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Abstract. The N-dimensional border collision normal form describes bifurcations of piecewise smooth systems. It is shown that there is an open set of parameters such that on one side of the bifurcation the map has a stable fixed point and on the other an attractor with Hausdorff dimension N. For generic parameters this attractor contains open sets and hence has topological dimension equal to N.

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1. Introduction

Piecewise smooth dynamical systems arise in many applications in control theory, the theory of electronic circuite and biological modelling [7]. They are characterized by regions of phase space in which the evolution is smooth separated by switching surfaces across which the evolution rule changes. Local analysis of special cases leads to the derivation of truncated systems called normal forms that capture much of the dynamics close to the situation being described. One such system is the border collision normal form which describes the local behaviour of a generic piecewise smooth map close to a bifurcation value of parameters at which a fixed point intersects a switching surface [18]. In the case of a two dimensional phase space the different changes of behaviour that can occur has been extensively studied [1, 2, 3, 7, 12, 16, 17, 18, 21]. Banerjee et al [3] show that stable periodic obits or fixed points can be destroyed in the bifurcation, with either new periodic orbits, or chaotic sets with fractal strucure being created after the bifurcation value of the parameter. Glendinning shows that a stable fixed point can be destroyed to create a fully two-dimensional attractor on open sets of parameters [14].

Higher dimensional phenomena are less well documented. The general normal form can be written down and it is possible to describe the existence and stability of fixed points [5, 8]. Glendinning and Jeffrey [15] show that these normal forms arise in the bifurcation of continuous time systems, and that after bifurcation there can be attractors with the same dimension as the ambient phase space [13]. However, those results do not consider the dynamics before the bifurcation, and the examples are such that no stable periodic orbits can exist in the system. The aim of this paper is to show that the same jump of dimension possible in two dimensions described in [14] occurs in general: there are border collision normal forms in $\mathbb{R}^N$ for which the dynamics on one side of the bifurcation is as simple as possible, a stable fixed point, whilst on the other side of the bifurcation there is an attractor with topological dimension equal to N. Moreover, this occurs generically in an open set of parameter values for the normal form.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_N)^T$ in $\mathbb{R}^N$, $N \geq 2$, then a border-collision system is the difference equation

$$\mathbf{x}(t+1) = F(\mathbf{x}(t)) = \begin{cases} A_0\mathbf{x}(t) + \mathbf{m} & \text{if } x_1(t) \leq 0 \\ A_1\mathbf{x}(t) + \mathbf{m} & \text{if } x_1(t) \geq 0 \end{cases} \tag{1}$$

where $t \in \mathbb{N}$, A_0 and A_1 are constant matrices, $\mathbf{m}$ is a constant vector, and the system is continuous across the surface $x_1 = 0$. For generic choices of A_0 and A_1 an affine change of coordinates can be used to transform the matrices into observer canonical form [5, 18] and the vector $\mathbf{m}$ to a vector that is zero apart from the first coordinate. In N dimensions this means that generically

$$A_k = \begin{pmatrix} -a_{k,1} & 1 & 0 & \cdots & 0 \\ -a_{k,2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{k,N-1} & 0 & 0 & \cdots & 1 \\ -a_{k,N} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

for $k = 0, 1$. The difference equation (1) with constants given by (2) is called the border collision normal form. By a linear change of variables the constant μ can be rescaled to the sign of μ and so in the analysis of the dynamics we may take $\mu = -1$ if $\mu < 0$ and $\mu = 1$ if $\mu > 1$ without loss of generality. The special bifurcation value $\mu = 0$ will not be considered here.

The main result of this paper is the following theorem.

Theorem 1 *There exists an open set* $\mathcal{U} \subset \mathbb{R}^{2N}$ *such that if* $(a_{0,1}, \ldots, a_{0,N}, a_{1,1}, \ldots a_{1,N}) \in \mathcal{U}$ *then the border collision normal form has a stable fixed point if* $\mu < 0$ *and an attractor with Hausdorff dimension equal to* N *if* $\mu > 0$. *This attractor has topological dimension equal to* N *generically in* $\mathcal{U}$.

No attempt will be made to optimize the set $\mathcal{U}$ of the theorem, it will be enough to choose the set

$$|a_{k,j}| < \epsilon, \ 1 \leq j < N, \ k = 0, 1; \ 0 < a_{1,N} < 2 - \epsilon; \ \tfrac{6}{11} < -a_{0,N} < \tfrac{10}{11} \tag{3}$$

for sufficiently small $\epsilon > 0$.

As in [14] the main theoretical results used are those of Buzzi [9, 11] and Tsujii [20]. However, in the two-dimensional case it is possible to provide a detailed description of carefully chosen regions and their images. In the higher dimensional cases there is a combinatoric proliferation of boundary points because a hypercube in $\mathbb{R}^N$ has 2^N vertices, and this makes the approach taken in [14] infeasible here. Instead an alternative, and in many ways simpler, strategy is taken.

In section two the simple stable fixed point is shown to exist if $\mu < 0$ and the remaining parameter values are in $\mathcal{U}$. In section three an invariant region is defined for $\mu > 0$ and in section four a criterion for expansion of a suitable iterate of the border collision normal form is determined. Theorem 1 follows from this expansion result together with the results of Buzzi and Tsujii.

2. The stable fixed point

If $\mu < 0$ then by rescaling variables we may take $\mu = -1$ without loss of generality. If there is a fixed point $(x_1^*, x_2^*, \dots, x_N^*)$ in $x_1 < 0$ with parameters satisfying (3) then it satisfies

$$a_{0,1}x_1^* + x_2^* - 1 = x^*, \quad x_{k+1}^* = x_k^* + O(\epsilon),\ k = 2, \dots N-1, \quad x_N^* = a_{0,N}x_1^*. \tag{4}$$

Let $-a_{0,N} = \omega > 0$, then $x_2^* = -\omega x_1^* + O(\epsilon)$ and so

$$x_1^* = -\frac{1}{1-\omega} + O(\epsilon).$$

Thus provided ω is less than one the fixed point exists for sufficiently small ϵ. The stability is determined from the eigenvalues of A_0, i.e. the roots of the characteristic polynomial

$$s^N + \sum_{r=0}^{N-1} a_{0,N-r}s^r = 0.$$

If $\epsilon = 0$ the characteristic equation is $s^N = \omega$ and so there are N roots with modulus equal to $\omega^{\frac{1}{N}}$ which is less than one. Hence for sufficiently small $\epsilon > 0$ the roots are all within the unit circle and the fixed point is stable.

3. An invariant region

Consider the rectangular region in $\mathbb{R}^N$ defined by

$$-(1+b_1) \le x_1 \le 1 + a_1, \quad -(2+b_k) \le x_k \le a_k, \quad k = 2, \dots, N. \tag{5}$$

with constants (a_k) and (b_k) positive and less than one, Then if $\mu = 1$ the images $(x_1', x_2', \dots, x_N')$ of points in this region satisfy

$$\begin{aligned} \min{}_k(a_{k,1}(1+b_1), \quad & -a_{k,1}(1+a_1)) - (2+b_2) + 1 < x_1' \\ & < \max{}_k(a_{k,1}(1+b_1), -a_{k,1}(1+a_1)) + a_2 + 1 \\ \min{}_k(a_{k,j}(1+b_1), \quad & -a_{k,j}(1+a_1)) - (2+b_{j+1}) < x_j' \\ & < \max{}_k(a_{k,j}(1+b_1), -a_{k,j}(1+a_1)) + a_{j+1} \end{aligned} \tag{6}$$

$j = 2, \dots, N-1$, whilst if $x_1 < 0$ and $\omega = -a_{0,N}$ then

$$-\omega(1+b_1) < x_N' < 0 \tag{7}$$

whilst if $x_1 > 0$ then

$$-(2-\delta)(1+a_1) < x'_N < 0 \tag{8}$$

with $0 \leq \delta < \epsilon$.

Comparison of (8) with (5) with $j = N$ and $\epsilon \to 0$ shows that $2a_0 < b_N$, and more generally using (6) $a_{k+1} < a_k$ and $b_{k+1} < b_k$, if the region is to be invariant. This motivates the following lemma. Again, no attempt has been made to optimize the sizes of the regions, the important feature is that the criteria are satisfied on an open set of parameter values.

Lemma 1 *Let*

$$a_k = 2(N+1-k)\epsilon \quad \text{and} \quad b_k = \frac{1}{20} - 2k\epsilon, \tag{9}$$

$k = 1, \ldots, N$. *If* $\epsilon > 0$ *is sufficiently small then (5) is an invariant region for the border collision normal form with* $\mu = 1$ *and coefficients satisfying (3).*

Proof: The proof is by direct verification using (5), (6) and (8). Choose $\epsilon > 0$ so that $b_1 > b_2 > \ldots > b_N > 2a_1$, i.e.

$$0 < \epsilon < \frac{1}{120N}.$$

Then $b_1 > a_1$ and so the maximum modulus of $a_{k,j}(1+b_1)$ and $a_{k,j}(1+a_1)$ is $\epsilon(1+b_1)$. Thus for x'_1 defined by (6) to be in the interval defined in (5)

$$1 + b_1 > 1 + b_2 + \epsilon(1+b_1), \quad 1 + a_1 > 1 + a_2 + \epsilon(1+b_1)$$

and since

$$b_1 - b_2 = a_1 - a_2 = 2\epsilon > (1+b_1)\epsilon$$

both these inequalities hold.

Similarly for the coordinates x'_k, $k = 2, \ldots, N-1$, the interval of possible values for the image is contained in the original region provided

$$2 + b_j > 2 + b_{j+1} + \epsilon(1+b_1) < 2, \quad a_j > a_{j+1} + \epsilon(1+b_1)$$

which hold for the same reason. Since $b_1\omega < 1$, $2 + b_n > \omega(1+b_1)$ so the iterate defined by (7) is in the defined region, and finally from (8), the region is invariant if

$$2 + b_N > 2(1+a_1),$$

i.e. $b_N > 2a_1$ as noted earlier.

□

4. Expansion

Piecewise affine expanding maps are maps defined on polyhedral regions (regions bounded by hyperplanes) such that in each region the map is expanding in some metric. General results of Buzzi [9, 11] and Tsujii [20] show that if a piecewise affine map is defined on a bounded polyhedral region in $\mathbb{R}^N$ then there is an attractor with an invariant measure which is absolutely continuous with respect to Lebesgue measure [11, 20] and hence has Hausdorff dimension equal to N. Moreover, for generic piecewise expanding maps the attractor contains open sets [9, 11]. The border collision normal form is not expanding in the Euclidean metric, although there are equivalent metrics for which induced maps are expanding. This creates some extra technical difficulties in [13, 14], and by using an appropriate induced map it is shown below that these difficulties can be avoided, making it possible to work with the Euclidean metric throughout the analysis.

The structure of the normal form matrices A_0 and A_1 of (2) is such that for small ϵ in (3) and high enough iterates of the map, expansion in the Euclidean metric is relatively straightforward. Let (α_r) be any countable set of real numbers, $r \in \mathbb{N}$, and let $M(r)$ denote the $N \times N$ matrix

$$M(r)_{ij} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \dots, N-1, \\ \alpha_r & \text{if } i = N,\ j = 1 \\ 0 & \text{otherwise} \end{cases}$$

then an elementary manipulation shows that the product

$$M(N) \dots M(1) = \text{diag } (\alpha_1, \dots \alpha_N)$$

and hence

$$M(2N)M(2N-1) \dots M(2)M(1) = \text{diag } (\alpha_1 \alpha_{N+1}, \dots, \alpha_N \alpha_{2N}). \tag{10}$$

If $\epsilon = 0$ then A_0 and A_1 are matrices of the form $M(r)$ with $\alpha_r = -a_{0,N}$ or $\alpha_r = -a_{1,N}$. Since $a_{1,N}^2$ is close to 4 and $|a_{0,N}a_{1,N}|$ is greater than $\frac{12}{11}$ to order ϵ for parameters satisfying (3), the only products of $2N$ matrices A_0 and A_1 which can produce contracting diagonal terms are those with A_0 as both the i^{th} term and the $(N+i)^{th}$ term of the product. The next lemma shows that this cannot happen for the parameters chosen here.

Lemma 2 *Consider the border collision normal form in $\mathbb{R}^N$ with $\mu = 1$ and parameters satisfying (3) with $\epsilon > 0$ sufficiently small. If $\mathbf{x}$ lies in the invariant region of Lemma 1 and $x_1 < 0$ then $F^N(\mathbf{x})$ lies in $x_1 > 0$.*

Proof: Let $\pi_k \mathbf{x}$ denote the k^{th} coordinate of $\mathbf{x}$. Then $x_1 = \pi_1(\mathbf{x}) < 0$ implies that $\pi_N(F(\mathbf{x})) = \omega x_1 < 0$, with $\omega = -a_{N,0} > 0$. Then $\pi_{N-1}(F^2(\mathbf{x})) = \omega x_1 + O(\epsilon)$ and continuing up the orbit $\pi_1(F^N(\mathbf{x})) = \omega x_1 + 1 + O(\epsilon)$. Since $x_1 \in (-\frac{21}{20}, 0)$ by Lemma 1, $|\omega x_1| < \frac{21}{20}\frac{10}{11} = \frac{21}{22}$, and so if ϵ is sufficiently small so that the order ϵ terms are less than $\frac{1}{20}$ in magnitude, $\pi_1(F^N(\mathbf{x})) > 0$.

□

Now consider the iterated map $G = F^{2N}$. G is itself a piecewise smooth map on domains that are polyhedral (as they are the preimages of polyhedral regions). The linear part of each component of F^{2N} is a product of matrices of the form $M(r) + O(\epsilon)$ with $\alpha_r \in \{\omega, -2\}$ and hence the linear parts are all ϵ-close to diagonal matrices with coefficients in $\{4, -2\omega\}$ as the third possibility, ω^2 cannot occur by Lemma 2. Hence each linear part is expanding in the Euclidean metric for sufficiently small $\epsilon > 0$ and so the results of Buzzi and Tsujii hold for G, so G has an absolutely continuous invariant measure for all parameter values in $\mathcal{U}$ and the support of this measure contains open sets generically. de Melo and van Strien [4] (see also [14]) show that this implies that the original map F also has an absolutely continuous invariant measure for all parameter values in $\mathcal{U}$ and it follows from Buzzi's result that these contain open sets generically [11, 14]. This completes the proof of the main theorem, Theorem 1, of this paper.

5. Conclusion

Theorem 1 shows that the two-dimensional result of [14] holds in higher dimensions. In particular, there is a transition from a stable fixed point to an attractor with the highest possible dimension in the ambient phase space for a generic subset of an open set of parameters. This poses questions about how complicated a bifurcation theorem for piecewise smooth system might be. For example, is it possible to find parameters for which there is a transition from a stable fixed point to an attractor with Hausdorff dimension d for every $d \in [0, N]$? There is also a danger that such bifurcation theorems will become little more than (long) lists of possibilities.

One thing is clear: the fact that this bifurcation occurs on open sets of parameter values shows that it is not exceptional, and that dimension reduction techniques will have limited application to certain piecewise smooth systems even if there are stable fixed points at some nearby parameter values.

The border collision normal form can be derived close to grazing-sliding bifurcations of periodic orbits, though in this case it is a lowest order truncation of the dynamical system [6, 15], so an important question is whether the existence of N-dimensional attractors in the truncated, piecewise linear, map implies the existence of N-dimensional attractors in the full map. There are results [10, 19] for piecewise real analytic maps, but the derivation of the border collision normal form from grazing-sliding bifurcations involves non-analytic terms and hence these results cannot be applied. Resolving this problem is important in assessing the implications of the results shown here for applications.

Finally it is unclear whether the restriction to generic parameters in $\mathcal{U}$ for the existence of attractors containing open sets is a real effect or an artefact of the methods of proof used by Buzzi.

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