper we shall demonstrate that $Alt(n)$ is a completion for all $n \geqslant 25$ from which Theorem 1.1 will then follow. Conder, in [2], investigated whether $G = Sym(n)$ is the automorphism group of some finite connected 5-arc transitive graph. Using the work of Lorimer [4], Conder rephrases his problem to ask whether there exists $H \cong Sym(4) \times \mathbb{Z}_2 \leqslant G$ and $a \in G$ such that

(i) $a^2 \in H$;

(ii) $G = \langle HaH \rangle$ and;

(iii) $[H : H \cap H^a] = 3$.

If such a subgroup $H$ and element $a$ can be found, then $Sym(n)$ is a completion of the Goldschmidt $G_3^1$-amalgam (with $P_1 = H, P_2 = H^a$ and $B = H \cap H^a$) which then implies that $Alt(n)$ is a completion of the Goldschmidt $G_3$-amalgam (see Lemma 2.2). Now Conder's method is to assemble various transitive representations for $Sym(4) \times \mathbb{Z}_2$ on $2, 3, 6, 12, 24$ and 48 points and then take appropriate combinations of these upon which a permutation that will fulfil the role of $a$ is defined. This is achieved by all $n$ of the form $n = 84b + 176c + 177d + 87$, where $b, c, d \in \mathbb{N}$. Since the Frobenius number of $\{84, 176, 177\}$- the largest integer such that it can't be formed as $84b + 176c + 177d$ for $b, c, d \geqslant 0$ - is 2743, it means that, taking into account the additional 87, for all $n \geqslant 2831$, $Sym(n)$ and $Alt(n)$ are respective completions of the Goldschmidt $G_3^1$- and $G_3$- amalgams.
We briefly survey what else is known about completions of the Goldschmidt $G_3$-amalgam. Which sporadic simple groups (with the sole exception of the Monster) are completions is settled in [6] and [7]. While in [5] for the classical groups in 3 dimensions we have that $SL_3(q)$ and $PSL_3(q)$ are completions of the Goldschmidt $G_3$-amalgam if and only if $q \notin \{4, 9\}$. For the unitary and orthogonal groups, $SU_3(q)$ and $PSU_3(q)$ are completions if and only if $q$ is odd and $q \notin \{3, 5\}$ and $SO_3(q)(\cong PSL_2(q))$ is a completion if and only if $q$ or $\sqrt{q}$ is a prime and $q \equiv \pm 1 (\text{mod } 8)$.
This paper is arranged as follows– Section 2 begins with two elementary results concerning completions of the Goldschmidt $G_3$-amalgam. Then we introduce a particular type of graph, the orbit graph $\mathcal{O}(P_1, P_2, \Omega)$. Here $P_1$ and $P_2$ are subgroups of $Sym(\Omega)$, the group of permutations on $\Omega$. This graph, courtesy of Lemma 2.5, can be used to determine whether $\langle P_1, P_2 \rangle$ is transitive on $\Omega$ or not. Definition 2.7 introduces the idea of twisting $P_1$ with respect to $(\Delta, \Psi)$ where $\Delta$ and $\Psi$ are certain $(P_1 \cap P_2)$-orbits of disjoint sets $\Sigma$ and $\Gamma$. This idea, via Lemma 2.8 plays a central role in the recursive construction, presented in Section 3, and which underlies the proof of Theorem 1.1. We end Section 2 with Theorem 2.9, a classical result of Jordan's.
In Section 3 we give a detailed account of a recursive construction for the case when $n \equiv 0 (\text{mod } 24)$. For the remaining congruences we list, in Section 4, the seed permutations which enable this construction to produce an example of $Alt(n)$ as a completion of the Goldschmidt $G_3$-amalgam. Further details on these seed permutations may be found in Sections 4 and 5.

## 2 Some preliminary results

Our first result concerns involutions and completions of the Goldschmidt $G_3$-amalgam

**Lemma 2.1.** *Suppose the group $G$ is a completion of the Goldschmidt $G_3$-amalgam $\mathcal{A}(P_1, P_2, B)$. Then the involutions in $P_1 \cup P_2$ are $G$-conjugate.*

*Proof.* As noted earlier, we shall assume $P_i \leqslant G, i = 1, 2$ with $P_1 \cap P_2 = B \cong Dih(8)$ and $P_i \cong Sym(4)$. Now, for $i = 1, 2$ $P_i$ has two conjugacy classes of involutions $C_i^{(1)}$ and $C_i^{(2)}$ with $B \cap C_i^{(1)} = O_2(P_i)^\sharp$. Since $O_2(P_1) \neq O_2(P_2)$ and $O_2(P_1) \cap O_2(P_2) \neq 1$, Lemma 2.1 follows. □

Next we see that a completion of the Goldschmidt $G_3^1$-amalgam, $\mathcal{A}(Sym(4) \times \mathbb{Z}_2, Sym(4) \times \mathbb{Z}_2, Dih(8) \times \mathbb{Z}_2)$, for $Sym(n)$ yields a completion of the Goldschmidt $G_3$-amalgam for $Alt(n)$.

**Lemma 2.2.** *If $Sym(n)$ is a completion of the Goldschmidt $G_3^1$-amalgam, then $Alt(n)$ is a completion of the Goldschmidt $G_3$-amalgam.*

*Proof.* Set $G = Sym(n)$ and $H = Alt(n)$. Let $P_1, P_2 \leqslant G$ be such that $P_i \cong Sym(4) \times \mathbb{Z}_2$ and $B = P_1 \cap P_2 \cong Dih(8) \times \mathbb{Z}_2$ with $\langle P_1, P_2 \rangle = G$. Also put $R_i = O^2(P_i) \cong Alt(4)$ and $Q_i = O_2(R_i) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Observe that $[B : B \cap H] = 2$ for otherwise $B \leqslant H$ and then $\langle P_1, P_2 \rangle \leqslant H \neq G$, a contradiction. Further we must have $n \geqslant 5$. Note that $R_i \leqslant H, i = 1, 2$ and that $Q_1$ and $Q_2$ normalize each other. Since $\mathcal{A}(P_1, P_2, B)$ is primitive, $Q_1 \neq Q_2$. From $Q_i \leqslant B \cap H, i = 1, 2$, we then see that $Q_1Q_2 = B \cap H$. So $Q_1Q_2$ has order 8 and either $Q_1Q_2$ is abelian or $Q_1Q_2 \cong Dih(8)$. If the former holds, then $Q_1Q_2 = C_{P_i}(Q_i) = O_2(P_i)$ for $i = 1, 2$, giving the impossible $O_2(P_1) = O_2(P_2)$. Therefore, $B \cap H = Q_1Q_2 \cong Dih(8)$. Since $[P_i : P_i \cap H] = 2$ for $i = 1, 2$, we have $P_i \cap H \cong Alt(4) \times \mathbb{Z}_2$ or $Sym(4)$. The former possibility doesn't contain a $Dih(8)$ subgroup and therefore $P_i \cap H \cong Sym(4)$. To complete the proof we must show that $H = \langle P_1 \cap H, P_2 \cap H \rangle$. Now $B$ normalizes $P_i \cap H, i = 1, 2$ and so

$$\langle P_1 \cap H, P_2 \cap H \rangle \trianglelefteq \langle P_1 \cap H, P_2 \cap H, B \rangle = G.$$

Because $n \geqslant 5$ $Alt(n)$ is simple, whence $H = \langle P_1 \cap H, P_2 \cap H \rangle$. □

In proving Theorem 1.1, for specified subgroups $P_1$ and $P_2$ of $G = Alt(n)$ we need to show that $\langle P_1, P_2 \rangle = G$. Not surprisingly, our first step is to prove that $\langle P_1, P_2 \rangle$ acts transitively on $\Omega = \{1, ..., n\}$. This leads us to consider a particular type of graph which we now define.

**Definition 2.3.** Suppose $G$ is a finite group acting on a finite set $\Omega$ with $P_1$ and $P_2$ being subgroups of $G$. Let $\{\Delta_1^{(1)}, ..., \Delta_\ell^{(1)}\}$ and $\{\Delta_1^{(2)}, ..., \Delta_k^{(2)}\}$ be, respectively, the $P_1$- and $P_2$-orbits on $\Omega$ (and we shall regard these sets as being disjoint). The orbit graph of $P_1$ and $P_2$, $\mathcal{O}(P_1, P_2, \Omega)$, has vertex set $\{\Delta_1^{(1)}, ..., \Delta_\ell^{(1)}, \Delta_1^{(2)}, ..., \Delta_k^{(2)}\}$ with distinct vertices $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ adjacent whenever their intersection contains a $(P_1 \cap P_2)$-orbit and the number of edges between them is the number of $(P_1 \cap P_2)$-orbits in $\Delta_i^{(1)} \cap \Delta_j^{(2)}$.

We observe that $\mathcal{O}(P_1, P_2, \Omega)$ is a bipartite graph and that $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ being adjacent is equivalent to $\Delta_i^{(1)} \cap \Delta_j^{(2)} \neq \varnothing$.

**Example 2.4.** *(i)* $\Omega = \{1, ..., 21\}, P_1 = \langle x, y, z \rangle, P_2 = \langle x, y, w \rangle$ *and* $B = P_1 \cap P_2 = \langle x, y \rangle$ *where*
$x := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)$;
$y := (1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(13, 17)(14, 18)(19, 20)$;
$z := (1, 11)(2, 12)(5, 9)(6, 10)(7, 8)(13, 15)(14, 16)(20, 21)$*; and*
$w := (1, 17)(4, 18)(5, 14)(6, 7)(8, 13)(10, 19)(11, 20)(16, 21)$.
*Here, the* $B$*-orbits on* $\Omega$ *are* $\Omega_1 := \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\Omega_2 := \{9, 10, 11, 12\}$, $\Omega_3 := \{13, 14, 17, 18\}$, $\Omega_4 := \{15, 16\}$, $\Omega_5 := \{19, 20\}$ *and* $\Omega_6 := \{21\}$*. The* $P_1$*-orbits are* $\Delta_1^{(1)} := \Omega_1 \cup \Omega_2$, $\Delta_2^{(1)} := \Omega_3 \cup \Omega_4$, $\Delta_3^{(1)} := \Omega_5 \cup \Omega_6$ *and* $P_2$*-orbits are* $\Delta_1^{(2)} := \Omega_1 \cup \Omega_3$, $\Delta_2^{(2)} := \Omega_2 \cup \Omega_5$, $\Delta_3^{(2)} := \Omega_4 \cup \Omega_6$*. So the orbit graph* $\mathcal{O}(P_1, P_2, \Omega)$ *is*

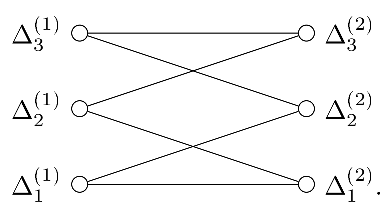


Figure 1

*(ii)* $\Omega = \{1, ..., 24\}, P_1 = \langle x, y, z \rangle, P_2 = \langle x, y, w \rangle$ *and* $B = P_1 \cap P_2 = \langle x, y \rangle$ *where*
$x := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)$;
$y := (1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(17, 24)(18, 19)(20, 21)(22, 23)$;
$z := (1, 9)(2, 10)(3, 20)(4, 19)(5, 14)(6, 13)(7, 23)(8, 24)(11, 21)(12, 22)(15, 18)(16, 17)$; *and*
$w := (1, 14)(2, 19)(3, 18)(4, 10)(5, 11)(6, 22)(7, 23)(8, 15)(9, 17)(12, 21)(13, 20)(16, 24)$.
*In this case, the* $B$*-orbits on* $\Omega$ *are* $\{1, ..., 8\}$, $\{9, ..., 16\}$ *and* $\{17, ..., 24\}$ *with both* $P_1$ *and* $P_2$ *acting transitively on* $\Omega$*. Thus* $\mathcal{O}(P_1, P_2)$ *is*

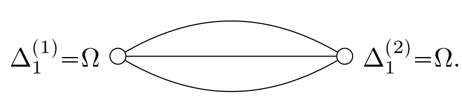


Figure 2

Example 2.4 (ii) plays a central role in proving Theorem 1.1.

**Lemma 2.5.** *Suppose $P_1, P_2 \leqslant Sym(\Omega)$, $\Omega$ a finite set. Then $\langle P_1, P_2\rangle$ is transitive on $\Omega$ if and only if $\mathcal{O}(P_1, P_2, \Omega)$ is connected.*

*Proof.* Let $\{\Delta_1^{(1)}, ..., \Delta_\ell^{(1)}\}$ and $\{\Delta_1^{(2)}, ..., \Delta_k^{(2)}\}$ be, respectively, the $P_1$- and $P_2$-orbits on $\Omega$. Suppose $\langle P_1, P_2\rangle$ acts transitively on $\Omega$ and let $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ be vertices of $\mathcal{O}(P_1, P_2, \Omega)$. Choose $\alpha \in \Delta_i^{(1)}$ and $\beta \in \Delta_j^{(2)}$. Then $\alpha^g = \beta$ for some $g \in \langle P_1, P_2\rangle$. Now $g = x_1x_2...x_p$ where $x_i \in P_1 \cup P_2$, and, as $x_1 \in P_1$ implies ${\Delta_i^{(1)}}^{x_1} = \Delta_i^{(1)}$, we may assume $x_1 \in P_2$. Put $\alpha^{x_1} = \gamma$ and let $\Delta_q^{(2)}$ be the $P_2$-orbit such that $\gamma \in \Delta_q^{(2)}$. Then $\alpha = \gamma^{x_1^{-1}} \in \Delta_i^{(1)} \cap \Delta_q^{(2)}$ whence $\Delta_i^{(1)}$ and $\Delta_q^{(2)}$ are adjacent in $\mathcal{O}(P_1, P_2, \Omega)$. Continuing in this fashion yields a path from $\Delta_i^{(1)}$ to $\Delta_j^{(2)}$. Therefore $\mathcal{O}(P_1, P_2, \Omega)$ is connected.
Now suppose that $\mathcal{O}(P_1, P_2, \Omega)$ is connected, and let $\alpha, \beta \in \Omega$. Let $\alpha \in \Delta_{i_1}^{(1)}, \beta \in \Delta_{i_k}^{(1)}$. We may find a sequence of $P_1$- and $P_2$-orbits in which the $\ell$th and $(\ell+1)$th orbits have non-empty intersections. This then gives $g \in \langle P_1, P_2\rangle$ for which $\alpha^g = \beta$, and so $\langle P_1, P_2\rangle$ is transitive on $\Omega$. □

**Hypothesis 2.6.** *Suppose $G$ and $H$ are subgroups of, respectively, $Sym(\Omega)$ and $Sym(\Gamma)$ where $\Omega$ and $\Gamma$ are finite sets which are disjoint. Assume that $G$ contains subgroups $Q_1$ and $Q_2$ with $C = Q_1 \cap Q_2$ and that $H$ contains subgroups $R_1$ and $R_2$ with $D = R_1 \cap R_2$. Further assume that $\theta_i : Q_i \to R_i, i = 1, 2$, are isomorphisms for which $\theta_1(x) = \theta_2(x)$ for all $x \in C$ and $Im(C) = D$. For $x \in Q_i,\ i = 1, 2$ we regard $x$ as a permutation on $\Omega\dot\cup\Gamma$ by letting $x$ act as the identity on $\Gamma$ and $\theta_i(x)$ as a permutation on $\Omega\dot\cup\Gamma$ by letting $\theta_i(x)$ act as the identity on $\Gamma$. Define subgroups $B, P_i,\ i = 1, 2$ of $Sym(\Omega\dot\cup\Gamma)$ by $P_i = \{x\theta_i(x) | x \in Q_i\}$ and $B = \{x\theta_i(x) | x \in C\}$.*

We sometimes refer to $P_1, P_2, B$ as being the concatenation of $Q_1, Q_2, C$ and $R_1, R_2, D$. Note that $P_1 \cong Q_1 \cong R_1, P_2 \cong Q_2 \cong R_2$ and $B \cong C \cong D$.

**Definition 2.7.** Assume Hypothesis 2.6 and let $\Delta$ and $\Psi$ be $B$-orbits of, respectively, $\Omega$ and $\Gamma$ which are isomorphic as $B$-sets. Now let $g = g(\Delta, \Psi) \in Sym(\Omega\dot\cup\Gamma)$ be an involution in $C_{Sym(\Omega\dot\cup\Gamma)}(B)$ which interchanges $\Delta$ and $\Psi$ and fixes all other points of $\Omega\dot\cup\Gamma$. Then we refer to the conjugate $P_1^g$ as *twisting* $P_1$ (with respect to $(\Delta, \Psi)$).

We note in Definition 2.7 that $P_1^g \cap P_2 \geqslant B$ (in situations we encounter we shall have $P_1^g \cap P_2 = B$).

**Lemma 2.8.** *Assume Hypothesis 2.6 holds, and use the notation given there. Let $g = g(\Delta, \Psi)$ where $\Delta$ and $\Psi$ are $B$-orbits of $\Omega$ and $\Gamma$ respectively which are isomorphic $B$-sets. Suppose $\langle P_1, P_2\rangle$ is transitive on both $\Omega$ and $\Gamma$. Then $\langle P_1^g, P_2\rangle$ is transitive on $\Omega\dot\cup\Gamma$ if and only if the edge in $\mathcal{O}(P_1, P_2, \Omega\dot\cup\Gamma)$ corresponding to $\Delta$ or the edge in $\mathcal{O}(P_1, P_2, \Omega\dot\cup\Gamma)$ corresponding to $\Psi$ is part of a cycle in $\mathcal{O}(Q_1, Q_2, \Omega)$, respectively, $\mathcal{O}(R_1, R_2, \Gamma)$.*

*Proof.* By Lemma 2.5, as $\langle P_1, P_2\rangle$ is transitive on $\Omega$ and $\Gamma$, $\mathcal{O}(Q_1, Q_2, \Omega)$ and $\mathcal{O}(R_1, R_2, \Gamma)$ are both connected graphs. Suppose $\Delta \subseteq \Delta_p^{(1)} \cap \Delta_q^{(2)}$ and $\Psi \subseteq \Psi_r^{(1)} \cap \Psi_s^{(2)}$ where $\Delta_p^{(1)}, \Delta_q^{(2)}$ are, respectively $P_1$- and $P_2$-orbits of $\Gamma$ and $\Psi_r^{(1)}, \Psi_s^{(2)}$ are respectively $P_1$- and $P_2$-orbits of $\Gamma$. So $\Delta_p^{(1)}$ and $\Delta_q^{(2)}$ are adjacent in $\mathcal{O}(Q_1, Q_2, \Omega)$ and $\Psi_r^{(1)}$ and $\Psi_s^{(2)}$ are adjacent in $\mathcal{O}(R_1, R_2, \Gamma)$. Since $g = g(\Delta, \Psi)$ fixes all points in $(\Omega \dot\cup \Gamma)\backslash(\Delta \cup \Psi)$, $P_1^g$ has the same orbits on $\Omega \dot\cup \Gamma$ as $P_1$ except $\Delta_p^{(1)}$ and $\Psi_r^{(1)}$ are replaced by, respectively, $\Delta_{p'}^{(1)} = (\Delta_p^{(1)}\backslash\Delta) \cup \Psi$ and $\Psi_{r'}^{(1)} = (\Psi_r^{(1)}\backslash\Psi) \cup \Delta$. Thus $\Delta \subseteq \Delta_q^{(2)} \cap \Psi_{r'}^{(1)}$ and $\Psi \subseteq \Psi_s^{(2)} \cap \Delta_{p'}^{(1)}$, whence $\Delta_q^{(2)}$ and $\Psi_{r'}^{(1)}$ are adjacent, as are $\Psi_s^{(2)}$ and $\Delta_{p'}^{(1)}$ in $\mathcal{O}(P_1, P_2, \Omega \dot\cup \Gamma)$. Since $\langle P_1^g, P_2\rangle$ is transitive on $\Omega \dot\cup \Gamma$ if and only if $\mathcal{O}(P_1^g, P_2, \Omega \dot\cup \Gamma)$ is connected, the lemma now follows. □

The following theorem is a classical one of Jordan's, which appears as Theorem 13.9 in [9]. This theorem suffices for our work. However, in [8], a generalization of this result is needed to deal with some of the other cases.

**Theorem 2.9.** *(Jordan) Let $G$ be a primitive permutation group of degree $n$ with a cycle of length $p^\lambda > 1$ where $p$ is prime. If $n > p^\lambda + 4$, then $G = Alt(n)$ or $G = Sym(n)$.*

## 3 A recursive construction

Put $G = Alt(n)$ and let $\Omega = \{1, ..., n\}$ be the standard $G$-set. Bearing in mind Lemma 2.1, we are looking for $G$-conjugate involutions which generate appropriate $Sym(4)$ subgroups of $G$. Thus we seek conjugate involutions $x, y, z$ and $w$ of $G$ such that

**Hypothesis 3.1.** *(i) $(xy)^2$ is conjugate to $x$;*

*(ii) $z \in N_G(\langle x, (xy)^2\rangle)$ and $zx = xz$;*

*(iii) $w \in N_G(\langle y, (xy)^2\rangle)$ and $wy = yw$;*

*(iv) $zy$ and $wx$ both have order 3; and*

*(v) $G = \langle x, y, z, w\rangle$*

These conditions will ensure that for $B = \langle x, y\rangle$, $P_1 = \langle x, y, z\rangle$ and $P_2 = \langle x, y, w\rangle$ we have $P_1 \cap P_2 = B \cong Dih(8)$ and $P_i \cong Sym(4)$, $i = 1, 2$.

As indicated earlier Example 2.4(ii) will be important, one reason being given in our next result

**Lemma 3.2.** *Let $x, y, z, w, P_1$ and $P_2$ be as in Example 2.4(ii). Then $P_i \cong Sym(4)$, $i = 1, 2$, $P_1 \cap P_2 \cong Dih(8)$ and $G = \langle P_1, P_2\rangle$. Thus $(P_1, P_2)$ exhibit $Alt(24)$ as a completion of the Goldschmidt $G_3$-amalgam.*

*Proof.* It is straightforward to verify Hypothesis 3.1 (i)-(iv) for $x, y, z, w$. So it remains to show that (v) holds. Since $\mathcal{O}(P_1, P_2, \Omega)$ is connected, Lemma 2.5 implies $\langle P_1, P_2\rangle$ acts transitively on $\Omega$. Calculation reveals that $h = [w, z]^x y[w, z]$ has cycle type $5^1.19^1$. So $h^5 \in \langle P_1, P_2\rangle$ is a 19-cycle. Hence, all the points of $\Omega$ in this 19-cycle either lie in the same block of a $\langle P_1, P_2\rangle$-system of imprimitivity or they all lie in separate blocks. So any $\langle P_1, P_2\rangle$-system of imprimitivity must either have block size at least 19 or at least 19 blocks. Since $\langle P_1, P_2\rangle$ acts transitively on $\Omega$ and $|\Omega| = 24$, we conclude that $\langle P_1, P_2\rangle$ acts primitively on $\Omega$. Furthermore, the presence of a 19-cycle, by Theorem 2.9, confirms $\langle x, y, z, w\rangle = G$, so proving the lemma. □

Now Example 2.4(ii) shows $\mathcal{O}(P_1, P_2, \Omega)$ where $\Omega = \{1, ..., 24\}$ has cycles and hence Lemma 2.8 becomes available to us. Put $G_m = Alt(m)$, acting on $\Omega_m = \{1, ..., m\}$ and suppose $Q_1^{(m)}, Q_2^{(m)}, C^{(m)}$ are subgroups of $G_m$ with $Q_i^{(m)} \cong Sym(4)$, $i = 1, 2$ and $Q_1^{(m)} \cap Q_2^{(m)} = C^{(m)} \cong Dih(8)$ with $\langle Q_1^{(m)}, Q_2^{(m)}\rangle = G_m$ (so $G_m$ is a completion of the Goldschmidt $G_3$-amalgam). Also let $H = Alt(24)$ with $H$ acting upon $\Gamma = \{m + 1, ..., m + 24\}$ with $R_1, R_2, D$ subgroups of $H$ as given in Example 2.4 ($R_i \cong Sym(4), R_1 \cap R_2 = D \cong Dih(8)$). Since the $D$-orbits on $\Gamma$ are regular, to employ Definition 2.7 we require that $C^{(m)}$ has a regular orbit on $\Omega_m$. So we further assume that this is the case, letting $\Delta = \{1, ..., 8\}$ be such a regular $C^{(m)}$-orbit and, restricted to $\Delta$, $x = (1, 2)(3, 4)(5, 6)(7, 8)$ and $y = (1, 8)(2, 3)(4, 5)(6, 7)$ (we can achieve this by conjugating the $Q_i^{(m)}$). Let $x_m, y_m, z_m, w_m$ be the generators for $G_m$ as given in Hypothesis 3.1 which give $Q_i^{(m)}$ and $C^{(m)}$. So $Q_1^{(m)} = \langle x_m, y_m, z_m\rangle, Q_2^{(m)} = \langle x_m, y_m, w_m\rangle$ and $C^{(m)} = \langle x_m, y_m\rangle$. The corresponding generators for $H$ are $x', y', z', w'$ where $R_1 = \langle x', y', z'\rangle, R_2 = \langle x', y', w'\rangle$ and $D = \langle x', y'\rangle$ and $\Psi = \{m+1, ..., m+8\}$ is a regular $D$-orbit upon which, restricted to $\Psi$, $x' = (m+1, m+2)(m+3, m+4)(m+5, m+6)(m+7, m+8)$, $y' = (m+1, m+8)(m+2, m+3)(m+4, m+5)(m+6, m+7)$. Put $\Omega = \Omega_m \dot\cup \Gamma$. Then we may define subgroups $P_1, P_2 \leqslant Sym(\Omega)$ and $B = P_1 \cap P_2$ as in Hypothesis 2.6. Thus we have $P_i = Sym(4)$ and $B \cong Dih(8)$. Now take $g(\Delta, \Psi) \in Sym(\Omega)$ where

$$g(\Delta, \Psi) = (1, m+1)(2, m+2)(3, m+3)(4, m+4)(5, m+5)(6, m+6)(7, m+7)(8, m+8)$$

Then $g(\Delta, \Psi)$ satisfies the conditions of Definition 2.7 and so we may twist $P_1$ with respect to $(\Delta, \Psi)$. Because of Lemma 2.8 we have that $\langle P_1^g, P_2\rangle$ is a transitive subgroup of $Sym(\Omega)$ which is actually a subgroup of $Alt(\Omega)$. Theorem 1.1 is proved (constructively) by beginning with an appropriate $G_m$ and appropriate $Q_1^{(m)}, Q_2^{(m)}, C^{(m)}$, and then "adding" copies of $H$. Thus the proof must consider twenty four cases given by the congruence of $n$ (mod 24) where $n = m + 24k$. For the full details we direct the reader to [8] or to the seed permutations given in Section 4 and here we will look at starting with $m = 48$ and repeatedly "adding" copies of $H$. We remark that recursive constructions of this ilk have also been employed in [1] and [6].

**Definition 3.3.** If we begin with $G = Alt(m)$ and recursively add on copies of $H$, then any points greater than $m$ will be denoted with numbers $\{1, ..., 24\}$

with an appropriate subscript attached. Explicitly, the number $m + 24i + j$ will be denoted $j_{i+1}$ for $j \in \{1, ..., 24\}$. In simpler terms, this will see the $i$th set of 24 points to be labelled $\{1_i, ..., 24_i\}$ for $1 \leqslant i \leqslant k$. The points $\{1, ..., m\}$ can be considered as having the subscript 0 attached, which will be useful in writing down the cycle decomposition of $z$ but not generally noted otherwise.
The labels $j_i$ and $m + 24(i-1) + j$ will be interchangeable depending on which is most convenient.

**Theorem 3.4.** *For $n \equiv 0 (\mathrm{mod}\ 24)$, $Alt(n)$ is a completion of the Goldschmidt $G_3$-amalgam.*

*Proof.* Starting at $m = 48$, we take the following elements in $G = Alt(48+24k)$:-

$$\begin{aligned}
x_{48+24k} :=& (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(22,23)(24,25)(28,29) \\
& (30,31)(32,33)(34,35)(36,37)(38,39)(40,41)(42,43) \\
& \prod_{i=1}^{k}(1_i,2_i)(3_i,4_i)(5_i,6_i)(7_i,8_i)(9_i,10_i)(11_i,12_i)(13_i,14_i)(15_i,16_i)(17_i,18_i) \\
& (19_i,20_i)(21_i,22_i)(23_i,24_i); \\
y_{48+24k} :=& (1,8)(2,3)(4,5)(6,7)(10,11)(13,17)(14,18)(19,20)(22,24)(26,27) \\
& (28,35)(29,30)(31,32)(33,34)(37,38)(40,44)(41,45)(46,47) \\
& \prod_{i=1}^{k}(1_i,8_i)(2_i,3_i)(4_i,5_i)(6_i,7_i)(9_i,16_i)(10_i,11_i)(12_i,13_i)(14_i,15_i) \\
& (17_i,24_i)(18_i,19_i)(20_i,21_i)(22_i,23_i); \\
z_{48+24k} :=& (1_k,38)(2_k,39)(5_k,36)(6_k,37)(7_k,8_k)(9,10)(13,16)(14,15)(20,21) \\
& (22,35)(23,34)(24,30)(25,31)(26,27)(28,29)(40,42)(41,43)(47,48) \\
& \prod_{i=1}^{k}(1_{i-1},9_i)(2_{i-1},10_i)(3_{i-1},20_i)(4_{i-1},19_i)(5_{i-1},14_i)(6_{i-1},13_i)(7_{i-1},23_i) \\
& (8_{i-1},24_i)(11_i,21_i)(12_i,22_i)(15_i,18_i)(16_i,17_i); \textit{and} \\
w_{48+24k} :=& (1,17)(4,18)(5,14)(6,7)(8,13)(10,19)(11,20)(16,21)(22,26)(24,27) \\
& (28,44)(31,45)(32,41)(33,34)(35,40)(37,46)(38,47)(43,48) \\
& \prod_{i=1}^{k}(1_i,14_i)(2_i,19_i)(3_i,18_i)(4_i,10_i)(5_i,11_i)(6_i,22_i)(7_i,23_i)(8_i,15_i)(9_i,17_i) \\
& (12_i,21_i)(13_i,20_i)(16_i,24_i).
\end{aligned}$$

For convenience we will denote $x_{48+24k}, y_{48+24k}, z_{48+24k}, w_{48+24k}$ simply as $x, y, z, w$ throughout this proof. Put $\Omega = \{1, ..., 48+24k\}$. We consider $H = \langle (zyx)^w, z, x \rangle \leqslant \langle x, y, z, w \rangle$ and show that $H$ fixes 19 and is transitive on $\Omega \backslash \{19\}$. Hence $\langle x, y, z, w \rangle$ is 2-transitive on $\Omega$. In fact, the generator $x$ is not necessary here but it will simplify the proof. First we look at $(zyx)^w$ and see that

$$
\begin{aligned}
(zyx)^w =&(19)(22)(24)(1,5,21)(2,21_1,16_1)(3,6_1,1_1)(4,8,15)(6,12_1,17_1)(7,5_1,2_1)\\
&(9,12,20)(10,11,16)(13,3_1,20_1)(14,24_1,7_1)(17,8_1,13_1)(18,9_1,4_1)\\
&(23,41,30)(25,34,40)(26,29,33)(27,44,45)(28,32,42)(31,35,48)\\
&(36,18_k,14_k)(37,38,43)(39,10_k,22_k)(46,15_k,11_k)(47,23_k,19_k)\\
&\prod_{i=1}^{k-1}(10_i,9_{i+1},4_{i+1})(11_i,24_{i+1},7_{i+1})(14_i,8_{i+1},13_{i+1})(15_i,3_{i+1},20_{i+1})\\
&(18_i,6_{i+1},1_{i+1})(19_i,21_{i+1},16_{i+1})(22_i,5_{i+1},2_{i+1})(23_i,12_{i+1},17_{i+1}).
\end{aligned}
$$

The following subgraphs of $\mathcal{O}(\langle(zyx)^w\rangle,\langle z\rangle,\Omega)$ will help to show that $H$ is transitive on $\Omega\backslash\{19\}$, where the orbits of $\langle(zyx)^w\rangle$ are on the left hand side and orbits of $\langle z\rangle$ are on the right. For the moment we assume $k\geqslant 2$, dealing with $k=0,1$ later.

Looking at Figure 3 and Figure 4, the following are subsets of $H$-orbits:

$\Delta_1 := \{1,3,5,9,10,11,12,13,16,18,20,21,1_1,3_1,4_1,6_1,9_1,20_1\}$,

$\Delta_2 := \{2,6,17,8_1,12_1,13_1,16_1,17_1,21_1\}$,

$\Delta_3 = \{4,8,14,15,24_1,7_1\}$,

$\Delta_4 := \{7,2_1,5_1\}$,

$\Delta_5 := \{22,23,24,25,26,27,28,29,30,31,32,33,34,35,37,38,40,41,42,43,44,45,47,48,1_k,6_k,19_k,23_k\}$,

$\Delta_6 := \{36,39,46,2_k,5_k,10_k,11_k,12_k,14_k,15_k,16_k,17_k,18_k,21_k,22_k\}$,

$\Delta_7 := \{7_k,8_k,13_k,24_k\}$,

$\Delta_8 := \{3_k,20_k\}$ and

$\Delta_9 := \{4_k,9_k\}$.

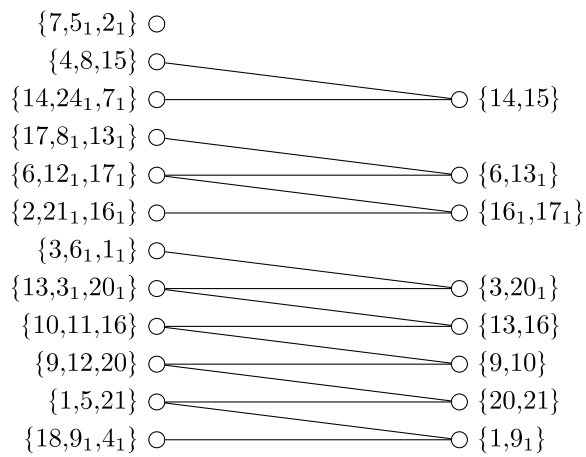


Figure 3

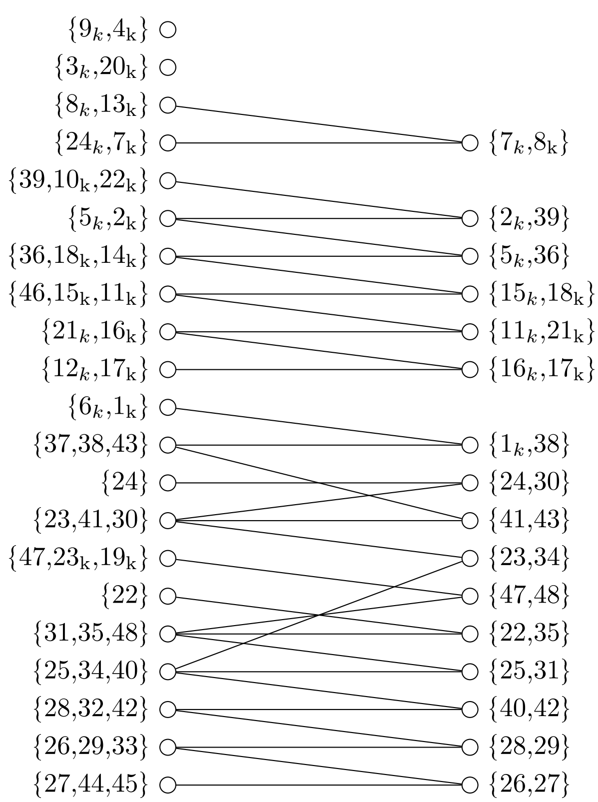


Figure 4

However, when also considering orbits of $\langle x \rangle$, we can connect these further as in Figure 5, which shows that $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ and $\Delta_5 \cup \Delta_6 \cup \Delta_7 \cup \Delta_8 \cup \Delta_9$ are subsets of $H$-orbits, with $\{1_k, ..., 24_k\} \subset \Delta_5 \cup \Delta_6 \cup \Delta_7 \cup \Delta_8 \cup \Delta_9$.

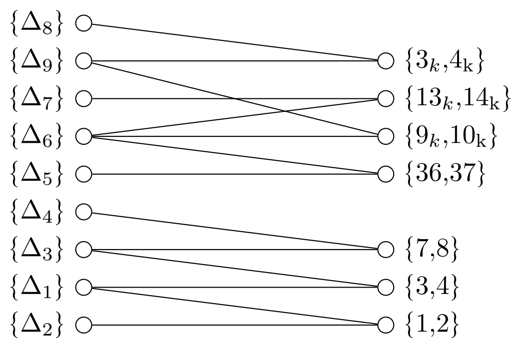


Figure 5

When $1 \leqslant i \leqslant k-1$, Figure 6 shows that the following are subsets of $H$-orbits:
$\Lambda_{i_1} := \{1_i, 6_i, 10_i, 14_i, 4_{i+1}, 8_{i+1}, 9_{i+1}, 13_{i+1}\}$,
$\Lambda_{i_2} := \{8_i, 11_i, 12_i, 13_i, 16_i, 17_i, 21_i, 22_i, 2_{i+1}, 5_{i+1}, 7_{i+1}, 24_{i+1}\}$,
$\Lambda_{i_3} := \{4_i, 9_i, 15_i, 18_i, 1_{i+1}, 3_{i+1}, 6_{i+1}, 20_{i+1}\}$,
$\Lambda_{i_4} := \{19_i, 23_i, 12_{i+1} 16_{i+1}, 17_{i+1}, 21_{i+1}\}$,
$\Lambda_{i_5} := \{7_i, 24_i\}$,
$\Lambda_{i_6} := \{3_i, 20_i\}$ and
$\Lambda_{i_7} := \{2_i, 5_i\}$.

Now, considering the orbits of $\langle x \rangle$ we can see that $\Lambda_i := \{1_i, ..., 24_i, 1_{i+1}\}$ is a subset of an $H$-orbit with the help of Figure 7. Clearly, there are more points with subscript $i+1$ that we could include, but all we require is that there is at least one, hence $1_{i+1} \in \Lambda_i \cap \Lambda_{i+1}$ and therefore, $\bigcup_{i=1}^{k-1} \Lambda_i \subset \Gamma$, where $\Gamma$ is an $H$-orbit.

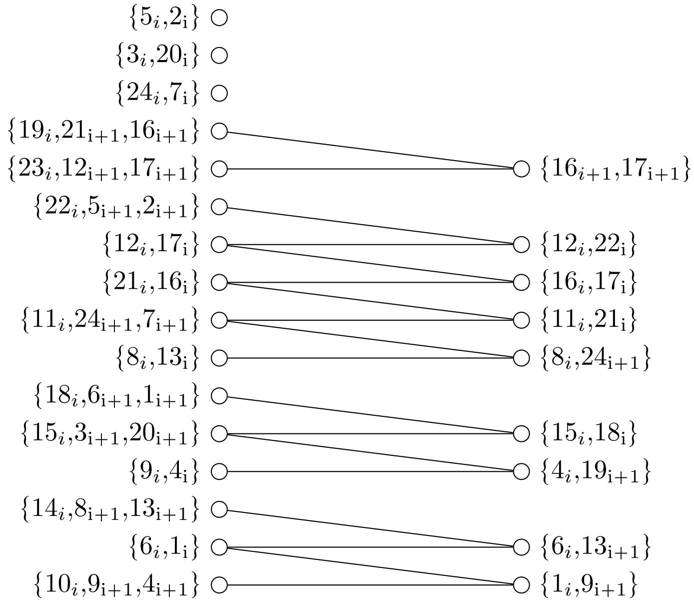


Figure 6

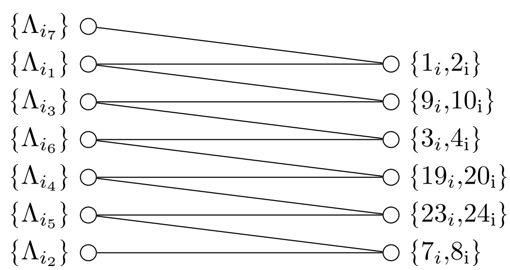


Figure 7

Since $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ contains points with subscript 1- all of which are in $\Lambda_1 \subseteq \Gamma$, we have $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \subseteq \Gamma$ also. Finally, $\Lambda_{k-1} \cap (\Delta_5 \cup \Delta_6 \cup \Delta_7 \cup \Delta_8 \cup \Delta_9) = \{1_k\}$ so $\Delta_5 \cup \Delta_6 \cup \Delta_7 \cup \Delta_8 \cup \Delta_9 \subseteq \Gamma$ and $\Gamma = \{1, ..., 48+24k\} \backslash \{19\}, \{19\}$ are $H$-orbits. Consequently $\langle x, y, z, w \rangle$ is 2-transitive when $k \geqslant 2$. For $k = 0$, consider $(zyx)^w z \in H$,

$$
\begin{aligned}
(zyx)^w z = &(19)(1, 36, 3, 17, 5, 20, 10, 11, 13, 15, 4, 7, 2, 48, 25, 23, 43, 6, 39, 18, 8, 14, 46,\\
&16, 9, 12, 21, 38, 41, 24, 30, 34, 42, 29, 33, 27, 44, 45, 26, 28, 32, 40, 31, 22, 35,\\
&47, 37).
\end{aligned}
$$

This is a 47-cycle, so clearly $H$ is transitive on all points except for 19.

When $k = 1$, we have

$$
\begin{aligned}
(zyx)^w z =\ &(19)(1,62,53,39,2,59,46,66,5,20,10,11,13,51,3,37,49,68,16,9,12,21,\\
&57,52,18)(4,72,56,6,70,50,71)(7,36,63,69,65,61,17,55,15,67,48,25,23,\\
&43,54,38,41,24,30,34,42,29,33,27,44,45,26,28,32,40,31,22,35,47)(8,\\
&14)(58,60,64).
\end{aligned}
$$

Here, $z$ contains the transpositions $(2,58),(4,67),(8,72)$ and $(60,70)$. These transpositions connecting the first and fifth, second and third, fourth and second and fifth and second cycles respectively- that is to say, they contain one point from each therefore proving that all of the points in both of those cycles must belong to the same $H$-orbit- to form the $H$-orbit $\{1,...,72\}\backslash\{19\}$.
So, for all $k \geqslant 0$ $\langle x,y,z,w\rangle$ is 2-transitive. Now we are in a position to demonstrate that $G = \langle x,y,z,w\rangle$, using the element $zw$.
Suppose that $k \neq 0$. Observing the cycle decomposition of $zw$ leads to us noticing that it follows patterns depending on the value of $k$ (mod 2). When $k \equiv 0(\text{mod } 2)$, we analyse the cycles in $zw$. Here, the underlined parts of the cycle are "repeated", only with the subscript on the points in subsequent or previous repeated parts being respectively higher or lower by the same amount with each repetition:
$(2, \underline{4_1, 2_2}, 4_3, 2_4, ......, 4_{k-3}, 2_{k-2}, \underline{4_{k-1}, 2_k}, 39, \underline{19_k, 10_{k-1}}, 19_{k-2}, 10_{k-1}, ......, 19_4,$
$10_3, \underline{19_2, 10_1})$
$(4, \underline{2_1, 4_2}, 2_3, 4_4, ..., 2_{k-3}, 4_{k-2}, \underline{2_{k-1}, 4_k}, \underline{10_k, 19_{k-1}}, 10_{k-2}, 19_{k-3}, ..., 10_4, 19_3, \underline{10_2,}$
$\underline{19_1}, 18)$
$(3, 13_1, 7, \underline{7_1}, 7_2, ..., 7_{k-1}, \underline{7_k}, 15_k, 3_k, 18_k, 8_k, \underline{23_k}, 23_{k-1}, ..., 23_2, \underline{23_1}, 6, 20_1)$
$(5, \underline{1_1, 17_2, 24_2, 15_1, 3_1, 13_2, 22_1, 21_1, 5_1}, 1_2, ..., 1_{k-1}, \underline{17_k, 24_k, 15_{k-1}, 3_{k-1}, 13_k, 22_{k-1},}$
$\underline{21_{k-1}, 5_{k-1}, 1_k}, 47, 43, 32, 41, 48, 38, \underline{14_k, 11_{k-1}, 12_{k-1}, 6_{k-1}, 20_k, 18_{k-1}, 8_{k-1}, 16_k,}$
$\underline{9_k}, 14_{k-1}, ..., 14_2, \underline{11_1, 12_1, 6_1, 20_2, 18_1, 8_1, 16_2, 9_2, 14_1}, 14, 15)$
$(22, 40, 42, 35, 26, 24, 30, 27),$
$(9, 19, 10),$
$(23, 33, 34),$
$(25, 45, 31),$
$(28, 29, 44),$
$(36, 11_k, 12_k, 6_k, 46, 37, 22_k, 21_k, 5_k),$
$(1, 17_1, 24_1, 13, 21, 11, 20, 16, 8, 16_1, 8, 16_1, 9_1).$
So, by counting the lengths of these cycles, we see $zw$ is an element of cycle type $(18k-7)^1.(2k+9)^1.(2(1+k))^2.12^1.9^1.8^1.3^4.1^1$ for which we can see the first two cycles are of odd length and the third will be divisible by 2 but not 4. Overall, this means that the only cycle of length divisible by 8 is the 8-cycle $(22, 40, 42, 35, 26, 24, 30, 27)$.
When $k \equiv 1(\text{mod } 2)$ we have the following cycles:
$(2, \underline{4_1, 2_2}, 4_3, ..., 4_{k-2}, \underline{2_{k-1}, 4_k}, \underline{10_k, 19_{k-1}}, 10_{k-2}, ..., 10_3, \underline{19_2, 10_1})$
$(4, \underline{2_1, 4_2}, 2_3, ..., 2_{k-2}, \underline{4_{k-1}, 2_k}, 39, \underline{19_k, 10_{k-1}}, 19_{k-2}, ..., 19_3, \underline{10_2, 19_1}, 18)$
$(6, 20_1, 3, 13_1, 7, \underline{7_1}, 7_2, ..., 7_{k-1}, \underline{7_k}, 15_k, 3_k, 18_k, 8_k, \underline{23_k}, 23_{k-1}, ..., 23_2, \underline{23_1})$
$(5, \underline{1_1, 17_2, 24_2, 15_1, 3_1, 13_2, 22_1, 21_1, 5_1}, 1_2, ......, 1_{k-1}, \underline{17_k, 24_k, 15_{k-1}, 3_{k-1}, 13_k,}$
$\underline{22_{k-1}, 21_{k-1}, 5_{k-1}, 1_k}, 47, 43, 32, 41, 48, 38, \underline{14_k, 11_{k-1}, 12_{k-1}, 6_{k-1}, 20_k, 18_{k-1}, 8_{k-1},}$

$\underline{16_k, 9_k, 14_{k-1}, ..., 14_2, 11_1, 12_1, 6_1, 20_2, 18_1, 8_1, 16_2, 9_2, 14_1}, 14, 15)$
$(22, 40, 42, 35, 26, 24, 30, 27)$,
$(9, 19, 10)$,
$(23, 33, 34)$,
$(25, 45, 31)$,
$(28, 29, 44)$,
$(36, 11_k, 12_k, 6_k, 46, 37, 22_k, 21_k, 5_k)$,
$(1, 17_1, 24_1, 13, 21, 11, 20, 16, 8, 16_1, 8, 16_1, 9_1)$.
All of this gives a cycle type of $(18k-7)^1.(2k+9)^1.(5+2(k-1))^1.(3+2(k-1))^1.12^1.9^1.8^1.3^4.1^1$, for which we can see that the first four cycles will all be of odd length and no others divisible by 8 other than the one 8-cycle $(22, 40, 42, 35, 26, 24, 30, 27)$.
In both cases, for some large odd number $f$, $(zw)^{4f} = (22, 26)(24, 40)(27, 35)(30, 42) =: c$ and $c^y = (24, 27)(22, 44)(26, 28)(29, 42)$ so $cc^y = (22, 28, 26, 44)(24, 40, 27, 35)(30, 29, 42)$ and $(cc^y)^4$ is a 3-cycle, sufficient by Theorem 2.9 to confirm that $G = \langle x, y, z, w\rangle$.
Finally, for $k = 0$ we calculate that $xzw = (1, 39, 17)(2, 47, 43, 35, 23, 40, 48, 38)$ $(3, 18, 4)(5, 46, 37, 14, 21, 11, 12, 20, 16)(6, 36, 7)(8, 13, 15)(10, 19)(22, 33, 41, 42, 32, 34, 26, 24, 45, 31, 27)(25, 30)(28, 44)$, which has cycle type $11^1.9^1.8^1.3^4.2^2.1^2$, therefore $(xzw)^{9.8}$ is an 11-cycle, sufficient by Theorem 2.9 to confirm that $G = \langle x, y, z, w\rangle$.
So for all $k \geqslant 0$, $G = \langle x, y, z, w\rangle$, which completes the proof of Theorem 1.1 in the case $n \equiv 0 (\text{mod } 24)$. $\square$

# 4 Seeds for the recursive construction

The following generators provide a starting point for the recursive method of adding 24 points for each case modulo 24. In each case $\{1, ..., 8\}$ is a regular $\langle x, y\rangle$-orbit with $x = (1, 2)(3, 4)(5, 6)(7, 8)$ and $y = (1, 8)(2, 3)(4, 5)(6, 7)$ on this orbit. So, we can take $\Delta = \{1, ..., 8\}$ and $\Psi = \{m+1, ..., m+8\}$ as the two orbits to twist by.
When $n \equiv 0(\text{mod } 24)$, we may start at $m = 24$ with:
$x := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)$;
$y := (1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(17, 24)(18, 19)(20, 21)(22, 23)$;
$z := (1, 10)(2, 9)(3, 20)(4, 19)(5, 13)(6, 14)(7, 23)(8, 24)(11, 17)(12, 18)(15, 22)(16, 21)$;
$w := (1, 10)(2, 22)(3, 23)(4, 14)(5, 15)(6, 19)(7, 18)(8, 11)(9, 21)(12, 17)(13, 24)(16, 20)$.
When $n \equiv 1(\text{mod } 24)$, we may start at $m = 25$ with:
$x := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)$;
$y := (1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(14, 15)(17, 21)(18, 22)(19, 20)(23, 24)$;
$z := (1, 12)(2, 11)(3, 4)(5, 10)(6, 9)(13, 14)(17, 19)(18, 20)(21, 22)(24, 25)$;
$w := (1, 17)(2, 3)(4, 18)(5, 22)(8, 21)(10, 24)(11, 23)(13, 16)(14, 19)(15, 20)$.
When $n \equiv 2(\text{mod } 24)$, we may start at $m = 26$ with:
$x := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)$;
$y := (1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(18, 19)(21, 25)(22, 26)(23, 24)$;
$z := (1, 8)(2, 7)(3, 5)(4, 6)(9, 10)(11, 18)(12, 17)(15, 20)(16, 19)(21, 24)(22, 23)(25, 26)$;

$w := (1,25)(4,26)(5,22)(6,7)(8,21)(9,14)(10,12)(11,13)(15,16)(17,20)(18,23)(19,24)$.
When $n \equiv 3 \pmod{24}$, we may start at $m = 27$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)$;
$y := (1,8)(2,3)(4,5)(6,7)(10,11)(14,15)(17,21)(18,22)(23,24)(25,26)$;
$z := (1,13)(2,14)(3,4)(5,15)(6,16)(9,10)(17,20)(18,19)(23,27)(25,26)$;
$w := (1,18)(2,3)(4,17)(5,21)(8,22)(10,24)(11,23)(14,26)(15,25)(19,27)$.
When $n \equiv 4 \pmod{24}$, we may start at $m = 28$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)$;
$y := (1,8)(2,3)(4,5)(6,7)(10,11)(14,15)(17,21)(18,22)(23,24)(25,26)$;
$z := (1,2)(3,10)(4,9)(7,12)(8,11)(15,16)(17,19)(18,20)(24,28)(26,27)$;
$w := (1,18)(2,3)(4,17)(5,21)(8,22)(10,23)(11,24)(14,25)(15,26)(20,27)$.
When $n \equiv 5 \pmod{24}$, we may start at $m = 29$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(18,19)(21,25)(22,26)(27,28)$;
$z := (1,18)(2,17)(5,20)(6,19)(7,8)(9,15)(10,16)(11,14)(12,13)(21,23)(22,24)(28,29)$;
$w := (1,2)(3,8)(4,6)(5,7)(9,22)(10,11)(12,21)(13,25)(16,26)(18,28)(19,27)(24,29)$.
When $n \equiv 6 \pmod{24}$, we may start at $m = 30$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)$
$(27,28)(29,30)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,13)(10,14)(15,22)(16,17)(18,19)(20,21)(23,30)(24,25)$
$(26,27)(28,29)$;
$z := (1,23)(2,24)(3,18)(4,17)(5,28)(6,27)(7,21)(8,22)(9,11)(10,12)(15,30)(16,29)$
$(19,25)(20,26)$;
$w := (1,13)(4,14)(5,10)(6,7)(8,9)(11,12)(15,16)(17,22)(18,20)(19,21)(23,24)$
$(25,30)(26,28)(27,29)$.
When $n \equiv 7 \pmod{24}$, we may start at $m = 31$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(17,24)(18,19)(20,21)(22,23)$
$(26,27)(29,30)$;
$z := (3,25)(4,26)(5,6)(7,27)(8,28)(9,15)(10,16)(11,14)(12,13)(17,20)(18,19)(21,23)$
$(22,24)(30,31)$;
$w := (1,24)(2,13)(3,12)(4,20)(5,21)(6,16)(7,9)(8,17)(10,18)(11,19)(14,23)(15,22)$
$(26,29)(27,30)$.
When $n \equiv 8 \pmod{24}$, we may start at $m = 32$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)(29,30)(31,32)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(17,24)(18,19)(20,21)(22,23)$
$(25,32)(26,27)(28,29)(30,31)$;
$z := (1,9)(2,10)(3,20)(4,19)(5,14)(6,13)(7,23)(8,24)(11,21)(12,22)(15,18)(16,17)$
$(25,32)(26,31)(27,29)(28,30)$;
$w := (1,2)(3,8)(4,6)(5,7)(9,22)(10,27)(11,26)(12,18)(13,19)(14,30)(15,31)(16,23)$
$(17,25)(20,29)(21,28)(24,32)$.
When $n \equiv 9 \pmod{24}$, we may start at $m = 33$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)$;

$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(18,19)(22,23)(25,29)(26,30)$
$(27,28)(31,32)$;
$z := (1,21)(2,22)(3,4)(5,23)(6,24)(9,18)(10,17)(13,20)(14,19)(15,16)(25,28)(26,27)$
$(29,30)(32,33)$;
$w := (2,29)(3,25)(4,5)(6,26)(7,30)(9,11)(10,16)(12,15)(13,14)(17,20)(18,27)$
$(19,28)(22,32)(23,31)$.
When $n \equiv 10 (\text{mod } 24)$, we may start at $m = 34$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)(29,30)(31,32)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(17,24)(18,19)(20,21)(22,23)$
$(26,27)(29,33)(30,34)(31,32)$;
$z := (1,3)(2,4)(5,8)(6,7)(9,26)(10,25)(13,28)(14,27)(15,16)(17,23)(18,24)(19,22)$
$(20,21)(29,32)(30,31)(33,34)$;
$w := (1,14)(2,19)(3,18)(4,10)(5,11)(6,22)(7,23)(8,15)(9,17)(12,21)(13,20)(16,24)$
$(25,28)(26,31)(27,32)(30,34)$.
When $n \equiv 11 (\text{mod } 24)$, we may start at $m = 35$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(18,19)(22,23)(25,29)(26,30)$
$(31,32)(33,34)$;
$z := (1,23)(2,24)(5,21)(6,22)(7,8)(9,19)(10,20)(13,17)(14,18)(15,16)(25,28)(26,27)$
$(31,32)(34,35)$;
$w := (1,2)(3,8)(4,6)(5,7)(9,16)(10,25)(11,29)(14,30)(15,26)(18,32)(19,31)(22,34)$
$(23,33)(27,35)$.
When $n \equiv 12 (\text{mod } 24)$, we may start at $m = 36$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(18,19)(21,29)(22,30)(23,31)$
$(24,32)(33,34)$;
$z := (1,8)(2,7)(3,5)(4,6)(11,20)(12,19)(13,14)(15,18)(16,17)(21,26)(22,25)(23,27)$
$(24,28)(33,35)$;
$w := (1,32)(4,31)(5,23)(6,7)(8,24)(9,30)(12,29)(13,21)(14,15)(16,22)(18,34)$
$(19,33)(26,36)(27,35)$.
When $n \equiv 13 (\text{mod } 24)$, we may start at $m = 37$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)(29,30)(31,32)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(17,24)(18,19)(20,21)(22,23)$
$(26,27)(30,31)(33,34)(35,36)$;
$z := (1,26)(2,25)(5,28)(6,27)(7,8)(9,12)(10,11)(13,15)(14,16)(17,29)(18,30)$
$(19,20)(21,31)(22,32)(33,37)(35,36)$;
$w := (1,14)(2,19)(3,18)(4,10)(5,11)(6,22)(7,23)(8,15)(9,17)(12,21)(13,20)(16,24)$
$(26,33)(27,34)(30,35)(31,36)$.
When $n \equiv 14 (\text{mod } 24)$, we may start at $m = 14$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$;
$y := (1,8)(2,3)(4,5)(6,7)(9,13)(10,14)$;
$z := (1,8)(2,7)(3,5)(4,6)(9,11)(10,12)$;

$w := (1,13)(4,14)(5,10)(6,7)(8,9)(11,12)$.

When $n \equiv 15 (\mathrm{mod}\ 24)$, we may start at $m = 15$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$;

$y := (1,8)(2,3)(4,5)(6,7)(10,11)(13,14)$;

$z := (1,2)(3,10)(4,9)(7,12)(8,11)(14,15)$;

$w := (1,6)(2,4)(3,5)(7,8)(10,13)(11,14)$.

When $n \equiv 16 (\mathrm{mod}\ 24)$, we may start at $m = 40$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32)(33,34)(35,36)(37,38)(39,40)$;

$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(17,24)(18,19)(20,21)(22,23)(25,32)(26,27)(28,29)(30,31)(33,40)(34,35)(36,37)(38,39)$;

$z := (1,3)(2,4)(5,8)(6,7)(9,25)(10,26)(11,21)(12,22)(13,30)(14,29)(15,18)(16,17)(19,28)(20,27)(23,31)(24,32)(33,35)(34,36)(37,40)(38,39)$;

$w := (1,38)(2,19)(3,18)(4,34)(5,35)(6,22)(7,23)(8,39)(9,10)(11,16)(12,14)(13,15)(17,33)(20,37)(21,36)(24,40)(25,26)(27,32)(28,30)(29,31)$.

When $n \equiv 17 (\mathrm{mod}\ 24)$, we may start at $m = 41$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32)(33,34)(35,36)$;

$y := (1,8)(2,3)(4,5)(6,7)(10,11)(13,17)(14,18)(19,20)(21,28)(22,23)(24,25)(26,27)(30,31)(33,37)(34,38)(39,40)$;

$z := (1,31)(2,32)(5,29)(6,30)(7,8)(9,24)(10,23)(11,28)(12,27)(13,15)(14,16)(19,20)(21,22)(33,35)(34,36)(40,41)$;

$w := (1,14)(2,3)(4,13)(5,17)(8,18)(10,19)(11,20)(15,16)(21,37)(24,38)(25,34)(26,27)(28,33)(30,39)(31,40)(36,41)$.

When $n \equiv 18 (\mathrm{mod}\ 24)$, we may start at $m = 18$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$;

$y := (1,8)(2,3)(4,5)(6,7)(10,11)(13,17)(14,18)(15,16)$;

$z := (1,8)(2,7)(3,5)(4,6)(9,10)(13,15)(14,16)(17,18)$;

$w := (1,17)(4,18)(5,14)(6,7)(8,13)(9,12)(10,15)(11,16)$.

When $n \equiv 19 (\mathrm{mod}\ 24)$, we may start at $m = 43$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(23,24)(25,26)(27,28)(29,30)(31,32)(33,34)(35,36)(37,38)$;

$y := (1,8)(2,3)(4,5)(6,7)(10,11)(15,17)(16,18)(19,20)(23,30)(24,25)(26,27)(28,29)(32,33)(35,39)(36,40)(41,42)$;

$z := (1,33)(2,34)(5,31)(6,32)(7,8)(9,27)(10,28)(11,23)(12,24)(13,16)(14,15)(20,21)(29,30)(35,37)(36,38)(42,43)$;

$w := (2,17)(3,15)(4,5)(6,16)(7,18)(10,19)(11,20)(14,22)(23,39)(26,40)(27,36)(28,29)(30,35)(32,41)(33,42)(38,43)$.

When $n \equiv 20 (\mathrm{mod}\ 24)$, we may start at $m = 20$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$;

$y := (1,8)(2,3)(4,5)(6,7)(10,11)(13,17)(14,18)(19,20)$;

$z := (1,2)(3,10)(4,9)(7,12)(8,11)(13,15)(14,16)(19,20)$;

$w := (1,14)(2,3)(4,13)(5,17)(8,18)(10,19)(11,20)(15,16)$.

When $n \equiv 21 (\mathrm{mod}\ 24)$, we may start at $m = 21$ with:

$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$;

$y := (1,8)(2,3)(4,5)(6,7)(10,11)(15,17)(16,18)(19,20)$;

$z := (1,12)(2,11)(3,4)(5,10)(6,9)(13,16)(14,15)(19,21);$
$w := (1,3)(2,8)(4,7)(5,6)(10,20)(11,19)(14,21)(16,18).$
When $n \equiv 22 (\text{mod } 24)$, we may start at $m = 22$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16);$
$y := (1,8)(2,3)(4,5)(6,7)(10,11)(13,17)(14,18)(19,20);$
$z := (1,11)(2,12)(5,9)(6,10)(7,8)(13,16)(14,15)(20,21);$
$w := (1,8)(2,14)(3,18)(6,17)(7,13)(10,20)(11,19)(15,22).$
When $n \equiv 23 (\text{mod } 24)$, we may start at $m = 47$ with:
$x := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$
$(25,26)(27,28)(29,30)(31,32)(33,34)(35,36)(37,38)(39,40)(41,42)(43,44);$
$y := (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15)(17,24)(18,19)(20,21)(22,23)$
$(25,32)(26,27)(28,29)(30,31)(33,40)(34,35)(36,37)(38,39)(42,43)(45,46);$
$z := (1,9)(2,10)(3,20)(4,19)(5,14)(6,13)(7,23)(8,24)(11,21)(12,22)(15,18)(16,17)$
$(25,43)(26,44)(29,41)(30,42)(31,32)(33,35)(34,36)(37,40)(38,39)(45,47);$
$w := (1,2)(3,8)(4,6)(5,7)(9,10)(11,16)(12,14)(13,15)(17,30)(18,35)(19,34)(20,26)$
$(21,27)(22,38)(23,39)(24,31)(25,33)(28,37)(29,36)(32,40)(42,45)(43,46).$

# 5 Magma code for generating completions

To use this code in the program MAGMA, one just needs to input the degree of the alternating group in question and it will output the generators $x, y, z$ and $w$.

```
G:=Alt(n);
if n mod 24 eq 0 then
m:=24;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24);seedy:=G!(1, 8)
(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(17, 24)(18, 19)(20, 21)
(22, 23); seedz:=G!(1, 10)(2, 9)(3, 20)(4, 19)(5, 13)(6, 14)(7, 23)(8, 24)
(11, 17)(12, 18)(15, 22)(16, 21); seedw:=G!(1, 10)(2, 22)(3, 23)(4, 14)
(5, 15)(6, 19)(7, 18)(8, 11)(9, 21)(12, 17)(13, 24)(16, 20);end if;

if n mod 24 eq 1 then
m:=25;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20);seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)
(10, 11)(14, 15)(17, 21)(18, 22)(19, 20)(23, 24); seedz:=G!(1, 12)(2, 11)
(3, 4)(5, 10)(6, 9)(13, 14)(17, 19)(18, 20)(21, 22)(24, 25); seedw:=G!
(1, 17)(2, 3)(4, 18)(5, 22)(8, 21)(10, 24)(11, 23)(13, 16)(14, 19)(15, 20)
;end if;

if n mod 24 eq 2 then
m:=26;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)
```

```
(13,14)(15,16)(17,18)(19,20)(21,22)(23,24); seedy:=G!(1,8)(2,3)(4,5)(6,7)
(9,16)(10,11)(12,13)(14,15)(18,19)(21,25)(22,26)(23,24); seedz:=G!(1,8)
(2,7)(3,5)(4,6)(9,10)(11,18)(12,17)(15,20)(16,19)(21,24)(22,23)(25,26);
seedw:=G!(1,25)(4,26)(5,22)(6,7)(8,21)(9,14)(10,12)(11,13)(15,16)(17,20)
(18,23)(19,24);end if;

if n mod 24 eq 3 then
m:=27;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)
(10, 11)(14, 15)(17, 21)(18, 22)(23, 24)(25, 26); seedz:=G!(1, 13)(2, 14)
(3, 4)(5, 15)(6, 16)(9, 10)(17, 20)(18, 19)(23, 27)(25, 26); seedw:=G!
(1, 18)(2, 3)(4, 17)(5, 21)(8, 22)(10, 24)(11, 23)(14, 26)(15, 25)(19, 27);
 end if;

if n mod 24 eq 4 then
m:=28;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)
(10, 11)(14, 15)(17, 21)(18, 22)(23, 24)(25, 26); seedz:=G!(1, 2)(3, 10)
(4, 9)(7, 12)(8, 11)(15, 16)(17, 19)(18, 20)(24, 28)(26, 27); seedw:=G!
(1, 18)(2, 3)(4, 17)(5, 21)(8, 22)(10, 23)(11, 24)(14, 25)(15, 26)(20, 27);
end if;

if n mod 24 eq 5 then
m:=29;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24); seedy:=G!(1, 8)
(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(18, 19)(21, 25)(22, 26)
(27, 28); seedz:=G!(1, 18)(2, 17)(5, 20)(6, 19)(7, 8)(9, 15)(10, 16)(11, 14)
(12, 13)(21, 23)(22, 24)(28, 29); seedw:=G!(1, 2)(3, 8)(4, 6)(5, 7)(9, 22)
(10, 11)(12, 21)(13, 25)(16, 26)(18, 28)(19, 27)(24, 29);end if;

if n mod 24 eq 6 then
m:=30;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)(29, 30);
seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 13)(10, 14)(15, 22)(16, 17)(18, 19)
(20, 21)(23, 30)(24, 25)(26, 27)(28, 29); seedz:=G!(1, 23)(2, 24)(3, 18)
(4, 17)(5, 28)(6, 27)(7, 21)(8, 22)(9, 11)(10, 12)(15, 30)(16, 29)(19, 25)
(20, 26); seedw:=G!(1, 13)(4, 14)(5, 10)(6, 7)(8, 9)(11, 12)(15, 16)(17, 22)
(18, 20)(19, 21)(23, 24)(25, 30)(26, 28)(27, 29); end if;

if n mod 24 eq 7 then
m:=31;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
```

```
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28);
seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(17, 24)
(18, 19)(20, 21)(22, 23)(26, 27)(29, 30); seedz:=G!(3, 25)(4, 26)(5, 6)
(7, 27)(8, 28)(9, 15)(10, 16)(11, 14)(12, 13)(17, 20)(18, 19)(21, 23)(22, 24)
(30, 31); seedw:=G!(1, 24)(2, 13)(3, 12)(4, 20)(5, 21)(6, 16)(7, 9)(8, 17)
(10, 18)(11, 19)(14, 23)(15, 22)(26, 29)(27, 30); end if;

if n mod 24 eq 8 then
m:=32;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28
)(29, 30)(31, 32); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)
(14, 15)(17, 24)(18, 19)(20, 21)(22, 23)(25, 32)(26, 27)(28, 29)(30, 31);
seedz:=G!(1, 9)(2, 10)(3, 20)(4, 19)(5, 14)(6, 13)(7, 23)(8, 24)(11, 21)
(12, 22)(15, 18)(16, 17)(25, 32)(26, 31)(27, 29)(28, 30); seedw:=G!(1, 2)
(3, 8)(4, 6)(5, 7)(9, 22)(10, 27)(11, 26)(12, 18)(13, 19)(14, 30)(15, 31)
(16, 23)(17, 25)(20, 29)(21, 28)(24, 32);end if;

if n mod 24 eq 9 then
m:=33;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28);
seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(18, 19)
(22, 23)(25, 29)(26, 30)(27, 28)(31, 32); seedz:=G!(1, 21)(2, 22)(3, 4)
(5, 23)(6, 24)(9, 18)(10, 17)(13, 20)(14, 19)(15, 16)(25, 28)(26, 27)(29, 30)
(32, 33); seedw:=G!(2, 29)(3, 25)(4, 5)(6, 26)(7, 30)(9, 11)(10, 16)(12, 15)
(13, 14)(17, 20)(18, 27)(19, 28)(22, 32)(23, 31);end if;

if n mod 24 eq 10 then
m:=34;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)
(29, 30)(31, 32); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)
(14, 15)(17, 24)(18, 19)(20, 21)(22, 23)(26, 27)(29, 33)(30, 34)(31, 32);
seedz:=G!(1, 3)(2, 4)(5, 8)(6, 7)(9, 26)(10, 25)(13, 28)(14, 27)(15, 16)
(17, 23)(18, 24)(19, 22)(20, 21)(29, 32)(30, 31)(33, 34); seedw:=G!(1, 14)
(2, 19)(3, 18)(4, 10)(5, 11)(6, 22)(7, 23)(8, 15)(9, 17)(12, 21)(13, 20)
(16, 24)(25, 28)(26, 31)(27, 32)(30, 34);end if;

if n mod 24 eq 11 then
m:=35;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28);
seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(18, 19)
(22, 23)(25, 29)(26, 30)(31, 32)(33, 34); seedz:=G!(1, 23)(2, 24)(5, 21)
(6, 22)(7, 8)(9, 19)(10, 20)(13, 17)(14, 18)(15, 16)(25, 28)(26, 27)(31, 32)
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(34, 35); seedw:=G!(1, 2)(3, 8)(4, 6)(5, 7)(9, 16)(10, 25)(11, 29)(14, 30)
(15, 26)(18, 32)(19, 31)(22, 34)(23, 33)(27, 35);end if;

if n mod 24 eq 12 then
m:=36;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28);
seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(18, 19)
(21, 29)(22, 30)(23, 31)(24, 32)(33, 34); seedz:=G!(1, 8)(2, 7)(3, 5)(4, 6)
(11, 20)(12, 19)(13, 14)(15, 18)(16, 17)(21, 26)(22, 25)(23, 27)(24, 28)
(33, 35); seedw:=G!(1, 32)(4, 31)(5, 23)(6, 7)(8, 24)(9, 30)(12, 29)(13, 21)
(14, 15)(16, 22)(18, 34)(19, 33)(26, 36)(27, 35);end if;

if n mod 24 eq 13 then
m:=37;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)
(29, 30)(31, 32); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)
(14, 15)(17, 24)(18, 19)(20, 21)(22, 23)(26, 27)(30, 31)(33, 34)(35, 36);
seedz:=G!(1, 26)(2, 25)(5, 28)(6, 27)(7, 8)(9, 12)(10, 11)(13, 15)(14, 16)
(17, 29)(18, 30)(19, 20)(21, 31)(22, 32)(33, 37)(35, 36); seedw:=G!(1, 14)
(2, 19)(3, 18)(4, 10)(5, 11)(6, 22)(7, 23)(8, 15)(9, 17)(12, 21)(13, 20)
(16, 24)(26, 33)(27, 34)(30, 35)(31, 36);end if;

if n mod 24 eq 14 then
m:=14;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(9, 13)(10, 14); seedz:=G!(1, 8)
(2, 7)(3, 5)(4, 6)(9, 11)(10, 12); seedw:=G!(1, 13)(4, 14)(5, 10)(6, 7)(8, 9)
(11, 12);end if;

if n mod 24 eq 15 then
m:=15;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(13, 14); seedz:=G!(1, 2)
(3, 10)(4, 9)(7, 12)(8, 11)(14, 15); seedw:=G!(1, 6)(2, 4)(3, 5)(7, 8)
(10, 13)(11, 14);end if;

if n mod 24 eq 16 then
m:=40;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)
(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40); seedy:=G!(1, 8)(2, 3)
(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(17, 24)(18, 19)(20, 21)(22, 23)
(25, 32)(26, 27)(28, 29)(30, 31)(33, 40)(34, 35)(36, 37)(38, 39); seedz:=G!
(1, 3)(2, 4)(5, 8)(6, 7)(9, 25)(10, 26)(11, 21)(12, 22)(13, 30)(14, 29)
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(15, 18)(16, 17)(19, 28)(20, 27)(23, 31)(24, 32)(33, 35)(34, 36)(37, 40)
(38, 39); seedw:=G!(1, 38)(2, 19)(3, 18)(4, 34)(5, 35)(6, 22)(7, 23)(8, 39)
(9, 10)(11, 16)(12, 14)(13, 15)(17, 33)(20, 37)(21, 36)(24, 40)(25, 26)
(27, 32)(28, 30)(29, 31);end if;

if n mod 24 eq 17 then
m:=41;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(21, 22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)
(33, 34)(35, 36); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(13, 17)(14, 18)
(19, 20)(21, 28)(22, 23)(24, 25)(26, 27)(30, 31)(33, 37)(34, 38)(39, 40);
seedz:=G!(1, 31)(2, 32)(5, 29)(6, 30)(7, 8)(9, 24)(10, 23)(11, 28)(12, 27)
(13, 15)(14, 16)(19, 20)(21, 22)(33, 35)(34, 36)(40, 41); seedw:=G!(1, 14)
(2, 3)(4, 13)(5, 17)(8, 18)(10, 19)(11, 20)(15, 16)(21, 37)(24, 38)(25, 34)
(26, 27)(28, 33)(30, 39)(31, 40)(36, 41);end if;

if n mod 24 eq 18 then
m:=18;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(13, 17)
(14, 18)(15, 16); seedz:=G!(1, 8)(2, 7)(3, 5)(4, 6)(9, 10)(13, 15)(14, 16)
(17, 18); seedw:=G!(1, 17)(4, 18)(5, 14)(6, 7)(8, 13)(9, 12)(10, 15)(11, 16);end if;

if n mod 24 eq 19 then
m:=43;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)
(35, 36)(37, 38); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(15, 17)(16, 18)
(19, 20)(23, 30)(24, 25)(26, 27)(28, 29)(32, 33)(35, 39)(36, 40)(41, 42);
seedz:=G!(1, 33)(2, 34)(5, 31)(6, 32)(7, 8)(9, 27)(10, 28)(11, 23)(12, 24)
(13, 16)(14, 15)(20, 21)(29, 30)(35, 37)(36, 38)(42, 43); seedw:=G!(2, 17)
(3, 15)(4, 5)(6, 16)(7, 18)(10, 19)(11, 20)(14, 22)(23, 39)(26, 40)(27, 36)
(28, 29)(30, 35)(32, 41)(33, 42)(38, 43);end if;

if n mod 24 eq 20 then
m:=20;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(13, 17)
(14, 18)(19, 20); seedz:=G!(1, 2)(3, 10)(4, 9)(7, 12)(8, 11)(13, 15)(14, 16)
(19, 20); seedw:=G!(1, 14)(2, 3)(4, 13)(5, 17)(8, 18)(10, 19)(11, 20)(15, 16);end if;

if n mod 24 eq 21 then
m:=21;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(15, 17)
(16, 18)(19, 20); seedz:=G!(1, 12)(2, 11)(3, 4)(5, 10)(6, 9)(13, 16)(14, 15)
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(19, 21); seedw:=G!(1, 3)(2, 8)(4, 7)(5, 6)(10, 20)(11, 19)(14, 21)(16, 18);end if;

if n mod 24 eq 22 then
m:=22;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16); seedy:=G!(1, 8)(2, 3)(4, 5)(6, 7)(10, 11)(13, 17)
(14, 18)(19, 20); seedz:=G!(1, 11)(2, 12)(5, 9)(6, 10)(7, 8)(13, 16)(14, 15)
(20, 21); seedw:=G!(1, 8)(2, 14)(3, 18)(6, 17)(7, 13)(10, 20)(11, 19)(15, 22);end if;

if n mod 24 eq 23 then
m:=47;
l:=n-m;a,k:=IsDivisibleBy(l,24); seedx:=G!(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)
(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)
(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40)(41, 42)(43, 44); seedy:=G!
(1, 8)(2, 3)(4, 5)(6, 7)(9, 16)(10, 11)(12, 13)(14, 15)(17, 24)(18, 19)
(20, 21)(22, 23)(25, 32)(26, 27)(28, 29)(30, 31)(33, 40)(34, 35)(36, 37)
(38, 39)(42, 43)(45, 46); seedz:=G!(1, 9)(2, 10)(3, 20)(4, 19)(5, 14)(6, 13)
(7, 23)(8, 24)(11, 21)(12, 22)(15, 18)(16, 17)(25, 43)(26, 44)(29, 41)
(30, 42)(31, 32)(33, 35)(34, 36)(37, 40)(38, 39)(45, 47); seedw:=G!(1, 2)
(3, 8)(4, 6)(5, 7)(9, 10)(11, 16)(12, 14)(13, 15)(17, 30)(18, 35)(19, 34)
(20, 26)(21, 27)(22, 38)(23, 39)(24, 31)(25, 33)(28, 37)(29, 36)(32, 40)
(42, 45)(43, 46);end if;

if l eq 0 then x:=seedx;y:=seedy;z:=seedz;w:=seedw;
else
seedz:=seedz^G!(1, n-23)(2, n-22)(3, n-21)(4, n-20)(5, n-19)(6, n-18)(7, n-17)(8, n-16);
x:=[];y:=[];z:=[];w:=[];
for i:=1 to k do x[i]:=G!(m+(i-1)*24+1, m+(i-1)*24+2)(m+(i-1)*24+3, m+(i-1)*24+4)
(m+(i-1)*24+5, m+(i-1)*24+6)(m+(i-1)*24+7, m+(i-1)*24+8)(m+(i-1)*24+9, m+(i-1)*24+10)
(m+(i-1)*24+11, m+(i-1)*24+12)(m+(i-1)*24+13, m+(i-1)*24+14)
(m+(i-1)*24+15, m+(i-1)*24+16)(m+(i-1)*24+17, m+(i-1)*24+18)
(m+(i-1)*24+19, m+(i-1)*24+20)(m+(i-1)*24+21, m+(i-1)*24+22)
(m+(i-1)*24+23, m+(i-1)*24+24);
y[i]:=G!(m+(i-1)*24+1, m+(i-1)*24+8)(m+(i-1)*24+2, m+(i-1)*24+3)
(m+(i-1)*24+4, m+(i-1)*24+5)(m+(i-1)*24+6, m+(i-1)*24+7)(m+(i-1)*24+9, m+(i-1)*24+16)
(m+(i-1)*24+10, m+(i-1)*24+11)(m+(i-1)*24+12, m+(i-1)*24+13)(m+(i-1)*24+14,m+(i-1)*24+15)
(m+(i-1)*24+17, m+(i-1)*24+24)(m+(i-1)*24+18, m+(i-1)*24+19)(m+(i-1)*24+20, m+(i-1)*24+21)
(m+(i-1)*24+22, m+(i-1)*24+23);
w[i]:=G!(m+(i-1)*24+1, m+(i-1)*24+14)(m+(i-1)*24+2, m+(i-1)*24+19)
(m+(i-1)*24+3, m+(i-1)*24+18)(m+(i-1)*24+4, m+(i-1)*24+10)(m+(i-1)*24+5, m+(i-1)*24+11)
(m+(i-1)*24+6, m+(i-1)*24+22)(m+(i-1)*24+7, m+(i-1)*24+23)(m+(i-1)*24+8, m+(i-1)*24+15)
(m+(i-1)*24+9, m+(i-1)*24+17)(m+(i-1)*24+12, m+(i-1)*24+21)(m+(i-1)*24+13, m+(i-1)*24+20)
(m+(i-1)*24+16, m+(i-1)*24+24);end for;
z[1]:=G!(1, m+9)(2, m+10)(3, m+20)(4, m+19)(5, m+14)(6, m+13)(7, m+23)(8, m+24)(m+11, m+21)
(m+12, m+22)(m+15, m+18)(m+16, m+17);
for i:=2 to k do z[i]:=G!(m+(i-2)*24+1, m+(i-1)*24+9)(m+(i-2)*24+2, m+(i-1)*24+10)
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(m+(i-2)*24+3, m+(i-1)*24+20)(m+(i-2)*24+4, m+(i-1)*24+19)(m+(i-2)*24+5, m+(i-1)*24+14)
(m+(i-2)*24+6, m+(i-1)*24+13)(m+(i-2)*24+7, m+(i-1)*24+23)(m+(i-2)*24+8, m+(i-1)*24+24)
(m+(i-1)*24+11, m+(i-1)*24+21)(m+(i-1)*24+12, m+(i-1)*24+22)(m+(i-1)*24+15, m+(i-1)*24+18)
(m+(i-1)*24+16, m+(i-1)*24+17); end for;
X:=Id(G);for i:=1 to k do X:=X*x[i];end for; Y:=Id(G);for i:=1 to k do Y:=Y*y[i];end for;
Z:=Id(G);for i:=1 to k do Z:=Z*z[i];end for; W:=Id(G);for i:=1 to k do W:=W*w[i];end for;
x:=seedx*X;y:=seedy*Y;z:=seedz*Z;w:=seedw*W;end if;
P1:=sub<G|x,y,z>;P2:=sub<G|x,y,w>;B:=sub<G|x,y>;
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