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Bifurcation from stable fixed point to two-dimensional attractor in the border collision normal form

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The border collision normal form is a family of continuous two-dimensional piecewise smooth maps describing dynamics close to a critical parameter at which a fixed point intersects the switching surface. It is well known that if the fixed point is stable on one side of the bifurcation point then after the bifurcation the system may have stable periodic orbits and/or chaotic attractors with a quasi-one dimensional structure (robust chaos). We show that it is also possible to have a robust transition from a stable fixed point to an attractor with topological dimension two, i.e. the highest dimension possible in the phase space.

Keywords: bifurcation, piecewise smooth dynamics, invariant measures, border collision, normal form.

1. Introduction

Many applications in control theory and many models in biology and mechanics involve sudden changes in the underlying description of the system. This may be due to crossing control thresholds which trigger a different system response, fast changes in gene expression or impacts respectively. This has led to a renewed interest in piecewise smooth dynamics, where the time evolution is defined by one dynamical system in one region of phase space and a different dynamical system in another region. The boundary between two such regions is called a switching surface, and further complications can be added by allowing 'jumps' on the switching surface. For example a bouncing ball reverses the direction of its velocity when it strikes the floor. A survey of the many different phenomena that can occur is given in di Bernardo *et al.* (2008).

In this paper we consider bifurcations that arise in maps with a switching surface across which the dynamics is continuous but not differentiable (so the Jacobian matrix changes discontinuously across the surface) when a fixed point intersects the surface at some critical parameter value. To lowest order the local dynamics is typically described by a piecewise linear (or more accurately piecewise affine) map called the border collision normal form, see Nusse and Yorke (1992). If $z \in \mathbb{R}^2$ then setting $z = \begin{pmatrix} x \\ y \end{pmatrix}$ the border collision normal form in $\mathbb{R}^2$ is

$$\begin{aligned} z_{n+1} &= F(z_n) \\ &= \begin{cases} f_L(z_n) = A_0 z_n + m & \text{if } x_n \leqslant 0 \\ f_R(z_n) = A_1 z_n + m & \text{if } x_n \geqslant 0 \end{cases} \end{aligned} \tag{1.1}$$

where

$$A_0 = \begin{pmatrix} t_0 & 1 \\ -d_0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} t_1 & 1 \\ -d_1 & 0 \end{pmatrix}, \tag{1.2}$$

and $m = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$. The line $x = 0$ is the switching surface and the map is continuous across this line.

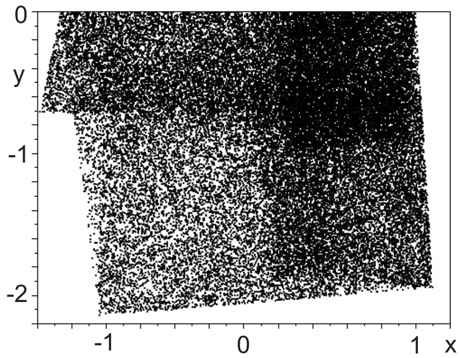

FIG. 1. Attractor of the border collision form with $\mu = 1$, $-d_0 = 0.85$, $d_1 = 1.95$, $t_0 = 0.05$ and $t_1 = 0.1$. The numerical verification that this is a two-dimensional attractor is given after the proof of Theorem 4.1 in section 4. 50000 iterates are shown.

The vector m and hence the real number μ is the bifurcation parameter; by rescaling z only the sign of μ matters in the study of dynamics, and so when describing the dynamics μ can be chosen in the set $\mu \in \{-1, 0, +1\}$ without loss of generality. We will concentrate on dynamics before and after the bifurcation, so the dynamics at the bifurcation point $\mu = 0$ will not be considered further here.

The border collision normal form has been studied by many groups, e.g. Banerjee & Grebogi (1999); Banerjee *et al.* (1998); Gardini (1992); Mira *et al.* (1996); Nusse and Yorke (1992); Young (1985). Banerjee & Grebogi (1999) show that if the bifurcating fixed point is stable then after the bifurcation there can be stable periodic orbits, but also that there can be *robust chaos* associated with intersections of stable and unstable manifolds of fixed points creating chaos with a one-dimensional unstable direction of the sort described in a more general setting by Young (1985).

On the other hand Dobryinskiy (1999); Glendinning (2014b); Glendinning and Jeffrey (2015) and Glendinning and Wong (2011) show that the border collision normal form can have attractors that are two-dimensional, i.e. the attractors contain open sets of phase space. However, the existence of these attractors has not been related to the dynamics on the 'other side' of the bifurcation. In all the examples that are known, the expansion required to generate the fully two-dimensional invariant sets implies that no stable fixed points or periodic orbits can exist – the assumption is that the determinants of the maps on each side of the switching surface has modulus greater than one and hence that the determinant of any iterate of the map has determinant with modulus greater than one and this precludes the existence of stable periodic orbits.

In this paper we prove the existence of bifurcations which have the simplest possible dynamics on one side of the bifurcation, so if the bifurcation parameter μ is negative there is a stable fixed point, whilst if μ is positive then the attractor has the highest possible dimension, namely the dimension of the ambient phase space, two. An example is shown in Figure 1. Moreover, since we use the results of Buzzi (1999, 2001) and Tsujii (2001), the attractor in $\mu > 0$ has an invariant measure which is absolutely continuous with respect to the two dimensional Lebesgue measure. The two-dimensional result is proved on an open set of the four-dimensional parameter space (d_0, d_1, t_0, t_1) and hence cannot be considered in any sense exceptional – it is one of the typical transitions that can occur in these families of maps. The existence and persistence of this transition is the main result of this paper. We conjecture that a similar result holds in n dimensions: that there are bifurcations from stable fixed points to n-dimensional attractors in the n-dimensional border collision normal form, though in this case the result will be generic rather than on open sets of parameter values due to constraints of the general results in higher dimensions.

In the next section we recall the results of Buzzi (1999, 2001) and Tsujii (2001) and define an equivalent metric that can be used to apply these results to the border collision normal form. In section three we describe a special choice of parameters for which the an induced map is expanding, and a particular case for which has dynamics is described by a finite Markov partition, so a much more detailed account of the numbers of periodic orbits and other properties can be deduced. In section four we show how to perturb the model of section three into an open set of parameter values so that (a) the system has a stable fixed point in $\mu < 0$ and (b) the expansion results of Buzzi and Tsujii can be used to prove the existence of an attractor with topological dimension two in $\mu > 0$, proving the result stated above.

2. The results of Buzzi and Tsujii

The main theoretical results used to prove the bifurcation result described in the introduction are those proved by Buzzi (1999, 2001) and Tsujii (2001) for more general piecewise expanding maps. However, since the border collision normal form is not expanding in the Euclidean metric a little work needs to be done to construct equivalent metrics in which it, or appropriate iterates of it, are expanding, so we follow a strategy used in Glendinning (2014a,b). Moreover, since we will be working with iterates of the original map, it is necessary to understand how statements about iterates of the map translate back to statements about the original map. This is done using the Induced Map Lemma (Lemma 2.1, below) which can be found in de Melo and van Strien (1993).

Let $\mathscr{D}$ be a polygonal region in $\mathbb{R}^2$, i.e. a compact connected region whose boundary is a finite union of straight line segments. Let $\mathscr{P}$ be a finite collection of non-intersecting open polygonal regions $\{P_i\}_{i=1}^m$ such that the union of the closures of these polygons is $\mathscr{D}$. Then a map $F : \cup P_i \to \mathscr{D}$ is a *piecewise affine map* if $F|_{P_i}$ is an affine map, $i = \{1,\dots,m\}$. If in addition there exists $\lambda > 1$ and a metric $d : \mathbb{R}^2 \to \mathbb{R}$ such that for each $i \in \{1,\dots,m\}$ $F|_{P_i}$ is expanding, i.e.

$$d(F(x),F(y)) \geqslant \lambda d(x,y) \quad \text{for all } x \in P_i$$

$i = 1,\dots,m$, then F is a *piecewise expanding affine map*. Note that although the border collision bifurcation is continuous the result stated below does not require continuity across the boundaries of the sets P_i.

THEOREM 2.1 (Buzzi (1999, 2001), Tsujii (2001)) Suppose F is a piecewise expanding affine map of a planar polygonal region $\mathscr{D}$. Then there exists an attractor in $\mathscr{D}$ such that F has an absolutely continuous invariant measure on the attractor and the attractor contains open sets.

The significance of the last statement is that the attractor has topological dimension equal to two.

In Glendinning (2014b) this result was applied to the border collision normal form by using the family of metrics defined for points $u = (x_1,y_1)$ and $v = (x_2,y_2)$ by a combination of the Euclidean one-dimensional distance $|.|$:

$$d_\alpha(u,v) = \alpha|x_1 - x_2| + |y_1 - y_2| \tag{2.1}$$

for $\alpha > 0$. The results of Buzzi (1999, 2001) and Tsujii (2001), and those of Glendinning (2014b) apply with some technical modifications in higher dimensions.

As noted earlier, the matrix A_0 will have eigenvalues inside the unit circle so that a fixed point in $x < 0$ is stable, therefore there is no hope that Theorem 2.1 can be applied directly to F in this context. However, it may apply to an induced map defined by different iterates of the original map in different polygonal domains.

A collection of disjoint open polygons $\mathscr{B} = \{B_k\}_1^M$ is a refinement of a partition $\mathscr{P} = \{P_i\}_1^m$ if the union of the closures of B_k is $\mathscr{D}$ and the closure of each set $P_i \in \mathscr{P}$ is the union of the closures of sets $B_k \in \mathscr{B}$. An induced map $G : \cup B_k \to \mathscr{D}$ is a map defined for each $j \in \{1, \ldots, M\}$ by

$$G(z) = F^{k(j)}(z) \quad \text{if } z \in B_j, \tag{2.2}$$

with $k(j) \geqslant 1$ and for each j, $F^r(B_j) \subseteq P_{\pi(r)}$ for $r = 1, \ldots, k(j)$, $\pi(r) \in \{1, \ldots, m\}$. Although F is continuous across the switching surface in the border collision normal form, induced maps can have discontinuities, hence the need to use the more general setting of Buzzi and Tsujii. In some situations it is unnecessary to define induced maps on the whole of $\mathscr{D}$ outside the boundaries of the sets (which are of measure zero).

Induced maps will be exploited in subsequent sections, but in this case Theorem 2.1 will apply to the induced map and not the original map. To understand how results for induced maps extend to the original map we use a result that can be found, for example, in de Melo and van Strien (1993).

LEMMA 2.1 (Induced Map Lemma) Suppose that a piecewise affine map F has an induced map G of the form (2.2). If G is a piecewise expanding affine map then F has an invariant measure absolutely continuous with respect to Lebesgue.

Proof: Since G is piecewise expanding it has an absolutely continuous invariant measure, μ. We need to show that this can be extended to an absolutely continuous invariant measure for the underlying map F. The argument follows de Melo and van Strien (1993), but is included for completeness.

Let μ_i be the measure μ restricted to the partition defining G, $\{B_k\}_1^M$, i.e. $\mu_i(A) = \mu(A \cap B_i)$ for any measureable set A and $\mu = \sum \mu_j$. Define the push-forward measure $G_*\mu = \mu(G^{-1}(A))$, so since μ is an invariant measure it satisfies $G_*\mu = \mu$, i.e.

$$\sum_{j=1}^{M} F_*^{k(j)} \mu_j = \sum_{j=1}^{M} G_* \mu_j = G_*\mu = \mu. \tag{2.3}$$

We look for a measure ν such that $F_*\nu = \nu$; this will be an invariant measure for F. Define

$$\nu = \sum_{j=1}^{M} \left(\sum_{r=0}^{k(j)-1} F_*^k \mu_j \right). \tag{2.4}$$

Since $F_*\mu$ is a linear operator on μ (it is the transfer operator), (2.4) implies that

$$\begin{aligned} F_*\nu &= \textstyle\sum_{j=1}^{M} \left(\sum_{r=1}^{k(j)} F_*^k \mu_j \right) \\ &= \textstyle\sum_{j=1}^{M} \left(\sum_{r=1}^{k(j)-1} F_*^k \mu_j \right) + \sum_{j=1}^{M} F_*^{k(j)} \mu_j. \end{aligned} \tag{2.5}$$

But by (2.3) the last term is just $\mu = \sum \mu_j$ and so reinserting this sum into the first term of (2.5) we recover (2.4), i.e. $F_*\nu = \nu$, so ν is an invariant measure for F.

Finally, suppose $\ell(A) = 0$, where ℓ is the standard Lebesgue measure. To prove that ν is absolutely continuous with respect to Lebesgue measure we need to show that $\nu(A) = 0$. Since μ is absolutely continuous, $\mu_i(A) = 0$ and hence $F_*^r \mu_i(A) = 0$ for all $r \geqslant 0$ as F is measureable. Thus $\nu(A) = 0$ as required and ν is absolutely continuous with respect to ℓ.

$\square$

3. A special case

In this section we consider the specific parameter values $t_0 = t_1 = 0$, $d_0 = 2$ and $d_0 = -\omega$, $\frac{1}{\sqrt{2}} < \omega < 1$.

If $\mu < 0$ there is a stable fixed point in $x < 0$. If $\mu > 0$ then without loss of generality let $\mu = 1$, and note that the square

$$S = \{(x,y) \mid -1 \leqslant x \leqslant 1,\ -2 \leqslant y \leqslant 0\}$$

is invariant. We will prove it contains a fully two-dimensional attractor.

LEMMA 3.1 Suppose $t_0 = t_1 = 0$, $d_0 = -\omega$, $\frac{1}{\sqrt{2}} < \omega < 1$ and $d_1 = 2$. If $\mu < 0$ then the border collision normal form has a stable fixed point in $x < 0$. If $\mu > 0$ then there is an attracting set $\mathscr{A} \subseteq S$ which contains open sets and has an absolutely continuous invariant measure.

Proof: If $\mu < 0$ this is a trivial calculation as noted above: there is a fixed point at $(\frac{\mu}{1-\omega}, \frac{\omega\mu}{1-\mu})$ and the eigenvalues of the Jacobian are $\pm\sqrt{\omega}$ which have modulus less than one, so the fixed point is stable.

If $\mu > 0$ consider (without loss of generality) the case $\mu = 1$. Partition S into four rectangles

$$\begin{aligned} L_1 &= \{(x,y) \mid -1 < x < 0,\ -1 < y < 0\}, \\ L_2 &= \{(x,y) \mid -1 < x < 0,\ -2 < y < -1\}, \\ R_1 &= \{(x,y) \mid 0 < x < 1,\ -1 < y < 0\}, \\ R_2 &= \{(x,y) \mid 0 < x < 1,\ -2 < y < -1\}. \end{aligned} \tag{3.1}$$

By direct calculation the images of the corners of these rectangles are:

$$\begin{array}{lll} (-1,-2) \to (-1,-\omega), & (-1,-1) \to (0,-\omega), & (-1,0) \to (1,-\omega), \\ (0,-2) \to (-1,0), & (0,-1) \to (0,0), & (0,0) \to (1,0), \\ (1,-2) \to (-1,-2) & (1,-1) \to (0,-2), & (1,0) \to (1,-2), \end{array} \tag{3.2}$$

and hence

$$f_L(L_1) \subseteq R_1, \quad f_L(L_2) \subseteq L_1, \quad f_R(R_1) \subseteq R_1 \cup R_2, \quad f_R(R_2) \subseteq L_2. \tag{3.3}$$

Now, if $z \in L_1$ then $f_L(z) \in R_1$ and so the second iterate of the border collision normal form is $f_R f_L(z) \in f_R(R_1) \subseteq R_1 \cup R_2$. Hence the third iterate is $f_R^2 f_L(z) \in R_1 \cup R_2 \cup L_2$.

Similarly, if if $z \in L_2$ then $f_L(z) \in L_1$ and so the second iterate of the border collision normal form is $f_L^2(z) \in f_L(L_1) \subseteq R_1$. Now we are back to the previous case and the fourth iterate is $f_R^2 f_L^2(z) \in R_1 \cup R_2 \cup L_2$.

Thus the induced map G defined by

$$G(z) = \begin{cases} f_R^2 f_L(z) & \text{if } z \in L_1 \\ f_R^2 f_L^2(z) & \text{if } z \in L_2 \\ f_R(z) & \text{if } z \in R_1 \cup R_2 \end{cases} \tag{3.4}$$

is well-defined. Iterates of G are iterates of the original border collision normal form (omitting some points on the trajectory) and and G is a piecewise affine map on the regions (as in the definition of piecewise affine maps in the previous section G is not defined on the boundaries of the regions). So G has an attractor in S and this attractor is a subset (possibly improper) of $\mathscr{A}$, the attractor of the border collision normal form. Thus if each component map is expanding the theorems of Tsujii and Buzzi hold and hence the attractor of G and hence $\mathscr{A}$ contains open sets.

There are three maps for which the expanding condition needs to hold simultaneously. The Jacobian of the map $f_R^2 f_L$ is

$$A_1^2 A_0 = \begin{pmatrix} 0 & -2 \\ -2\omega & 0 \end{pmatrix}$$

and so if $z_k = (x_k, y_k) \in L_1$, $k = 1, 2$ then

$$d_\alpha(G(z_1), G(z_2)) = 2\alpha|y_1 - y_2| + 2\omega|x_1 - x_2| = \tfrac{2\omega}{\alpha}\left(\alpha|x_1 - x_2| + \tfrac{\alpha^2}{\omega}|y_1 - y_2|\right)$$

and so provided $\alpha^2 > \omega$

$$d_\alpha(G(z_1), G(z_2)) \geqslant \tfrac{2\omega}{\alpha} d_\alpha(z_1, z_2)$$

so G is expanding on L_1 in the metric d_α provided

$$\sqrt{\omega} < \alpha < 2\omega. \tag{3.5}$$

Now consider $z \in L_2$. The Jacobian of G is

$$A_1^2 A_0^2 = \begin{pmatrix} -2\omega & 0 \\ 0 & -2\omega \end{pmatrix}$$

and so a simple calculation shows that G is expanding on L_2 in the metric d_α provided $2\omega > 1$ which is true by assumption. Finally consider G in $R_1 \cup R_2$. It has Jacobian A_1 and hence

$$d_\alpha(G(z_1), G(z_2)) = \alpha|y_1 - y_2| + 2|x_1 - x_2| = \tfrac{2}{\alpha}\left(\alpha|x_1 - x_2| + \tfrac{\alpha^2}{2}|y_1 - y_2|\right).$$

Hence G is expanding on $R_1 \cup R_2$ provided $\alpha^2 > 2$ and $\frac{2}{\alpha} > 1$ or

$$\sqrt{2} < \alpha < 2. \tag{3.6}$$

Comparing (3.5) and (3.6) we see that G is expanding in all regions provided $2\omega > \sqrt{2}$, or $\omega > 1/\sqrt{2}$, which is the condition specified in the lemma.

Thus G is a piecewise expanding map in S and the lemma follows from the results of Tsujii and Buzzi together with the induced map lemma stated earlier.

□

In the special boundary case $\omega = 1$ the dynamics with $\mu = 1$ can be described precisely since (3.2) implies that (3.3) holds with equality in all the inclusions and hence there is a finite Markov partition.

4. Expansion on open sets of parameters

In this section we show that the expansion argument of the proof of Lemma 3.1 extend to an open set of parameter values. Unfortunately, the regions defining the partition cannot be left independent of parameters, so the calculations are a little less elegant although many of the calculations are essentially the same. The first step is to understand how small perturbations of the linear part of the map change the conditions required for expansion in d_α for some $\alpha > 0$.

Let

$$m = \min\left(\frac{|d|}{|b|}, \frac{|d| - |a| + \sqrt{(|d| - |a|)^2 + 4|bc|}}{2|b|}\right)$$

and let M denote the maximum of the same two quantities.

LEMMA 4.1 Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then A is expanding in the metric d_α if either

$$M < \alpha < \frac{|c|}{(1+|a|)}, \quad \text{or if } |a| > 1 \text{ and } \frac{|c|}{|a|} < \alpha < m.$$

Note that the first case will be applied if $|a|$ and $|d|$ are small whilst the second will be applied if $|b|$ and $|c|$ are small.

Proof: For all $u, v \in \mathbb{R}^2$, $d_\alpha(Au, Av) = d_\alpha(A(u-v), 0)$ so we may look at the expansion of $z \neq 0$ from 0. Then if $z = (x, y)$

$$\begin{aligned} d_\alpha(Az, 0) \quad &= \alpha|ax+by| + |cx+dy| \\ &\geqslant \alpha(|by| - |ax|) + (|cx| - |dy|) \end{aligned}$$

using the crude inequality $|u+v| \geqslant |u| - |v|$. Taking the constants out of the modulus signs will give the modulus of the constants, so to simplify expressions we will work with positive constants and interpret these as the modulus of the constants in the general case. Thus

$$d_\alpha(Az, 0) \geqslant (c - \alpha a)|x| + (\alpha b - d)|y| \tag{4.1}$$

so if $c - \alpha a > 0$ and $\alpha b - d > 0$, i.e. $\frac{d}{b} < \alpha < \frac{c}{a}$, then

$$d_\alpha(Ax, 0) \geqslant \frac{c - \alpha a}{\alpha}\left(\alpha|x| + \frac{\alpha(\alpha b - d)}{c - \alpha a}|y|\right). \tag{4.2}$$

Hence if $\alpha > 0$ can be chosen so that

$$\frac{c - \alpha a}{\alpha} > 1 \quad \text{and} \quad \frac{\alpha(\alpha b - d)}{c - \alpha a} \geqslant 1$$

then the right hand side of (4.2) is greater than or equal to $\lambda d_\alpha(z, 0)$ with $\lambda > 1$ so the linear part A is expanding in the metric d_α. With a little algebra this becomes the condition stated in the lemma.

On the other hand, if $c - \alpha a < 0$ and $\alpha b - d < 0$, i.e. $\frac{c}{a} < \alpha < \frac{d}{b}$, then the equivalent of (4.1) is

$$d_\alpha(Az, 0) \geqslant \frac{\alpha a - c}{\alpha}\left(\alpha|x| + \frac{\alpha(d - \alpha b)}{\alpha a - c}|y|\right).$$

and so provided

$$\frac{\alpha a - c}{\alpha} > 1 \quad \text{and} \quad \frac{\alpha(d - \alpha b)}{\alpha a - c} \geqslant 1$$

the map is expanding, giving the second set of conditions in the lemma.

□

THEOREM 4.1 There exists $\varepsilon > 0$ such that if $\frac{8}{11} < -d_0 < \frac{10}{11}$, $0 < 2 - d_1 < \varepsilon$ and $0 < t_0, t_1 < \varepsilon$ then the planar border collision normal form has a stable fixed point if $\mu < 0$ and a two-dimensional attractor with an absolutely continuous invariant measure if $\mu > 0$.

Before giving the proof two observations are worth making. First, the region defined is open, emphasizing the comments that this is not an exceptional occurrence in the border collision normal form.

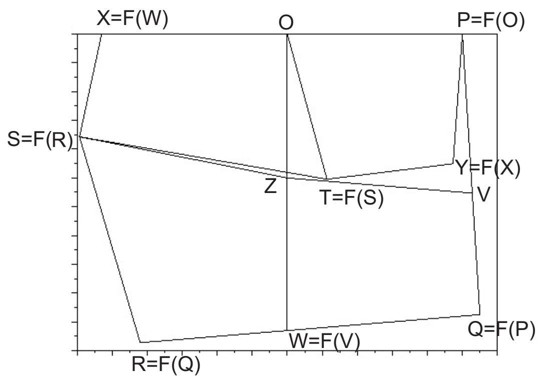

FIG. 2. Invariant region, partition and images for the same parameters as Figure 1. T lies inside the region $OPVZ$.

Second, no attempt has been made to optimise the region of validity for the result here as this would only add to the algebraic complexity of the argument without adding to the conclusion.

Proof of Theorem 4.1: As in the previous section write $\omega = -d_0$. Also write $d_1 = 2-\delta$ with $0<\delta<\varepsilon$. Figure 2 shows the geometry of the regions and points defined in the proof.

If $\mu<0$ the fixed point in $x<0$ exists and is stable provided $t_0 < 1-\omega < \delta$ and so we require $\varepsilon < \frac{1}{11}$ although we may need to choose ε smaller later.

If $\mu>0$ then without loss of generality set $\mu=1$. The proof in this region is in three steps: the construction of a polygonal bounding region; the definition of an induced map, and then the verification that the induced map is piecewise expanding so that the Induced Map Lemma of section two (Lemma 2.1) can be applied.

The bounding region: Let $O=(0,0)$, $P=F(O)=(1,0)$ and $Q=F(P)=(1+t_1,-2+\delta)$. Since $t_1>0$ this is in $x>0$ near $(1,-2)$ and hence, writing quantities as column vectors for clarity,

$$R = F(Q) = \begin{pmatrix} -1+\delta+t_1(1+t_1) \\ -2+\delta-2t_1+\delta t_1 \end{pmatrix}$$

near $(-1,-2)$ as shown in Figure 2. Since $\delta+t_1(1+t_1) < 2\varepsilon+\varepsilon^2 < 1$, R lies in $x<0$ and so if $R=(R_1,R_2)$ then $S=F(R)$ is given by

$$S = \begin{pmatrix} -1+\delta+R_1t_0-t_1(2-\delta) \\ R_1\omega \end{pmatrix} \tag{4.3}$$

near $(-1,-\omega)$. The line between Q and R is $Q+\lambda\overrightarrow{QR}$ i.e.

$$\begin{pmatrix} 1+t_1 \\ -2+\delta \end{pmatrix} + \lambda \begin{pmatrix} -2+\delta+t_1^2 \\ t_1(-2+\delta) \end{pmatrix}$$

and so this line intersects the x-axis with $\lambda = (1+t_1)/(2-\delta-t_1^2)$ at $W=(0,W_2)$ where

$$W_2 = -2+\delta+\frac{(-2+\delta)t_1(1+t_1)}{2-\delta-t_1^2}.$$

Thus W is close to $(0,-2)$. Also $X = F(W) = (X_1, 0) = (1 + W_2)$, i.e.

$$X_1 = -1 + \delta + \frac{(-2+\delta)t_1(1+t_1)}{2-\delta-t_1^2}. \tag{4.4}$$

The geometry of these points is shown in Figure 2. One feature to note is that the sketch indicates that if the x- and y-coordinates of any point is denoted by subscripts 1 and 2 respectively, then $S_1 < X_1 < R_1$. This is not hard to verify for sufficiently small ε from the equations. Hence $OWRSX$ is convex (and $OPQW$ is clearly convex, as is $PQRSX$ which is the union of the two).

Invariance of $PQRSX$. The region $PQRSX$ will be the bounding region on which the induced map is defined. To establish that it is invariant we will consider the two convex components $OPQW$ in $x \geqslant 0$ and $WRSXO$ in $x \leqslant 0$ separately.

The first is easy: the image of the region $OPQW$ is simply the region bounded by the images of the corners, i.e $PQRX$ which is contained in $PQRSX$ by convexity.

The image of the second, $WRSXO$, is also the region bounded by the straight lines between images of the corners, i.e. $XSTYP$ where $T = F(S)$ and $Y = F(X)$, so provided both of these points lie in $PQRSX$ then the invariance is proved. This cann be shown for sufficiently small $\varepsilon > 0$ direct calculation again and the details are omitted here.

Definition of an induced map: Let $Z = (0,-1)$, so $F(Z) = O$. Then if $V = F^{-1}(W)$ on PQ close to $(1,-1)$ a simple calculation shows $V_1 = |W_2|/(2-\delta)$ and $V_2 = -1 - t_1 V_1 < -1$. Hence $V_2 < -1 < T_1$ for ε sufficiently small, and so T lies above the line ZV and the line ST is above SV. Now let

$$R_1 = OPVZ, \quad R_2 = ZVQW, \quad L_1 = XOZS, \quad L_2 = SZWR$$

as shown in Figure 2. Then clearly

$$\begin{array}{ll} F(R_1) = R_2, & F(R_2) = XOWR \subseteq L_1 \cup L_2, \\ F(L_1) = YPOT \subseteq R_1, & F(L_2) = TOXS \subseteq L_1 \cup R_1. \end{array} \tag{4.5}$$

This should be compared with the $\varepsilon = 0$ case of the previous section (3.3): only the last inclusion is different, so L_2 splits into two (polygonal) regions, one that behaves as L_1 and the other that maps to L_1 as was the case in (3.3). Now define the induced map

$$G(x) = \begin{cases} F(x) & \text{if } x \in R_1 \cup R_2 \\ F^3(x) & \text{if } x \in L_1 \\ F^3(x) & \text{if } x \in L_2 \text{ and } F(x) \in R_1 \\ F^4(x) & \text{if } x \in L_2 \text{ and } F(x) \in L_1 \end{cases} \tag{4.6}$$

Then by (4.5) G is piecewise affine on each of its components and maps the polygonal region into itself.

G is expanding: By inspection of the induced map G of (4.6) the linear parts are A_1, $A_1^2 A_0$ and $A_1^2 A_0^2$ for F, F^3 and F^4 respectively, so to apply the results summarized in section 2 we need to show that there exists α such that all of these maps are expanding in the metric d_α. However, each of them is ε-close to the corresponding matrices in section 3, so the same criteria for expansion hold up to order ε and hence we have expansion for sufficiently small ε (note that the lower bound on ω, $8/11$, was chosen so that $\sqrt{2} < 2\omega$, one of the criteria for the existence of positive α for the metric d_α). Thus the argument of section two together with Lemma 4.1 show that if ε is sufficiently small then there exists $\alpha > 0$ such that each of the maps defining G in (4.6) is expanding in d_α.

Completion of the proof: The Theorem now follows since the induced map G is a piecewise expanding affine map in d_α for some $\alpha \in (\sqrt{2}, \frac{16}{11})$ and hence by the theorems of Tsujii and Buzzi has an attractor with an absolutely continuous invariant measure and containing an open set. By the Induced Map Lemma (Lemma 2.1) this implies that F has an absolutely continuous invariant measure and since the attractor of the induced map is clearly contained in the attractor of the original map, this latter attractor contains an open set too.

□

Although the precise algebraic form of the region of the four-dimensional parameter space has been left implicit, the results of Lemma 4.1 make it relatively simple to verify numerically whether an induced map is expanding, and this together with numerical verification of the inclusions (4.5) imply that a given set of parameters produces a two-dimensional attractor.

For the example of Figure 1, $-d_0 = 0.85$, $d_1 = 1.95$, $t_0 = 0.05$ and $t_1 = 0.1$, Figure 2, which uses the same parameters, shows that the inclusions (4.5) hold, although the parameters are close to a boundary of validity for these inclusions as T is close to the line ZV. The matrices of the induced map (4.6) are easy to calculate numerically and the criteria of Lemma 4.1 can be checked numerically. For A_1 this shows (using the first criterion) that G is expanding in d_α if

$$1.298 < \alpha < 1.771$$

where here and below the lower boundary is rounded up and the upper is rounded down. For $A_1^2 A_0$, the Jacobian matrix for G^3, the criterion (again using the first criterion) is

$$1.112 < \alpha < 1.646$$

and for $A_1^2 A_0^2$, the Jacobian matrix for G^4, (using the second criterion)

$$0.152 < \alpha < 4.632.$$

Thus if α in the (non-empty) intersection of these three intervals the G is expanding. Hence the attractor shown in Figure 1 is two-dimensional.

If $\mu < 0$ then at these parameters there is a fixed point in $x < 0$ since $1 - t_0 - \omega > 0$, and its stability is determined by the eigenvalues of A_0 which, by numerical calculation, are approximately $(-0.897, 0.947)$. These have modulus less than one so the fixed point is stable.

5. Conclusion

The two-dimensional border collision normal form is an important model of deterministic hybrid dynamics, and we have shown that as the standard parameter μ varies there is a jump from an attracting fixed point to an attractor which has the dimension of the ambient phase space as μ passes through zero.

The results of Buzzi and Tsujii, and hence the proof of Theorem 5, hold for piecewise affine maps, whilst the bifurcation analysis derives maps in this class by ignoring higher order correction terms. It is not yet clear how this changes the description of attractors presented here.

In higher dimensions a normal form can be defined as in di Bernardo (2003) and di Bernardo *et al.* (2011). We conjecture that the equivalent result will hold in this more general case for generic sets of parameters. This would imply an even greater jump in dimension on passing through the bifurcation point, lending weight to the suggested ‘curse of dimensionality’ of Glendinning and Jeffrey (2015) for piecewise smooth bifurcations.

It is worth stressing the importance of the fact that the result applies to an open set of parameters. It is not an 'exceptional' phenomenon, but part of the lexicon of typical behaviours that can be expected from systems undergoing a border collision bifurcation.

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