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Analysis of optimal liquidation in limit order books *

James W. Blair, Paul V. Johnson, & Peter W. Duck†

Abstract

In this paper we study the optimal trading strategy of a passive trader who is trading in the limit order book. Using a combined approach of accurate numerical methods and asymptotical analysis we examine the problem using different stochastic processes to model the asset price, as well as introducing a proportional resilience for the limit order book. This results in more complex equations to solve than when examined under the case of standard Brownian motion, allowing us to perform interesting analytical (asymptotic) analysis which adds insight into the solution space. Under Geometric Brownian Motion, we reduce the resulting four-dimensional Hamilton-Jacobi-Bellman partial differential equation (PDE) to a novel three-dimensional non-linear PDE, as well as rescaling the variables to reduce the number of input parameters by two. We use numerical methods to solve the PDE before asymptotically examining it in several limits, with each approach informing and confirming the other. We find the transition from a time-varying solution to a perpetual-type solution results in the development of singular behaviour, and this transition is examined in some detail. Finally we emphasise the adaptability of our proposed methodologies by implementing the same methods on a mean-reverting process for the asset price. Throughout the paper we also analyse the resulting trading strategies from a financial perspective. The trading strategies we develop are asset-price dependent, which to our knowledge is a unique concept in the passive optimal trading literature, and is arguably more realistic.

Optimal liquidation; Asymptotic analysis; Stochastic optimal control; Algorithmic trading; High-frequency trading

1 Introduction

Optimal liquidation (execution) consists of selling (buying) a large amount of an asset before a specified time, while obtaining the best price under some specified risk criteria. Generally speaking, liquidation (selling) and execution (buying) are interchangeable for most methods by simply changing initial conditions and constraints. In this paper we focus on the optimal liquidation of a portfolio of assets, although adapting the method to fill a portfolio would be straightforward.

Algorithmic trading has exploded in recent years, with reports of 50-77% of trading volume in the US coming from computerised algorithms, see SEC (2010). Various algorithms have been investigated extensively by both academics and institutional traders; these algorithms focus on being the most profitable, the least risky, or a trade-off between the two. The main focus of previous literature is split between the modelling of aggressive and passive optimal trading strategies. Aggressive trading focuses on finding the optimal rate to trade by filling orders while passive trading focuses on placing orders into the Limit Order Book (LOB) and waiting for an aggressive trader to fill those orders. Optimal trading of large orders to control the trading costs was originally proposed for aggressive trading to find an optimal balance between reducing market impact (by trading slow) and reducing market risk (by trading fast). The

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first models were developed around the turn of the millennium by Bertsimas et al. (1999) and Almgren and Chriss (2001). These models focused on trading in the market book, in which the objective was to maximise the efficient frontier of the investor’s wealth. Both models used linear impact functions for discrete time trading and developed static trading strategies, meaning the strategies were known before trading had begun and thus were independent of the asset price, which was driven by a standard Brownian motion with drift. Extensions to this framework include: Almgren (2003) who considered continuous time trading with non-linear impact functions, Almgren and Lorenz (2007) who allow one update of the asset price at a fixed time and from this created dynamical strategies based on this update, Lorenz and Almgren (2011) who allow continuous, but discrete, updates of the share price to develop dynamical trading strategies, Schied and Schöneborn (2009) who maximise general utility functions, rather than a mean-variance trade-off, to develop price dependent strategies, Schied et al. (2010) who maximise the investor’s utility using CARA utility functions to find deterministic trading strategies and Forsyth et al. (2012) who used numerical methods to solve the problem under Geometric Brownian Motion (GBM). More recently, the concept of dark pools have been introduced into the literature. Kratz and Schöneborn (2013) consider a trader who can trade in the market book (with price impact) and in a dark pool (no price impact), with execution occurring randomly in the dark pool according to a Poisson process.

The framework of Almgren and Chriss (2001) was further developed for trading in the LOB, which is a better replication of real-life trading than trading in the market book. In the latter case, assets can be bought or sold at the same price, that being the ‘fair’ asset price. However, in reality there is a difference between the price that assets can be bought and sold. The difference between the highest price that a buyer is willing to pay for an asset and the lowest price for which a seller is willing to sell it is known as the bid-ask spread. The LOB includes the highest bid price, the lowest ask price as well as all offers lower than the best bid price and higher than the best ask price. Obizhaeva and Wang (2013), which has been a preprint paper since 2004, introduced aggressive trading in the LOB in which a block shaped density for the LOB was assumed. The temporary market impact function used in market book models, in which only one trade was affected, was replaced by a transient market impact function, a decreasing function of time representing the LOB refilling post trade. Alfonsi et al. (2010) considered general shape functions for the LOB and tackled the problem through the use of Lagrange multipliers, rather than through the use of dynamical programming, as carried out by Obizhaeva and Wang (2013). Kharroubi and Pham (2010) considered a continuous time setting but, to better replicate real life, allowed only for discrete time trading by introducing a lag variable tracking the time interval between successive trades, under a GBM diffusion process. This paper was followed up by Guilbaud et al. (2010) who discussed numerical (finite difference) methods used to solve this type of problem. Bouchard et al. (2011) examined the use of different algorithms depending on the preferences of a trader. For a full review of aggressive trading methods the authors recommend the review paper of Schied and Slynko (2011). The dynamics of the LOB have also been examined and methods have been suggested to make it mathematically tractable, see for example Cont et al. (2010) and Cont and De Larrard (2013).

Passive trading on the other hand does not involve the trade-off between market impact and market risk. Market impact is replaced by a new type of risk: the risk of non-execution. In passive trading, orders are placed into the LOB and are only filled when met by an aggressive trader’s order. The further into the LOB the trades are placed, the higher the payoff for the trader but with a lower probability of the order being filled. Given this trading scenario, the filling of an order is not guaranteed prior to the terminal time, and hence there is a risk of non-execution. As the trader can continuously amend his asking price, the price at which the asset is sold is controlled by the trader. However, the trader cannot control the ‘fair’ price of the asset, which is driven by a stochastic process, and thus market risk is still present. This type of model was first suggested by Ho and Stoll (1981) but lay dormant in the literature for some
time until Avellaneda and Stoikov (2008) revisited this problem with some modifications. The methods used by these latter authors do not explicitly consider the LOB but instead consider the statistical properties of its liquidity, reducing the dimension of the problem to a level that is more amenable to work with. Under the objective of maximising the investor’s utility, with a general Brownian Motion driving the asset price, they derive a Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) and examine its asymptotics. Guéant et al. (2012a) reduce this HJB PDE to a system of first-order ordinary differential equations (ODEs) by introducing a constraint on the inventory that the investor can hold. In a similar framework, Guéant et al. (2012b) examined the problem for a single sided LOB, one in which assets could only be sold, while introducing a terminal time penalty for any remaining inventory, before examining a steady-state solution. Bayraktar and Ludkovski (2012) who developed a framework for portfolio liquidation for a risk-neutral investor in which the objective is to maximise the expected revenues of sales. These authors considered a one-sided LOB, similar to Guéant et al. (2012b), but introduced a power-law intensity, as opposed to the negative exponential intensity which has dominated the literature. Guéant and Lehalle (2013) developed the framework of the previous paper Guéant et al. (2012b) by considering general intensity shapes for the Poisson process. Cartea and Jaimungal (2013b) used a similar set-up to Avellaneda and Stoikov (2008) but under a high frequency trading (HFT) framework. Their objective was to maximise terminal wealth while penalising inventory deviations from zero and terminal inventory holdings, properties which comply with institutional HFT as outlined by Brunetti et al. (2011).

In this paper we expand on the above framework, in which the asset price was modelled as a standard Brownian motion with drift, by introducing a novel (and perhaps more realistic) concept to the optimal trading literature, that being the use of more general diffusion processes for the asset price. It was over a decade after Almgren and Chriss (2001) published their seminal paper, which examines trading in the market book rather than the limit order book with a similar time frame in mind, that Forsyth et al. (2012) examined the same problem under GBM. It has been argued through empirical evidence that GBM is favourable in modelling asset prices over arithmetic Brownian motion (see Osborne, 1959). GBM can be used for both long and short trading horizons, and avoids the fallacy of negative price scenarios that can appear in standard Brownian motion. Although it is argued that under small time horizons standard Brownian motion approximates GBM, many of the problems in optimal execution consider infinite horizon problems (see for example Guéant et al. (2012b) and Schied and Schöneborn (2009)) to which this approximation is no longer valid. Additionally, the rescaling of variables we use in section 2 makes our model applicable for various time scales. As opposed to Guéant et al. (2012b) and Cartea and Jaimungal (2013b), in which they reduced the HJB PDE to a system of ODEs, using GBM (and mean-reverting processes) will disable us from using similar methods. Although the equations are more complex to solve, this could be a reason GBM (and more general diffusion processes) was avoided in the initial framework. Having more complex equations allows us to do what we consider to be interesting analytical (asymptotic) analysis, finding analytical solutions in various limits, and discovering the development of a singularity when a steady-state framework is examined. We have thus taken a novel combined approach in solving the problem which involves both theoretically (analytically/asymptotically) examining the problem and using numerical methods to obtain solutions, with each approach informing and confirming the other. Using more general diffusion processes will result in asset-dependent trading strategies, as opposed to the asset-independent strategies found by Guéant et al. (2012b) and Cartea and Jaimungal (2013b), among others. Asset-dependent strategies appear more realistic from a financial perspective, as the additional amount you ask for the asset should depend on the price of the asset itself. This also results in not selling the asset for a negative price (effectively paying someone to take the asset), which can occur in the work of Guéant et al. (2012b) and Cartea and Jaimungal (2013b), amongst others. Additionally, we model the control parameter, the additional amount we ask for the asset also referred to
as the optimal trading strategy, as a proportional quantity of the asset price, rather than an absolute amount as modeled by Guéant et al. (2012b). This not only complicates the model but produces interesting (and arguably more realistic) results while doing so. Having an asset-dependent control parameter will induce some transparency in our results, given the optimal trading strategy will be expressed as a percentage of the asset. The use of a proportional optimal control results in a proportional resilience parameter for the limit order book, a novel concept in the literature. As we will see shortly, the methods we introduce in this paper are not constrained to the use of GBM as the driving process for the asset price.

Contrasting frameworks for obtaining optimal trading strategies have also been developed. To name a few, Cartea and Jaimungal (2013a) developed a hidden Markov model to understand the key behaviour of stock dynamics resulting in an optimal tick-by-tick trading strategy that an investor who uses limit orders to profit from the bid-ask spread should follow. Hult and Kiessling (2010) modeled the entire LOB as a high-dimensional Markov chain. Optimal trading strategies were discussed in the theory of Markov decision processes, and a value iteration procedure was presented which enables optimal strategies to be found numerically.

Finally, combinations of limit order book and market book trading have been investigated in the literature. Huitema (2013) derive a model where a trader, whose assets follow standard Brownian motion, can trade in both the market book (with the rate of trading impacting the price) and the limit order book (with a Poisson process with controlled intensity driving the execution of orders), and solves the resulting PIDE numerically. Guilbaud and Pham (2013b) develop a model where passive and aggressive trading is possible, in which limit orders can be placed at best quote or best quote improved by one tick and occur according to a Cox process, while aggressive trading can only occur at discrete times. This is further developed in Guilbaud and Pham (2013a) by examining a pro-rata micro-structure, rather than the usual price/time priority structure.

The layout of this paper is as follows: in section 2 we introduce the problem, suggesting an ansatz solution which reduces the HJB PDE to a non-linear PDE before deriving boundary conditions and providing numerical solutions. In section 3 we discuss a small-time-to-termination solution for the problem, examined using perturbation theory. Section 4 considers a perpetual-type solution for the problem. In section 5 we examine the same problem but with the use of a mean-reverting process, that being the Cox et al. (1985) (CIR) process, as the diffusion process for the asset price, replacing the GBM. This not only emphasises the adaptability of our methods but also produces interesting results given that the CIR process has previously been quite extensively suggested for the diffusion process of an asset price, most notably commodities (see Linetsky, 2004). We conclude in section 6.

2 Problem Formulation

We consider an investor who wishes to maximise his expected utility, given a portfolio of assets to liquidate, before a specified terminal time, \( T \). We assume at time \( t = 0 \) that the investor starts with an initial inventory of \( q(0) \) assets, in which \( q \) takes positive integer values, and an initial wealth \( X(0) \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a filtration, \((\mathcal{F}_t, t \in [0, T])\). We assume the ‘fair’ asset price \( S(t) \) follows a GBM, and so the diffusion process is defined as

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW(t)\tag{1}
\]

with \( \mu \) as the relative drift, \( \sigma \) as the relative volatility and \( W(t) \) as a Wiener process which is \( \mathcal{F}_t \) measurable.

The investor will continuously post orders into the ask side of the LOB for price \( S^a(t) \) which is \( \delta = \delta(t, X, q, S) \) percent greater than the ‘fair’ asset value \( S(t) \), i.e.:

\[
S^a(t) = S(t) (1 + \delta) \tag{2}
\]
This is another distinct aspect to our approach as Guéant et al. (2012b), and others, do not model the control as a percent of the asset price but instead have asking price $S^a(t) = S(t) + \delta$.

Asset sales follow a time-dependent Poisson process, $N(t)$, which is $\mathcal{F}_t$ measurable and independent of $W(t)$:

$$dq(t) = -dN(t).$$

for $q(t) > 0$, thus assuming the trader becomes inactive after liquidating and does not short sell. For each occurrence of a jump (sale), the investor’s wealth increases by the amount that asset was sold for, i.e. $S(t)(1 + \delta)$. The dynamics of the wealth is given by

$$dX(t) = S(t)(1 + \delta) dN(t),$$

where $N(t)$ is the same Poisson process as before. Therefore, when a jump occurs, the values of $q(t)$ and $X(t)$ change simultaneously, according to (3) and (4) respectively. We are thus assuming sales are of unitary size, consistent with that of previous literature. $N(t)$ has intensity $\Lambda(\delta)$ which takes the form:

$$\Lambda(\delta) = \lambda e^{-\lambda (\frac{S^a-S}{S})} = \lambda e^{-\lambda \delta}$$

for some positive constants $\lambda$ and $\ell$. The liquidity of the market is described by the intensity of the Poisson process. If no additional amount is added to the ‘fair’ asset value then the rate at which the assets are sold is $\Lambda(0)$, which for the case of (5) is equal to $\lambda$. Given the negative exponential form of (5) there is less liquidity for assets sold for prices higher than their ‘fair’ value and as such the probability of execution is lower. The parameter $\ell$ can be interpreted as the exponential decay factor for the fill rate of orders placed away from the ‘fair’ price (see Cartea and Jaimungal, 2013b), i.e. how quickly or slowly the demand changes as we move further into the LOB and thus how quickly or slowly the probability of execution decreases, which is proportional to the price of the asset. This is different than the standard, absolute decay used in previous literature. For ease of reference we shall denote $\ell$ as the resilience of the LOB. Justification for using the form of (5) for the intensity of the Poisson processes is described thoroughly in Avellaneda and Stoikov (2008) which is supported by empirical evidence by Gopikrishnan et al. (2000) and Maslov and Mills (2001) for the distribution of the size of the market orders and by Gabaix et al. (2006), Weber and Rosenow (2005) and Potters and Bouchaud (2003) for the change in price following a market order.

The objective is to liquidate this portfolio before some final time $T$. Assets that are not liquidated before this time will be sold in the market for their ‘fair’ value. The utility, $\Phi(\cdot)$, we seek to maximise, takes the form of a negative exponential function and as such the investor has constant absolute risk aversion (CARA) defined by

$$A(W) = -\frac{\Phi_W(W)}{\Phi(W)},$$

which for the case of the exponential utility family is constant and equal to the risk aversion parameter, $\gamma$, noting the subscript in (6) represents the derivative and $W$ represents the investor’s wealth. This form of the utility function is consistent with the previous literature of Avellaneda and Stoikov (2008) and Guéant et al. (2012b). We define our value function, $u(t, X, q, S)$, as the maximum expected utility at time $t$:

$$u(t, X, q, S) = \sup_{(\delta_t)_{t\in[0,T]}\in\mathcal{A}} \mathbb{E} \left[ -e^{-\gamma (X(T) + q(T)S(T))} \right],$$

where $\gamma > 0$ is the risk aversion characterising the investor and $\mathcal{A} \in (-1, \infty)$ is the set of admissible trading strategies. Assuming the trader starts with some non-negative wealth, $x_0$, and with some positive quantity of inventory, $q_0$, the term $\gamma (X + qS)$ is strictly positive. Therefore the objective function, and by definition the value function, are bound, with $u \in (-1, 0)$. We
should note that the lower bound of the admissible strategy occurs naturally due to the investor never wanting to sell his asset for a negative price, implying he would be paying someone to take the asset from him, and hence it does not have to be explicitly implemented. Using the form of the asking price \( S^a = S + \delta \), under a standard Brownian motion, trading strategies are independent of the asset price and hence selling for a negative price can occur, which is a fundamental flaw of previous models.

Given the optimisation problem of (7), a HJB equation can be derived by applying the Bellman (1957) principle of optimality and using Itô’s lemma:

\[
\begin{align*}
&u_t(t, X, q, S) + \mu S u_S(t, X, q, S) + \frac{1}{2} \sigma^2 S^2 u_{SS}(t, X, q, S) \\
&+ \sup_{\delta} \left[ \lambda e^{-\delta} (u(t, X + S (1 + \delta), q - 1, S) - u(t, X, q, S)) \right] = 0,
\end{align*}
\]
with conditions:  
\[
\begin{align*}
&u(T, X, q, S) = \Phi(X, q, S) = e^{-\gamma X + q S} \\
&\text{and} \\
&u(t, X, 0, S) = \Phi(X, 0, S) = e^{-\gamma X}.
\end{align*}
\]
A derivation of the HJB PDE (8) can be found in the appendix A.1 along with a verification theorem in appendix A.2 verifying that the solution of the HJB PDE (8) is in fact the solution of the original optimisation problem (7).

2.1 Reduction of the problem

The problem as stated in (8) is a four-dimensional HJB PDE. Guéant et al. (2012b) and Cartea and Jaimungal (2013b) both reduced their (similar) problems to a system of ODEs by suggesting an ansatz form of the solution which assumes the solution can be separated into two functions, one involving variables \( X, S \) and \( q \) the other involving variables \( q \) and \( t \), with the PDE being reduced to a \( t \)-dependent ODE, thus arriving with trading strategies independent of \( X \) and \( S \). Due to the use of GBM, as opposed to the arithmetic Brownian motion used by Guéant et al. (2012b) and Cartea and Jaimungal (2013b), reducing the problem to a system of ODEs is no longer possible. However, we can still make use of the form of the utility function to make a significant reduction to the complexity of the problem.

We begin by assuming an ansatz solution

\[
\begin{align*}
u(t, X, q, S) &= e^{-\gamma X} f(\tau, q, S),
\end{align*}
\]
where we also use a change in the time variable \( \tau = T - t \) so that we are now solving forward in \( \tau \) rather than backward in \( t \). Using this form of the solution we obtain:

\[
\begin{align*}
-e^{-\gamma X} f_{\tau}(\tau, q, S) + \mu S e^{-\gamma X} f_S(\tau, q, S) + \frac{1}{2} \sigma^2 S^2 e^{-\gamma X} f_{SS}(\tau, q, S) \\
+ \sup_{\delta} \left[ \lambda e^{-\delta} \left( e^{-\gamma X} f(\tau + S (1 + \delta), q - 1, S) - e^{-\gamma X} f(\tau, q, S) \right) \right] = 0.
\end{align*}
\]

We can solve for the optimal control by differentiating the supremum with respect to the optimal control (\( \delta \)) and setting the result equal to zero which locates the stationary point.

---

2In some of the literature, such as Guéant et al. (2012b), a terminal penalty is included such the assets are sold at a discount of their actual price at the terminal price, while others, such as Avellaneda and Stoikov (2008) and Guéant et al. (2012a), neglect inclusion of this terminal penalty. Inclusion of the penalty is quite trivial but in terms of both solving this problem numerically and investigating it asymptotically the terminal penalty would make little difference to both the difficulty of the methods used and the results obtained. We have therefore chosen to neglect it.
Solving this we obtain:

\[ \delta(\tau, q, S) = \frac{1}{S\gamma} \ln \left( \frac{(\gamma S + l) f(\tau, q - 1, S)}{lf(\tau, q, S)} \right) - 1, \] (13)

which we notice is independent of \( X \), hence confirming the use of our ansatz solution. We also notice the optimal strategy is a function of \( S \) which is a unique addition to the literature for this general class of problem. Using the form of (13) we find the asking price of the asset to be

\[ S_a(\tau) = \frac{1}{\gamma} \ln \left( \frac{(\gamma S + l) f(\tau, q - 1, S)}{lf(\tau, q, S)} \right). \] (14)

To confirm with previous literature, notably Guéant et al. (2012b), we will keep our focus on the optimal control \( (\delta) \) when examining our results.

Substituting (13) into (12) and cancelling common factors we obtain the following non-linear PDE

\[-f_\tau(\tau, q, S) + \mu S f_S(\tau, q, S) + \frac{1}{2} \sigma^2 S^2 f_{SS}(\tau, q, S) - \lambda e^{l/\gamma} S f(\tau, q, S) S^{\gamma/l} = 0, \] (15)

with

\[ f(0, q, S) = -e^{-\gamma q S}, \] (16)

and

\[ f(\tau, 0, S) = -1. \] (17)

When discussing results we shall now refer to \( f(\tau, q, S) \) as the value function\(^3\).

2.2 Rescaling of the PDE

We shall now perform a rescaling (effectively a non-dimensionalisation) of this PDE to eliminate several of the parameters. We use the following change of variables:

\[ \tilde{\tau} = \lambda \tau, \quad \tilde{S} = S\gamma, \quad \tilde{\mu} = \frac{\mu}{\lambda}, \quad \tilde{\sigma} = \frac{\sigma}{\sqrt{\lambda}}. \] (18)

\(^3\)If we were to follow the framework of Guéant et al. (2012b) and use an absolute control parameter, so our asking price would be \( S^a(t) = S(t) + \delta \), we would still have an asset-dependent trading strategy, given by

\[ \delta(\tau, q, S) = \frac{1}{\gamma} \ln \left( \frac{(\gamma S + l) f(\tau, q - 1, S)}{lf(\tau, q, S)} \right) - S, \]

and would still result arrive with (12) and (14) if we used a proportional resilience, so (5) would take the form

\[ \Lambda(\delta) = \lambda e^{-l(\tilde{S}^a - \tilde{S})} = \lambda e^{-\tilde{\delta}}, \]

If we weren’t to use a proportional resilience, i.e.

\[ \Lambda(\delta) = \lambda e^{-l(S^a - S)} = \lambda e^{-\delta} \]

the trading strategies we obtain would take the form

\[ \delta(\tau, q, S) = \frac{1}{\gamma} \ln \left( \frac{(\gamma + l) f(\tau, q - 1, S)}{lf(\tau, q, S)} \right) - S, \]

which would leave to a non-linear PDE analogous to (15) with trading strategies that are asset price-dependent.
All the new variables of (18) are then dimensionless. This results in the non-linear PDE:

\[-f_{\tilde{\tau}}(\tilde{\tau}, q, \tilde{S}) + \mu S f_{\tilde{S}}(\tilde{\tau}, q, \tilde{S}) + \frac{1}{2} \sigma^2 S^2 f_{\tilde{S}\tilde{S}}(\tilde{\tau}, q, \tilde{S}) - e^{-q \tilde{S}} f_{\tilde{S}}(\tilde{\tau}, q, \tilde{S}) \tilde{S} + \frac{1}{2} \sigma^2 S^2 f_{\tilde{S}\tilde{S}}(\tilde{\tau}, q, \tilde{S}) + 1 = 0,\]  

(19)

with

\[f(0, q, \tilde{S}) = -e^{-q \tilde{S}}\]  

(20)

\[f(\tilde{\tau}, 0, \tilde{S}) = -1.\]  

(21)

(19) is now dimensionless as are the conditions (20) and (21). The optimal trading strategy in dimensionless form is then

\[\delta(\tilde{\tau}, q, \tilde{S}) = \frac{1}{\tilde{S}} \ln \left( \left( \frac{\tilde{S} + l}{\tilde{S}} \right) f(\tilde{\tau}, q - 1, \tilde{S}) \right) - 1.\]  

(22)

To make this problem well-posed we must consider boundary conditions for \(\tilde{S} = 0\) and \(\tilde{S} \to \infty\). For \(\tilde{S} = 0\) we take the limit of (19) which results in

\[f_{\tilde{\tau}}(\tilde{\tau}, q, 0) = 0.\]  

(23)

Given (23) and (20) the \(\tilde{S} = 0\) boundary condition we arrive at is

\[f(\tilde{\tau}, q, 0) = -1,\]  

(24)

and so

\[\delta(\tilde{\tau}, q, 0) = f_{\tilde{S}}(\tilde{\tau}, q, 0) - f_{\tilde{S}}(\tilde{\tau}, q - 1, 0) + \frac{1}{l} - 1.\]  

(25)

For \(\tilde{S} \to \infty\), \(f(\tilde{\tau}, q, \tilde{S}) \to 0\) although for numerical purposes we assume a (softer) Neumann boundary condition

\[\frac{\partial f}{\partial \tilde{S}} \to 0.\]  

(26)

which, from investigation, we found to be satisfactory. From a financial perspective we can interpret this as the investor not being able to become any more satisfied when he has an asset worth an infinite amount of money.

To solve (19) we use a finite difference scheme. We have tested both implementing implicit differences on the derivative terms while taking the non-linear term as an explicit term, thus negating the need to use an iterative scheme, and using an iterative Crank-Nicolson scheme, with each method confirming the other. For the former we expected our method to exhibit \(O\left(\Delta \tilde{\tau}, \Delta \tilde{S}^2\right)\) convergence which we found was the case; for the latter we expected \(O\left(\Delta \tilde{\tau}^2, \Delta \tilde{S}^2\right)\) convergence which we also confirmed, where \(\Delta \tilde{\tau}\) and \(\Delta \tilde{S}\) are the grid sizes in \(\tilde{\tau}\) and \(\tilde{S}\) respectively. A number of calculations were performed on a transformed grid \(Y = \ln \tilde{S}\), which did have some advantages for certain calculations, but also exhibited some disadvantages (including the need for two, rather than one, domain truncation parameters). However the results thus obtained did provide a useful check of the results from the untransformed \((\tilde{S})\) grid.
2.3 Numerical Examples

We shall focus on the behaviour of the optimal trading strategy, \( \delta \), given that it is a transformation of the value function and, from a financial perspective, is more transparent than examining the value function per se. The parameter values used are: \( \bar{\mu} = 0.04, \bar{\sigma} = 0.4, l = 25 \) and \( T = 2 \) which comply with Avellaneda and Stoikov (2008), and the empirical discussions within. The results can be seen in figure 1 for \( q = 1 \). Looking first at the optimal strategy over time, figure 1(a), for various values of \( \bar{S} \) we notice that the general behaviour is decreasing for increasing \( \bar{S} \). This is due to the CARA characteristic of the utility function. Given the investor exhibits constant absolute risk aversion, the investor will dislike the higher absolute volatility present as \( \bar{S} \) increases. He will thus opt to sell the asset quicker, so as to lock in the current price and avoid risk. What is particularly vivid for larger values of \( \bar{S} \) but also true for smaller values, is that as \( \bar{\tau} \) increases the optimal trading strategy tends to a perpetual (time-independent) type solution. This can be seen in the dot-dash line tending to a constant value around \( \bar{\tau} \sim 0.3 \). It can also be seen in figure 1(b) that the solution is tending to a perpetual type solution as we can see the values for \( \bar{\tau} = 1 \) are close to those at \( \bar{\tau} = 2 \), with the two lines are almost identical for \( \bar{S} > 10 \). We notice in both figure 1(a) and figure 1(b) that as we perturb away from the terminal time solution there is interesting behaviour and as such a small-\( \bar{\tau} \) solution is also of some interest. Given the optimal strategy \( \delta \) is a transformation of \( f ( \bar{\tau}, q, \bar{S} ) \) the function \( f \) exhibits similar behaviour as the optimal strategy \( \delta \) does in figure 1. Therefore in the following sections we shall investigate both the small-\( \bar{\tau} \) solution and perpetual solution for \( f \).

The remainder of this section consists of examining trading strategies for varying amounts of inventory, \( q \), and varying parameter values. We examine these at a time close to the terminal time, \( \bar{\tau} = 0.01 \), and at (an earlier time), \( \bar{\tau} = 2 \). The reason for doing so is that the solution varies rapidly as \( \bar{\tau} \) initially increases before tending to a perpetual solution. Therefore, showing the solutions at both times give a good insight into the overall behaviour of the solution.

The properties of the optimal trading strategy for \( q > 1 \) are similar to those of \( q = 1 \). Figure 2 gives an indication of how the optimal trading strategy behaves for various values of \( q \) for a single time step around the terminal time (\( \bar{\tau} = 0.01, \) figure 2(a)) and at an earlier time (\( \bar{\tau} = 2, \) figure 2(b)). The optimal trading strategy is \( q \) independent at \( \bar{\tau} = 0 \), which can be confirmed.

Figure 1: Optimal strategy for investor with one asset remaining, with \( \bar{\mu} = 0.04, \bar{\sigma} = 0.4 \) and \( l = 25 \).
by substituting (20) into (22) which results in:

$$\delta(\tilde{\tau}=0,q,\tilde{S}) = \frac{1}{\tilde{S}} \ln \left( \frac{(\tilde{S}+l)}{l} \right).$$  \hspace{1cm} (27)

We have included a plot of (27) in figure 2(a) to highlight the significant difference in solutions from a small deviation in time, and as \(q\) is increased.

To conclude the discussion on the numerical examples we shall briefly describe how the optimal trading strategy changes in relation to changes to the various parameters in the model. In figure 3 the optimal strategy for various times has been plotted with each parameter, \(\tilde{\mu}\), \(\tilde{\sigma}\) and \(l\) being varied, as well as a ‘base’ case which is calculated using the same parameters as stated above. As can be seen the same properties hold at both ends of the time spectrum for a given parameter variation.

As the drift, \(\tilde{\mu}\), increases the investor will be more happy to hold on to the asset as he expects the price to rise and thus his wealth to rise. He therefore asks for a higher price for the asset over assets with lower drift. As the volatility, \(\tilde{\sigma}\), increases the investor’s asking price decreases. Understandably, an investor who is risk-averse dislikes higher volatility as it brings a level of risk in stock price movement and thus will wish to sell quicker than if volatility was lower. As we decrease \(l\), the resilience parameter for the LOB, we see the investor asks for a higher price when his asset price is low and a lower price when his asset price is high. This switch is centered around whether the investor is selling the asset above par or at a discount, i.e. \(\delta\) greater than zero or less than zero, which occurs as \(\tilde{S}\) is increased. When selling above par a lower \(l\) signifies that the investor can place his asset further into the LOB without significantly reducing his probability of sale. When the investor is eager to sell he will sell the asset for a discount. In order to do so under lower \(l\) the investor must sell at a high discount in order to increase his probability of sale by a significant amount. The opposite is true for larger \(l\); when selling above par the probability of selling significantly decreases while the probability of selling significantly increases when selling at a discount; this can be seen in figure 3(b). However if we were to consider \(\tilde{\tau} = 0.01\) for higher values of \(\tilde{S}\) we would see the curve representing lower \(l\) cross the curve representing the ‘base’ case.
Figure 3: Optimal strategy under varying parameters. The base case has parameter values: $\tilde{\mu} = 0.04$, $\tilde{\sigma} = 0.4$ and $l = 25$.

3 Small-time-to-termination solution

In the light of the results from section 2.3 we now visit the concept of a small-$\tilde{\tau}$ solution. From the numerical results we see that there is a lot of ‘activity’ (which mirrors financial interest) taking place around $\tilde{\tau} = 0$. This is often the case in applied mathematics with financial applications, as, it is well known that options priced using the Black-Scholes-Merton framework close to expiry (see Evans et al., 2002) can also exhibit rapid changes in value.

In this regime the investor is more worried about not being able to sell the asset for any price higher than its current price or, when the asset is already at a high price, the investor is worried that volatility could cause a decrease in his asset price before the terminal time. This is a property of the investor’s risk-aversion.

This behaviour around this regime presents an interesting asymptotic problem. The behaviour (shape) of the optimal trading strategy is similar to that of the value function if examined much more closely. From (13) we see that the optimal strategy is derived by a log transformation of the value function, hence the similarity in shape. Given this information, the same interesting small-time-to-termination solution is present in the value function as is in the optimal trading strategy.

To examine the small-time-to-termination solution we shall expand $f\left(\tilde{\tau}, q, \tilde{S}\right)$ as follows

$$f\left(\tilde{\tau}, q, \tilde{S}\right) = f_0\left(q, \tilde{S}\right) + \tilde{\tau}f_1\left(q, \tilde{S}\right) + \tilde{\tau}^2f_2\left(q, \tilde{S}\right) + O\left(\tilde{\tau}^3\right).$$

The $O\left(\tilde{\tau}^0\right)$ term, $f_0\left(q, \tilde{S}\right)$, is merely equal to the terminal condition given by (20). By substituting (28) into (19) we can find an analytical solution for $f_1\left(q, \tilde{S}\right)$ by collecting the $O\left(\tilde{\tau}^0\right)$ terms

$$f_1\left(q, \tilde{S}\right) = \left(-\tilde{\mu}\tilde{S}q + \frac{1}{2}\tilde{\sigma}^2\tilde{S}^2q^2 - \frac{\tilde{\delta}}{\tilde{S} + l} \left(\frac{l}{\tilde{S} + l}\right)^{\frac{1}{2}}\right) f_0\left(q, \tilde{S}\right),$$

(29)
Figure 4: Comparison of ‘exact’ optimal trading strategy and asymptotic expansion for $\tilde{S}$ near zero and $\tilde{S}$ near 10 using a two-term and three-term model, with parameters $\tilde{\mu} = 0.04, \tilde{\sigma} = 0.4$ and $l = 5$.

and from the $O(\tilde{\tau}^1)$ terms

$$
f_2(q, \tilde{S}) = \frac{1}{2} \left( \tilde{\mu} \tilde{S} f_{1\tilde{S}}(q, \tilde{S}) + \tilde{\sigma}^2 \tilde{S}^2 f_{1\tilde{S}\tilde{S}}(q, \tilde{S}) - \frac{1}{\tilde{S} + l} \left( \frac{l}{\tilde{S} + l} \right)^{\frac{\tilde{\tau}}{l}} \right) \times \left( (l + \tilde{S}) f_1(q, \tilde{S}) - l f_1(q - 1, \tilde{S}) e^{-\tilde{S}} \right).
$$

(30)

$f_{1\tilde{S}}$ and $f_{1\tilde{S}\tilde{S}}$ are the first and second derivative of $f_1$ respectively which can both be easily calculated analytically; therefore (29) and (30) have fully analytical solutions.

We shall now discuss the accuracy of this asymptotic expansion, along with its limitations. The values we use for our parameters are the same as that of section 2.3 but with $l = 5$, for reasons that will be explained later. In figure 4 we have plotted the solution of the optimal trading strategy along with its asymptotic approximations for two values of $\tilde{S}$ which was calculated by substituting (28) into (22). We can see that in both figures the three-term approximation is a good approximation for the solution for $\tilde{\tau}$ of $O(1)$ while the two-term approximation is valid for smaller $\tilde{\tau}$. We have tested this for various $q$ values and these observations still hold true.

This approximation is surprisingly accurate for a range of parameter values. However, there are restrictions to how well the model holds depending on the parameter values used. For higher values of $\tilde{S}$ and $l$ the solution of $f(\tilde{\tau}, q, \tilde{S})$ diverges faster from the small-time-to-termination solution due to the stronger presence of the non-linear term, hence the use of the smaller $l$ in our numerical example. In figure 5 we have plotted the optimal strategy against $\tilde{S}$ for increasing $\tilde{\tau}$. We can see that as $\tilde{\tau}$ increases the range of $\tilde{S}$ for which the approximation is accurate decreases. It can also be seen that for $\tilde{\tau} = 0.1$ the difference between the full numerical and three-term asymptotic approximation is negligible. If $l$ were to be reduced even further the asymptotic approximation remains valid for a larger range of $\tilde{\tau}$ values.

We shall now move on to examine what happens when $\tilde{\tau} \to \infty$, i.e. the “perpetual” state.
Figure 5: Comparison of ‘exact’ optimal trading strategy and three-term asymptotic expansion for increasing $\tilde{\tau}$, with $\tilde{\tau} = \{0.1, 0.5, 1\}$ and parameter values $\tilde{\mu} = 0.04, \tilde{\sigma} = 0.4$ and $l = 5$.

4 Perpetual Solution

We saw from figure 1 that it is possible that perpetual (time-independent) solutions exist. Indeed, Guéant et al. (2012b) also found such solutions (in certain parameter regimes). In this section we shall study the perpetual solution in some detail. For the perpetual solution we require $f_{\tilde{\tau}}(\tilde{\tau}, q, \tilde{S}) \to 0$ as $\tilde{\tau} \to \infty$. In making this assumption, we are implicitly assuming that the terminal time in which the assets must be sold is sufficiently distant for us to neglect the time variation. Then (19) becomes

$$\tilde{\mu} \tilde{S} f_{\tilde{S}}(q, \tilde{S}) + \frac{1}{2} \tilde{\sigma}^2 \tilde{S}^2 f_{\tilde{S}\tilde{S}}(q, \tilde{S}) - \frac{e^{\tilde{\tau} f_{\tilde{S}}(q, \tilde{S})}}{\tilde{S} + l} \left( \frac{lf_{\tilde{S}}(q, \tilde{S})}{(\tilde{S} + l) f(q - 1, \tilde{S})} \right)^{\frac{\tilde{\tau}}{\tilde{\sigma}}} = 0,$$

with

$$f(0, \tilde{S}) = -1.$$  

To solve this non-linear ODE (31) we must consider two boundary conditions, one for $\tilde{S} = 0$ and one for $\tilde{S} \to \infty$. We know from the non-steady-state problem that $f(\forall \tilde{\tau}, q, \tilde{S} = 0) = -1$ and thus $f(q, \tilde{S} = 0) = -1$ for the perpetual case. For $\tilde{S} \to \infty$ we use the same boundary condition as used in section 2.2 for the full problem, that being a Neumann boundary condition as given by (26). We solve (31) using a Newton iterative method, see Press et al. (2009). The optimal trading strategy takes the form:

$$\delta(q, \tilde{S}) = \frac{1}{\tilde{S}} \ln \left( \frac{(\tilde{S} + l) f(q - 1, \tilde{S})}{lf_{\tilde{S}}(q, \tilde{S})} \right) - 1.$$  

We plot the value function (figure 6(a)) and optimal trading strategy (figure 6(b)) for $q \in \{1, 2, 3, 4\}$ with the same parameter values as used in section 2.3. This solution was tested against the full numerical solution for the non-perpetual (time-varying) case. We notice that the optimal strategy for the perpetual case is larger for small $\tilde{S}$ than for the full problem. This is for two reasons. The first being that $\delta_{\tilde{\tau}} > 0$ for small $\tilde{S}$ (as can be seen in figure 4(a)) which is due to the prospect of the drift increasing the value of the asset dominating over the prospect
of the volatility decreasing the value. The second is due to the presence of a singularity about \( \tilde{S} = 0 \), which we will now discuss.

We found that there are convergence issues for certain parameter values. Examining \( f_{\tilde{S}} \) for small \( \tilde{S} \) it appeared that a singularity was present for \( \tilde{S} \ll 1 \). On closer inspection using asymptotic balancing suggests that in this limit, the solution behaves as

\[
f(q, \tilde{S}) \approx -1 + c(q) \tilde{S}^\beta
\]  

for some positive constant \( \beta \) and \( c(q) \geq 0 \), and so the solution is bounded, provided

\[
\beta = 1 - \frac{2\tilde{\mu}}{\tilde{\sigma}^2} > 0,
\]

which gives us the requirement

\[
\tilde{\mu} < \frac{\tilde{\sigma}^2}{2}
\]  

is necessary for a solution to exist, given \( \beta > 0 \). We have verified this constraint numerically (later in this section we address the issue of parameters outside this region). This constraint is analogous to that found by Guéant et al. (2012b) in which they derived (in their notation)

\[
\mu < \frac{\gamma \sigma^2}{2}
\]

to be a necessary condition for a solution to exist for the perpetual case under standard Brownian motion. We notice there is a lack of the \( \gamma \) parameter in our constraint in comparison to Guéant et al. (2012b). Although we have scaled out \( \gamma \) in our model, even without this rescaling we still arrive at a constraint that is independent of \( \gamma \).

As can be seen in figure 6(b) we notice a singularity is apparently present in the optimal trading strategy. We can make an asymptotic approximation for the optimal trading strategy for \( \tilde{S} \ll 1 \) given as:

\[
\delta(q, \tilde{S}) \approx K(q) \tilde{S}^{\beta-1}
\]

which is found by taking the limit of (33) as \( \tilde{S} \to 0^+ \), using the form (34) for the value function, and noting that \( c(q) > c(q-1) \). Given the latter point we expect \( K(q) \) to be a decreasing function of \( q \), tending to zero as \( q \to \infty \), which can be seen in figure 7.
Figure 7: $K(q)$ variation with $q$ for $S \rightarrow 0$ with parameter values: $\tilde{\mu} = 0.04, \tilde{\sigma} = 0.4$ and $l = 1$. We set $Y_{\text{min}} = -20$.

In the following we have set $l = 1$; larger values of $l$ lead to a general dominance of the non-linear term, and rapid valuation variations close to $\tilde{S} = 0$, both of which conceal valuation behaviours, and thereby make insights thereof more difficult to discern.

It is of some interest to inspect how (quickly) the time-varying solution approaches the perpetual state (assuming (35) is satisfied). Figure 8 shows sample results for the difference between the former and latter states, indicating a quite rapid asymptotic approach in this case (which, indeed, generally was a canonical observation found in most calculations). However, although this difference is diminishing, quite rapidly, it is clear that there is a maximum deviation at decreasing values of $\tilde{S}$ as $\tilde{\tau}$ increases. It is clear that the $O(\tilde{S}^\beta)$ in the perpetual state as $\tilde{S} \rightarrow 0$ leads to singular behaviour, whilst the numerical results of the time-varying solution very strongly point to the non-linear term in (19) being negligible in this regime, and so a series ($\tilde{S} \rightarrow 0$) solution of the linearised form of this PDE\(^4\) takes the form

$$f \left( \tilde{\tau}, q, \tilde{S} \right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \left(q\tilde{S}\right)^n e^{\left(n\tilde{\mu} + \frac{1}{2}n(n-1)\tilde{\sigma}^2\right)\tilde{\tau}}. \quad (39)$$

Although this is clearly a divergent series (for sufficiently large $n$), nonetheless it does highlight the likely occurrence of (very) short $\tilde{S}$ scales as $\tilde{\tau}$ increases (and suitable truncations of this series does lead to results consistent with the full numerical system). The $n = 1$ term leads to the result that

$$f_{\tilde{S}}(\tilde{\tau}, q, 0) = qe^{\tilde{\mu}\tilde{\tau}}, \quad (40)$$

which clearly suggests the development of an (exponentially thin) region close to $\tilde{S} = 0$, together with an increasing magnitude for this derivative (as opposed to a diminishing value for $|f|$), which we surmise leads ultimately with a connection to (34). Outside of this region, the valuation is effectively that predicted by the perpetual solution.

A natural question that arises is what is the situation if the values of $\tilde{\mu}$ and $\tilde{\sigma}$ do not comply with (36)? Figure 9 shows such a case (corresponding to $\beta = -0.92$), and it is very clear (particularly when compared with previous results), that a time-independent state is not

\(^4\)It is possible to find an exact solution to this linear system, satisfying the appropriate conditions at $\tilde{\tau} = 0$, namely

$$f(\tilde{\tau}, q, Y) = \frac{-1}{\tilde{\sigma} \sqrt{2\pi \tilde{\tau}}} \int_{-\infty}^{\infty} \exp \left(-qe^{(Y-p)} - \frac{(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)\tilde{\tau}}{2\tilde{\sigma}^2} + p(\frac{1}{2} - \frac{\tilde{\mu}}{\tilde{\sigma}^2}) - \frac{p^2}{2\tilde{\sigma}^2}\right) dp,$$

with $Y = \ln \tilde{S}$.
being approached, rather that the non-negligible valuation is confined to a diminishingly small regime, close to $\tilde{S} = 0$.

### 4.1 Perpetual solution for large $q$

To conclude our examination of the perpetual problem we shall look at the case of large $q$. Examination of figure 6 suggests that as $q$ is increased the solutions tend to collapse on to a single distribution (a conclusion supported by a number of numerical experiments conducted by the authors). Further to that we have seen in figure 7 that $K(q) \to 0$ as $q \to \infty$. Examining the non-linear term we find

$$\lim_{q \to \infty} \left( \frac{f(q, \tilde{S})}{f(q-1, \tilde{S})} \right) \to 1.$$  \hfill (41)

Using (41) we can reduce (31) to a linear ODE which is now independent of $q$,

$$\tilde{\mu} \tilde{S} f_{\tilde{S}}(\tilde{S}) + \frac{1}{2} \tilde{\sigma}^2 \tilde{S}^2 f_{\tilde{S}^2}(\tilde{S}) - \frac{e^l \tilde{S} f(\tilde{S})}{\tilde{S} + l} \left( \frac{l}{\tilde{S} + l} \right)^{\frac{l}{\tilde{S}}} = 0,$$  \hfill (42)

in which we shall refer $f(\tilde{S})$ as the $q \to \infty$ solution, with boundary conditions $f(\tilde{S} = 0) = -1$ and $f(\tilde{S} \to \infty) \to 0$, as used on the non-linear ODE previously. We have been unable to find an analytical solution for (42) but it is straightforward to solve numerically using finite difference methods (for example). Remembering the explicit form of the optimal control given by (33), we can see that under the assumption of (41) the optimal trading strategy is given by

$$\delta(\tilde{S}) = \frac{1}{\tilde{S}} \ln \left( \frac{\tilde{S} + l}{l} \right) - 1.$$  \hfill (43)

The solution is shown in figure 10 for the same parameter values as used for figure 6. In figure 10(a) we can see that as $q$ is increased the solutions tend to the $q \to \infty$ solution. The optimal trading strategy is given by (43) and can be seen in figure 10(b). We notice how, as with the
value function, the solutions of the optimal trading strategies converge to the $q \to \infty$ optimal trading strategy as $q$ is increased. It can be seen that the solution for $\delta$ at $S \to 0^+$ is near zero for the $q \to \infty$ solution while singular for the solutions for finite $q$. If we were to increase $q$ significantly, the coefficient of the singularity in this limit decreases towards zero, in line with our observations above regarding $K(q \to \infty)$.

5 Optimal liquidation of mean-reverting assets

In this section we consider using a mean-reverting diffusion process for the asset price, so as to emphasise the adaptability and advantages of using numerical methods to solve this class of problem. The process we shall use is the CIR process, which was first suggested in 1985 by Cox et al. (1985) as an extension to the Vasicek (1977) model. The attractiveness of the CIR model is its avoidance of the possibility of negative values. It was originally suggested for the modelling of interest rates but has since been suggested for other models such as stochastic volatility models and, of particular interest to us for this framework, commodities (see Linetsky, 2004).

An asset $S(t)$ following the CIR process solves the following stochastic differential equation

$$dS(t) = \kappa (\zeta - S(t)) dt + \sigma \sqrt{S(t)} dW(t)$$

with $\kappa$ as the mean-reversion speed, $\zeta$ is the long-term mean asset price and $\sigma \sqrt{S(t)}$ is the absolute volatility. Following the same problem formulation as we have in section 2 but with the CIR process for the asset price, we can derive a similar HJB PDE. Furthermore we use the change of variables

$$\tilde{\tau} = \lambda (T - t), \quad \tilde{S} = S \gamma, \quad \tilde{\zeta} = \zeta \gamma, \quad \tilde{\kappa} = \frac{\kappa}{\lambda}, \quad \tilde{\sigma} = \sigma \sqrt{\frac{\gamma}{\lambda}}$$

(45)

to derive the dimensionless non-linear PDE:

$$-f_{\tilde{\tau}}(\tilde{\tau}, q, \tilde{S}) + \tilde{\kappa} (\tilde{\zeta} - \tilde{S}) f_{\tilde{S}}(\tilde{\tau}, q, \tilde{S}) + \frac{1}{2} \tilde{\sigma}^2 \tilde{S} f_{\tilde{S} \tilde{S}}(\tilde{\tau}, q, \tilde{S})$$

$$- e^{\tilde{\lambda} \tilde{S}} f_{\tilde{\lambda}}(\tilde{\tau}, q, \tilde{S}) \left( \frac{1 f_{\tilde{\tau}}(\tilde{\tau}, q, \tilde{S})}{\tilde{S} + l} f_{\tilde{S}}(\tilde{\tau}, q, \tilde{S} - 1, \tilde{S}) \right)^{\frac{\gamma}{\lambda}} = 0,$$

(46)
Figure 10: Perpetual solution for large $q$. In figure 10(a) the solid line represents the solution of (42) while the broken lines represent the solutions of (31) for $q = \{10, 20, 30, 40, 50\}$, beginning with $q = 10$ which is furthest from the $q \to \infty$ solution. We can see that as $q$ is increased the solutions converge to the $q \to \infty$ solution. The same can be seen for the optimal strategy in figure 10(b). The parameter values used: $\tilde{\mu} = 0.04, \tilde{\sigma} = 0.4$ and $l = 25$.

with

\[
f(0, q, \tilde{S}) = -e^{-q\tilde{S}},
\]

(47)

\[
f(\tilde{\tau}, 0, \tilde{S}) = -1.
\]

(48)

Note that the optimal trading strategy takes the same form as (22).

For the CIR model it is necessary to be careful when establishing boundary conditions as the concept of positivity and non-negativity come into play, the difference being the inclusion of zero to the asset-price domain. Under certain conditions the CIR model is defined on a strictly positive domain, while when these condition are not satisfied the CIR model has a positive probability of reaching zero. The boundary behaviour of (44) has been studied in great detail by Feller (1951) in which he found that the CIR model is defined on a positive domain if (in our notation)

\[
2\tilde{\kappa}\tilde{\zeta} \geq \tilde{\sigma}^2.
\]

(49)

is satisfied for most realistic parameter values applicable to finance, and as such we shall focus only on this regime in this paper.

For the boundary condition at $\tilde{S} = 0$ we implement the degenerate form of the PDE (46) resulting in

\[
-f_{\tilde{\tau}}(\tilde{\tau}, q, 0) + \tilde{\kappa}\tilde{\zeta} f_{\tilde{S}}(\tilde{\tau}, q, 0) = 0,
\]

(50)

which could be seen as the boundary behaviour, as $\tilde{S}$ never reaches zero. Indeed, (50) implies that $f(\tau, q > 0, \tilde{S} = 0)$ approaches zero as $\tilde{\tau}$ increases since $f_{\tilde{S}}(\tilde{\tau} > 0, q, 0) > 0$, then as $\tilde{S} \to 0$, $\delta = O(\frac{1}{\tilde{S}})$.

For the case of $\tilde{S} \to \infty$ we found from numerical investigation that the same boundary condition as used for the GBM case, that being a Neumann condition given by (26), is consistent and satisfactory.

5.1 Numerical results

Several of the properties found in the results for the GBM model are mirrored in the CIR model and thus we shall not repeat them. However a unique property for the mean-reverting process
Figure 11: Value function and optimal trading strategy under CIR process at time $\tilde{\tau} = 2$. The parameters take the values: $q = 1$, $\zeta = 10$, $\tilde{\kappa} = 2.5$, $\tilde{\sigma} = 0.8$ and $l = 25$. Note the value function is not concave over the whole domain and as such the investor switches from being risk-averse to risk-seeking.

We can observe that the optimal trading strategy grows significantly as $\tilde{S}$ approaches zero (in line with our comments above). This is due to the mean-reverting characteristic of the asset price. When $\tilde{S}$ is near zero the investor expects the future value of the asset to revert to the long-term mean, $\tilde{\zeta}$. He will thus ask for a large multiple of the current asset price. Figure 12, which is the solution of (46) for $\tilde{S} = 0$, reinforces the explanation of why the optimal trading strategy increases in this region and also as $\tilde{\tau}$ increases.

The risk-seeking behaviour observed in figure 11(a) is more pronounced for higher values of the resilience, $l$, and the speed of reversion, $\tilde{\kappa}$. This is due to the investor decreasing his asking price more predominantly as the probability of execution decreases more rapidly for larger $l$ and the asset price decreases faster to the long-term mean $\tilde{\zeta}$ for large values of $\tilde{\kappa}$.

Similar to the case under GBM, the CIR model has interesting characteristics close to the terminal time which we shall investigate in the next section. However, unlike the GBM model, the CIR model does not tend to a perpetual type solution as $\tilde{\tau} \to \infty$, but rather the value diminishes towards zero, as evidenced by figure 12. Therefore we cannot construct a non-trivial perpetual solution in the same sense as we could for the GBM case. From a financial perspective, when $\tilde{S}$ is small (and less than $\tilde{\zeta}$) the investor asks for a relatively high price for the asset given it is expected to increase and revert back to the its long-term mean, $\tilde{\zeta}$. The investor’s value function thus approaches zero as $\tilde{\tau}$ increases.
Figure 12: Value function at $\tilde{S} = 0$. The parameters take the values: $q = 1, \zeta = 10, \kappa = 2.5, \sigma = 0.8$ and $l = 25$. Notice the approach of $f(\tilde{\tau}, q, \tilde{S} = 0)$ to zero as $\tilde{\tau}$ increases, which reinforces the growth in $\delta$ as $\tilde{S} \to 0$ as seen in figure 11(b).

5.2 Small-time-to-termination solution

To examine the small-$\tilde{\tau}$ solution we use the same perturbation method as used in section 3. Substituting (28) into (46) we collect the $O(\tilde{\tau}^0)$ and $O(\tilde{\tau}^1)$ terms in which we find the values for $f_1(q, \tilde{S})$ and $f_2(q, \tilde{S})$ to be

$$f_1(q, \tilde{S}) = \left( -\kappa (\zeta - \tilde{S}) q + \frac{1}{2} \sigma^2 \tilde{S} q^2 - \frac{\tilde{S}}{\tilde{S} + l} \left( \frac{l}{\tilde{S} + l} \right) \right) f_0(q, \tilde{S})$$

and

$$f_2(q, \tilde{S}) = \frac{1}{2} \left( \kappa (\zeta - \tilde{S}) f_1(q, \tilde{S}) + \frac{1}{2} \sigma^2 \tilde{S} f_1(q, \tilde{S}) - \frac{1}{\tilde{S} + l} \left( \frac{l}{\tilde{S} + l} \right) \right)$$

$$\times \left( (l + \tilde{S}) f_1(q, \tilde{S}) - lf_1(q - 1, \tilde{S}) e^{-\tilde{S}} \right)$$

(52)

A comparison of the full numerical solution against a two-term and three-term asymptotic can be found in figure 13. The same parameter values as in section 5.1 are used. We see the asymptotic approximation is of $O(1)$. Note that in this case, both $|f_1|$ and $|f_2|$ are both numerically large, which explains the initial rapid variation in values close to the terminal time, especially as $\tilde{S} \to 0$.

6 Conclusion

In this paper we have developed a novel approach to the optimal trading in the LOB problem. We suggest using GBM as the driving process for the asset along with an asset proportional control parameter (trading strategy) and asset proportional resilience. This is distinct from the standard Brownian motion and non-proportional control and resilience used by Guéant et al. (2012b). The trading strategies we found were variable with respect to the asset price, as opposed to the trading strategies found by Guéant et al. (2012b) which were found to be constant for all asset values. The former is a characteristic we feel would be more realistic in the finance industry.
Figure 13: Comparison of ‘exact’ value function and optimal trading strategy against a two-term and three-term asymptotic expansion for $\tilde{S} = \tilde{\zeta}$ for the CIR model. The parameters take the values: $\tilde{\zeta} = 10, \tilde{\kappa} = 2.5, \tilde{\sigma} = 0.8$ and $\tilde{l} = 5$.

Focusing on the problem, we reduced the four-dimensional HJB PDE to a three-dimensional non-linear PDE by finding an explicit form of the optimal control in terms of the reduced variable function. We also used a change of variables (non-dimensionalisation) which eliminated two parameters from the model. We used numerical methods to solve this problem, noting that the three-dimensional PDE approach is much less computational expensive than solving the full HJB PDE and is thus much more attractive from an algorithmic trading perspective.

We investigated both the small-time-to-termination solution and the perpetual solution, which provided us with a deeper understanding of the solution topology. The small-$\tilde{\tau}$ solution provided trading strategies which could be calculated extremely quickly and thus would be especially useful for a high-frequency trading framework in which little time is left to liquidate the portfolio and a quick solution is needed. The perpetual solution on the other hand provided trading strategies which could be implemented for an investor with a long time remaining before expiry, as well as an interesting asymptotical problem. We found a constraint for the perpetual case similar (but different from) that found by Guéant et al. (2012b) under standard Brownian motion.

Solving this problem numerically has led to the development of a methodology which can easily be generalised. This was highlighted in section 5 by changing the diffusion process for the asset price from a GBM to a mean-reverting process, that being the CIR process. Implementing other such mean-reverting processes such as the Uhlenbeck and Ornstein (1930) process, the Dixit and Pindyck (1994) process or the Schwartz (1997) process would also be straightforward using our numerical methods. As well as changing the stochastic process driving the asset price, numerical methods allow us to expand on this problem in many forms to make it more mathematically interesting and financially realistic. Examples include introducing another form of stochasticity such as including stochastic resilience, as suggested but not implemented by Cartea et al. (2011), or allow trading of multiple correlated assets, the latter which will be the topic of a future paper.
A HJB PDE and Verification Theorem

We will now discuss the derivation of the HJB PDE (8) using the Bellman (1957) Principle of Optimality and Itô’s Lemma followed by a verification theorem which is used to prove that the solution we obtain from the HJB PDE (8) is in fact the solution of the optimal control problem (7) we wish to solve.

A.1 Deriving HJB PDE using Bellman principle

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a filtration, \((\mathcal{F}_t, t \in [0, T])\). Define the Levy process \(L(t)\) adapted to the filtration \(\mathcal{F}_t\) as:

\[
\begin{bmatrix}
\frac{dS(t)}{dt} \\
\frac{dq(t)}{dt} \\
\frac{dX(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
\mu S(t) \\
0 \\
0
\end{bmatrix} dt +
\begin{bmatrix}
\sigma S(t) dW(t) \\
-dN(t) \\
S(t)(1 + \delta) dN(t)
\end{bmatrix}.
\]

We define the value function \(u(t, X, q, S)\) as

\[
u(t, X, q, S) = \sup_{\delta_t \in [0, T]} \mathbb{E}\left[ \Phi(X(T), q(T), S(T)) \right],
\]

where \(\mathcal{A} \in (-1, \infty)\) is the set of admissible trading strategies and \(\Phi(\cdot)\) is the form of the utility function. Bellman’s Principle of Optimality reads:

\[
u(t, X, q, S) = \sup_{\delta_t \in \mathcal{A}} \mathbb{E}\left[ u(t + h, X(t + h), q(t + h), S(t + h)) \right].
\]

The intuition behind Bellman’s Principle is that the supremum of the value function is found over \([t, T]\) if we find the supremum over \([t, t + h]\) and then continue optimally over \([t + h, T]\) with the state values at time \(t + h\) as the initial values.

To derive the dynamical programming equation we will first assume \(\delta(s) = \delta\) for \(s \in [t, t + h]\), i.e. the control parameter is constant over the interval \([t, t + h]\). From (55) we have:

\[
u(t, X, q, S) \geq \mathbb{E}\left[ u(t + h, X(t + h), q(t + h), S(t + h)) \right].
\]

Subtracting \(u(t, X, q, S)\) from both sides gives

\[
0 \geq \mathbb{E}\left[ u(t + h, X(t + h), q(t + h), S(t + h)) - u(t, x, q, s) \right].
\]

Assuming \(u \in C^{1,2}([0, T], \mathbb{R}^+)\) in \(t\) and \(S\) respectively, we now apply Itô’s lemma to \(u(t + h, X(t + h), q(t + h), S(t + h)) - u(t, x, q, s)\):

\[
u(t + h, X(t + h), q(t + h), S(t + h)) - u(t, X(t), q(t), S(t)) =
\]

\[
+ \int_t^{t+h} u_t(s, X(s),q(s),S(s)) ds + \int_t^{t+h} u_s(s, X(s),q(s),S(s)) dS(s)
\]

\[
+ \frac{1}{2} \int_t^{t+h} u_{ss}(s, X(s),q(s),S(s)) d[S,S](s)
\]

\[
+ \int_t^{t+h} (u(s, X(s) + \Delta X(s),q(s) + \Delta q(s),S(s)) - u(s, X(s),q(s),S(s))) dN(s),
\]

ignoring quadratic variation terms which equal zero. Here we are assuming continuity in the value function, and not the Levy process, and for further insight we refer the reader to Øksendal.
and Sulem (2005). Evaluating the quadratic variation $d[S,S](s) = \sigma^2 S(s)^2 ds$, we can expand (58) to

$$u(t+h, X(t+h), q(t+h), S(t+h)) - u(t, X(t), q(t), S(t)) = \int_t^{t+h} \left( u_t(s, X(s-), q(s-), S(s)) + \mu S(s) u_S(s, X(s-), q(s-), S(s)) + \frac{1}{2} \sigma^2 S(s)^2 u_{SS}(s, X(s-), q(s-), S(s)) + (u(s, X(s-) + \Delta X(s-), q(s-) + \Delta q(s-), S(s)) - u(s, X(s-), q(s-), S(s))) \Lambda(\delta) \right) ds$$

$$+ \int_t^{t+h} \sigma S(s) u_S(s, X(s-), q(s-), S(s)) dW(s) + \int_t^{t+h} (u(s, X(s-) + \Delta X(s-), q(s-) + \Delta q(s-), S(s)) - u(s, X(s-), q(s-), S(s))) d\tilde{N}(s),$$

(59)

with subscript denoting the partial derivative and $\tilde{N}(t)$ denoting a compensating Poisson process with intensity $\Lambda(\delta)$.

Substituting (59) into (57), taking the expectation, dividing by $h$ and letting $h \to 0$ we obtain:

$$0 \geq \mathcal{L}u(t, X, q, S) + \Lambda(\delta) (u(t, X + S(1+\delta), q - 1, S) - u(t, X, q, S)),$$

(60)

where we define the generator $\mathcal{L}$ of the Levy process as

$$\mathcal{L}u(t, X, q, S) = u_t(t, X, q, S) + \mu S(t) u_S(t, X, q, S) + \frac{1}{2} \sigma^2 S(t)^2 u_{SS}(t, X, q, S).$$

(61)

Here we assume the final two integrals are martingales, which we will show true in the verification theorem below.

(60) holds through for any $\delta \in \mathcal{A}$. Assume now that the optimal control is given by the Markov control policy $\delta^* = \delta^*(s, X^*(s), q^*(s), S(s))$. Here $X^*(s)$ and $q^*(s)$ denote the optimal state process controlled under $\delta^*$. Using the optimal policy and making the same assessment as above we obtain

$$0 = \mathcal{L}u(t, X, q, S) + \Lambda(\delta^*) (u(t, X + S(1+\delta), q - 1, S) - u(t, X, q, S)).$$

(62)

Combining (60) and (62) we obtain

$$0 = \sup_{\delta \in \mathcal{A}} [\mathcal{L}u(t, X, q, S) + \Lambda(\delta(t)) (u(t, X + S(1+\delta), q - 1, S) - u(t, X, q, S))].$$

(63)

As $u(t, X, q, S)$ is already defined as a supremum, defined by (54), we can take the generator processes outside the supremum so (63) becomes

$$0 = \mathcal{L}u(t, X, q, S) + \sup_{\delta \in \mathcal{A}} [\Lambda(\delta) (u(t, X + S(1+\delta), q - 1, S) - u(t, X, q, S))].$$

(64)

with

$$u(T, X, q, S) = \Phi(X, q, S),$$

(65)

and

$$u(t, X, 0, S) = \Phi(X, 0, S),$$

(66)

for a given function $\Phi(\cdot)$ which denotes the form of the utility function.
A.2 Verification theorem

We shall now show that the solution we obtain from solving (64) is in fact the solution of (54). Define

\[ \phi(t, X, q, S) := \mathbb{E}[\Phi(X(T), q(T), S(T))] , \tag{67} \]

with \( \phi \in C^{1,2}([0, T), \mathbb{R}^+ ) \) in \( t \) and \( S \) respectively. \( \Phi(X, q, S) = -e^{-\gamma(X+qS)} \) for \( q \in \mathbb{Z}^+ \), \( \gamma, X, S \in \mathbb{R}^+ \) and hence \( \Phi \in (-1, 0) \). By definition \( \phi \in (-1, 0) \) and hence it satisfies a polynomial growth condition, i.e. there exist constants \( C \in \mathbb{R}^+ \) and \( k \in \mathbb{Z}^+ \) such that

\[ |\phi(t, X, q, S)| \leq C \left( 1 + |X + qS|^k \right) , \quad \forall (t, X, q, S) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ . \]

Let \( \phi^* (t, X, q, S) \) be the solution of

\[ 0 = \mathcal{L}\phi(t, X, q, S) + \sup_{\delta \in A} [\Lambda(\delta) \phi(t, X + \delta, q - 1, S) - \phi(t, X, q, S)] , \tag{68} \]

with \( \mathcal{L} \) as the generator processes defined by (61). We now want to show that \( \phi^* (t, X, q, S) \) is identical to the solution obtained by solving (54).

We begin by applying a Taylor’s expansion to \( \phi^* (T, X, q, S) \) which gives

\[ \phi^* (T, X, q, S) = \phi^* (t, X, q, S) + \int_t^T d\phi^* (s, X, q, S) . \tag{69} \]

Since \( \phi \in C^{1,2}([0, T), \mathbb{R}^+ ) \) in \( t \) and \( S \) we have for all \( (t, X, q, S) \in [0, T] \times \mathbb{R}^+ \times \mathbb{Z}^+ \times \mathbb{R}^+ , u \in [t, T) \), and any stopping time \( \tau \) valued in \([t, \infty)\), by Itô’s formula

\[
\begin{align*}
\phi^* (u \land (t + h), X(u \land (t + h)), q(u \land (t + h)), S(u \land (t + h))) - \phi^* (t, X(t), q(t), S) = & \\
& \int_t^{u \land (t + h)} \left( \phi^* (s, X, q, S) + \mu S(s) \phi^*_{\sigma S}(s, X, q, S) + \frac{1}{2} \sigma^2 S(s)^2 \phi^*_{SS}(s, X, q, S) + \right) ds \\
& + \int_t^{u \land (t + h)} \sigma S(s) \phi^*_{\sigma S}(s, X(s), q(s), S(s)) dW(s) \\
& + \int_t^{u \land (t + h)} \sigma S(s) \phi^*_{\sigma S}(s, X(s), q(s), S(s)) d\tilde{N}(s),
\end{align*}
\]

with \( x \land y = \max(x, y) \). We choose \( \tau = \tau_n = \inf \{ u \geq t : \int_t^u |\sigma S(s) \phi^*_{\sigma S}(s, X(s), q(s), S(s))|^2 ds \geq n \} \) and notice that \( \tau_n \to \infty \) when \( n \) goes to infinity. The stopped process

\[
\int_t^{u \land \tau_n} \sigma S(s) \phi^*_{\sigma S}(s, X(s), q(s), S(s)) dW(s), \quad t \leq u \leq T,
\]

is then a martingale and

\[
\mathbb{E} \left[ \int_t^T (S(s) \phi^*_{\sigma S}(s, X(s), q(s), S(s)))^2 ds \right] < \infty.
\]

As the value function is bounded between -1 and 0, and noticing that \( \delta \) is bounded from below and hence \( \Lambda(\delta) \) is bounded we have that

\[
\mathbb{E} \left[ \int_t^T |\phi^* (s, X(s) + \Delta X(s), q(s) + \Delta q(s), S(s), \Lambda(\delta) ds | ds \right] < \infty,
\]

and

\[
\mathbb{E} \left[ \int_t^T |\phi^* (s, X(s), q(s), S(s), \Lambda(\delta) ds | ds \right] < \infty,
\]

24
so the final integral is a martingale with expectation zero.

We take the expected value of both sides of (69). The first integral goes to zero given it is identical to (68), as do the final two integrals as they have been shown to be martingales. This results in

$$
E \left[ \Phi \left( X^*, q^*, S \right) \right] = \phi^* \left( t, X, q, S \right),
$$

(70)
as \( \phi^* \left( T, X, q, S \right) = \Phi \left( X^*, q^*, S \right) \). If \( \delta \) was chosen arbitrarily we would have

$$
E \left[ \Phi \left( X, q, S \right) \right] \leq \phi^* \left( t, X, q, S \right).
$$

(71)

Thus we have

$$
\sup_{\delta \in A} E \left[ \Phi \left( X(T), q(T), S(T) \right) \right] = E \left[ \Phi \left( X^*, q^*, S \right) \right] = \phi^* \left( t, X, q, S \right) = u(t, X, q, S).
$$

(72)

We have thus shown that the solution found in our HJB PDE is the solution of the original optimisation problem. For further reading on the theory of optimal control the authors recommend Øksendal and Sulem (2005) and Pham (2009).

References


