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The Indian Schema Analogy Principles

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Abstract

We investigate the status within Unary Pure Inductive Logic of a family of analogy principles suggested by the so called Indian Schema from Gotama's Nyāyasūtra showing that they all follow from the symmetry principle of Atom Exchangeability. Their status under the weaker assumptions of Constant and Predicate Exchangeability and Strong Negation are also investigated.

Key words: Indian Schema, Nyāyasūtra, Analogy, Pure Inductive Logic, Logical Probability, Rationality, Uncertain Reasoning.

Introduction

In the Nyāyasūtra of Gotama[1], the founding Indian logic text from c.200BCE-150CE, the author aims to delineate in five terse aphorisms (Sūtras 3.1.32,3.1.34-37) a scheme for right reasoning. Subsequently numerous commentators, most

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[1]Aka Gautama, Akṣapāda.

notably Vātsyāyana (c.375CE-450CE), Uddyotakara (6th century CE), Gaṅgeśa (12th century CE), added their own explanations and developments, as well as incorporating revisions from other Indian Schools of Philosophy. When H.T.Colebrooke first introduced this Nyāya system of philosophy to the Victorian public in 1824 the pattern of reasoning he called the Hindu Syllogism, subsequently dubbed the Indian Schema, was exemplified by a handful of examples,[2] most prominently the Smoke-Fire example:[3]

(a) *Where there is smoke there is fire, like in the kitchen.*

(b) *There is smoke on the hill.*

(c) *Therefore there is fire on the hill.*

How exactly such examples should be understood, in their classical Sanskrit context or within the framework of contemporary notions of reasoning (e.g. deductive, default, case based, etc.) has been the subject of much debate, see [3] for a sample. In particular it is a moot point why the instance 'like in the kitchen' is present at all since the earlier 'Where there is smoke there is fire' would seem to render it redundant. Certainly its featuring there was taken by some Victorians to dismiss the Indian Schema as simply invalid analogical reasoning, from particular to particular.

In a previous paper, [11], we mentioned possible grounds for supposing that Gotama's original intention for line (a) (the udāharaṇa) was just to cite an instance of smoke in a kitchen being the result of a fire and that the later explicit introduction of the universal by the Buddhist logician Dharmakīrti, see Oetke's [10], or possibly his predecessor Dignāga (c.480CE-540CE), see Ganeri's [5, page 38],[4] represented a shift from analogical to deductive reasoning. Viewed in this way then the Indian Schema becomes:

(a) *When there was smoke in the kitchen there was fire.*

(b) *There is smoke on the hill.*

(c) *Therefore there is fire on the hill.* $(\star)$

Whether or not this was Gotama's original intention, we continued in [11] to argue that nevertheless, and not withstanding the Victorians' rebuke, the 'reasoning' in this version can be justified as *rational* within the context of Pure Inductive

[2] For a quick introduction see J.Ganeri's [4].

[3] This is the (last) three line form meant for reasoning for oneself, see [13].

[4] Ganeri also points out in this paper that Vātsyāyana (c.375CE-450CE) had already made a significant step in this direction.

Logic (hereafter PIL) – see [12]. Our interest in this paper is to investigate more fully some of the analogical principles of probability assignment resulting from various possible formalizations of ($\star$). Before doing so we need to briefly recall the framework of PIL as presented, for example, in [12].

The Pure Inductive Logic Context

Pure Inductive Logic as described in [12] is conventionally set within a predicate language L with a finite set of relation symbols and countably many constant symbols $a_1, a_2, a_3, \ldots$ and no function symbols nor equality. Let SL denote the set of sentences of L_q and let $QFSL$ denote the quantifier free sentences in L.

A probability function on L is a function $w : SL \to [0, 1]$ such that for $\theta, \phi, \exists x\, \psi(x) \in SL$,

(i) If $\models \theta$ then $w(\theta) = 1$.

(ii) If $\theta \models \neg\phi$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$.

(iii) $w(\exists x\, \psi(x)) = \lim_{n\to\infty} w\left(\bigvee_{i=1}^{n} \psi(a_i)\right)$.

From these all the expected properties of probability follow (see [12, Proposition 3.1]), in particular if $\theta \models \phi$ then $w(\theta) \leq w(\phi)$.

Given such a w we set the conditional probability function $w(\cdot \mid \cdot) : SL \times SL \to [0, 1]$ to be a function such that for $\theta, \phi \in SL$ with $w(\phi) > 0$,

$$w(\theta \mid \phi) = \frac{w(\theta \wedge \phi)}{w(\phi)}\ .$$

In practice it will be convenient to identify

$$w(\theta \mid \phi) = c \ \text{ with } \ \ w(\theta \wedge \phi) = cw(\phi)$$

since this avoids separating the cases when $w(\phi)$ is zero and non-zero.

In PIL we are at this stage of its development primarily interested in elucidating 'rationality constraints' on w in the case when the symbols of L are entirely uninterpreted. So if w is to represent a 'rational' assignment of probabilities to the sentences of L in the absence of any particular meaning of, and information about, the constants and the predicates what properties in addition to (i)-(iii) should w satisfy?

Numerous such constraints, usually in the form of principles that w should obey, have been proposed based on various intuitions of what 'rational' might mean but the most forceful, going back to Johnson [7] and Carnap [1] (or see Carnap's Axioms for Inductive Logic at [14, p973]), are those based on symmetry, the idea being that it would be irrational of w to break existing symmetries in the language. At its simplest level this has been understood as saying that if we have an isomorphism of the symbols of a language then the probability assigned to a sentence should be the same as that assigned to its symbol-wise image under that isomorphism, because the isomorphism provides, or witnesses, a symmetry between sentences and their images.

The most obvious example of such a symmetry is when the isomorphism simply permutes the constant symbols and leaves the relation symbols fixed. In this case the requirement of preserving symmetries, that is of assigning the same probability to a sentence as to its isomorphic image, amounts to:

The Constant Exchangeability Principle, Ex

If $\theta \in SL$ and the constant symbol a_j does not appear in θ then $w(\theta) = w(\theta')$ where θ' is the result of replacing each occurrence of a_i in θ by a_j.

Analogous to Constant Exchangeability but this time permuting relation symbols of the same arity gives:

The Predicate Exchangeability Principle, Px

If the relation symbols Q, R of L have the same arity and R does not appear in $\theta \in SL$ then $w(\theta) = w(\theta')$ where θ' is the result of replacing each occurrence of Q in θ by R.

Satisfying these two principles is widely viewed as necessary for w to be considered rational. A third symmetry condition is based on the idea that since the context is supposed to be entirely uninterpreted there is symmetry between R and $\neg R$,[5] just in the same way as there is between heads and tails when we toss a coin. This yields:

The Strong Negation Principle, SN

$w(\theta) = w(\theta')$ where θ' is the result of replacing each occurrence of the relation symbol R in θ by $\neg R$.

Until fairly recently Inductive Logic has been almost entirely concerned with unary languages, that is where all the relation (or predicate) symbols have a single argument. In this case there is a further widely accepted[6] symmetry principle, Atom Exchangeability. To wit let L_q be the language whose only relation (i.e. predicate)

[5] Since $\neg\neg R \equiv R$. Again there is an underlying, albeit more complicated, isomorphism.
[6] Since it holds for the members of Carnap's Continuum of Inductive methods

symbols are the unary $P_1, P_2, \ldots, P_q$ and let $\alpha_1(x), \alpha_2(x), \ldots, \alpha_{2^q}(x)$ be the *atoms* of L_q, that is the formulae of the form

$$P_1^{\epsilon_1}(x) \wedge P_2^{\epsilon_2}(x) \wedge \ldots \wedge P_q^{\epsilon_q}(x)$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_q \in \{0,1\}$ and $P_i^\epsilon = P_i$ if $\epsilon = 1$, $\neg P_i$ if $\epsilon = 0$.

The Atom Exchangeability Principle, Ax

If σ is a permutation of $\{1, 2, \ldots, 2^q\}$ then

$$w\left(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = w\left(\bigwedge_{i=1}^{n} \alpha_{\sigma(h_i)}(a_i)\right) \tag{1}$$

In the case of Ax then the 'symmetry' is between possible complete descriptions of constants – knowing which atom a particular a_i satisfies tells us everything there is to know about a_i as such. [For more on the purported 'rationality' of this principle see [12, p87].] Notice that both Px and SN follow from Ax.

In the case of this purely unary language L_q Constant Exchangeability, Ex, has two consequences which we shall be needing later. The first is de Finetti's Representation Theorem (in the context of this paper). To explain this let

$$\mathbb{D}_{2^q} = \{\langle x_1, x_2, \ldots, x_{2^q}\rangle \,|\, x_i \geq 0, \sum_i x_i = 1\}$$

and for $\vec{x} \in \mathbb{D}_{2^q}$ let

$$w_{\vec{x}}\left(\bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = \prod_{i=1}^{n} x_{h_i}.$$

As shown in [12] $w_{\vec{x}}$ extends to a probability function on L_q which satisfies Ex. Indeed every probability function on L_q satisfying Ex is a convex combination of these $w_{\vec{x}}$:

de Finetti's Representation Theorem

If w is s probability function on L_q satisfying Ex then there is a countably additive normalized measure μ on $\mathbb{D}_{2^q}$ such that for any $\theta(a_1, a_2, \ldots, a_n) \in SL_q$,

$$w(\theta(a_1, a_2, \ldots, a_n)) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta(a_1, a_2, \ldots, a_n))\, d\mu,$$

and conversely.

We refer to μ as the de Finetti prior of w. If w additionally satisfies Ax then it may be assumed that the measure μ is invariant under permutations of the coordinates, see [12, Chapter 14].

A consequence of this theorem, due to Gaifman, [2] or see [12, p71], which we shall also need later is:

The Extended Principle of Instantial Relevance, EPIR

For $\theta(a_1, a_2, \ldots, a_n), \phi(a_1) \in SL_q$,

$$w(\phi(a_{n+2}) \,|\, \phi(a_{n+1}) \wedge \theta(a_1, a_2, \ldots, a_n)) \geq w(\phi(a_{n+2}) \,|\, \theta(a_1, a_2, \ldots, a_n)). \qquad (2)$$

Two particular probability functions, $c_\infty^{L_q}$ and $c_0^{L_q}$, will figure in some of the results which follow.

Carnap's $c_\infty^{L_q}$ (aka m^*) equals $w_{\vec{x}}$ when

$$\vec{x} = \langle 2^{-q}, 2^{-q}, \ldots, 2^{-q} \rangle \in \mathbb{D}_{2^q}.$$

This probability function has the property that for any sentence $\theta(a_1) \in SL_q$ and $n \geq 0$,

$$c_\infty^{L_q}(\theta(a_{n+1}) \,|\, \theta(a_1) \wedge \theta(a_2) \wedge \ldots \wedge \theta(a_n)) = c_\infty^{L_q}(\theta(a_{n+1})). \qquad (3)$$

That is, $c_\infty^{L_q}$ denies any inductive support, the probability it gives to $\theta(a_{n+1})$ is unaffected by the evidence that $\theta(a_i)$ held for $i = 1, 2, \ldots, n$ no matter how large n may be.

Carnap's $c_0^{L_q}$ equals

$$2^{-q} \sum_{i=1}^{2^q} w_{\vec{e_i}}$$

where $\vec{e_i} \in \mathbb{D}_{2^q}$ is the vector with 1 in the ith coordinate and 0's elsewhere. This probability function has the property that for any sentence $\theta(a_1)$ and $n > 0$,

$$c_0^{L_q}(\theta(a_{n+1}) \,|\, \theta(a_1) \wedge \theta(a_2) \wedge \ldots \wedge \theta(a_n)) = c_0^{L_q}(\theta(a_{n+1}) \,|\, \theta(a_1)) = 1.^7 \qquad (4)$$

In this case then $c_0^{L_q}$ derives the maximal possible inductive support for $\theta(a_{n+1})$ on the basis of a single given $\theta(a_1)$. So as the notation already hints it is at the other end of the 'inductive scale' from $c_\infty^{L_q}$.

On account of the above properties neither $c_\infty^{L_q}$ nor $c_0^{L_q}$ are particularly widely favoured choices for blank slate probability assignments in the absence of any intended interpretation.

[7] Recall the convention regarding zero divisors in conditional probabilities.

Unary Formalizations

To make the forthcoming results more immediate we will write S, F, h, k for P_1, P_2, a_1, a_2.[8]

As argued in [11] for w a probability function on L_2[9] we might formalize ($\star$) in one of the forms[10,11]

$$w(F(h) \mid (S(k) \to F(k)) \wedge S(h)) > 1/2, \tag{5}$$

$$w(F(h) \mid (S(k) \leftrightarrow F(k)) \wedge S(h)) > 1/2, \tag{6}$$

$$w(F(h) \mid S(k) \wedge F(k) \wedge S(h)) > 1/2, \tag{7}$$

since a probability of more than 1/2 could, in the uninterpreted, ceteris paribus, context of PIL, be taken as a justification for opting for $F(h)$ in the sense that it must be more probable than $\neg F(h)$. In [11] we showed that with the exception of $w = c_\infty^{L_2}$, any w satisfying Ex+Px+SN must also satisfy each of (5), (6) and (7); for $w = c_\infty^{L_2}$ the inequality in each of (5), (6) and (7) becomes equality.[12] So excluding $w = c_\infty^{L_2}$, each of (5), (6) and (7) is at least as rational as Ex+Px+SN.

Hence the only way one could argue that while accepting Ex+Px+SN, ($\star$) as formalised by (5), (6) or (7) was not justified in PIL, would be if one held that $c_\infty^{L_2}$ was an acceptable choice. But then since, from (3),

$$c_\infty^{L_2}(F(a_{n+1}) \mid F(a_1) \wedge F(a_2) \wedge \ldots, \wedge F(a_n)) = c_\infty^{L_2}(F(a_{n+1})) = 1/2$$

one would also have to argue that even under the evidence of $F(a_1), F(a_2), \ldots, F(a_n)$ the conclusion that $F(a_{n+1})$ was not justified, no matter how large n was.

Buoyed by the apparent success that (5), (6) and (7) follow from Ex+Px+SN (unless $w = c_\infty^{L_2}$) one might naturally raise the question whether the belief that there is fire on the hill should not be greater the more kitchen fires one has experienced. In other words should not (5), (6) and (7) for w a probability function on L_2 be enhanced to

[8] Since we will always have Ex+Px the choices of suffices here are not important.

[9] We could equally take L_q ($q \geq 2$) here in place of L_2.

[10] We shall consider other possibilities later.

[11] In [11] we employed $\geq$ rather than $>$ but the strict inequality obviously carries more weight. Since the equality occurs only in very special cases, as discussed below, we prefer to adopt the strict inequality here.

[12] If instead we had taken L_q in place of L_2 the exceptions would have been those w which equal $c_\infty^{L_2}$ when restricted to SL_2.

$$w(F(h)\,|\,\bigwedge_{i=1}^{n+1}(S(k_i)\to F(k_i))\wedge S(h)) > w(F(h)\,|\,\bigwedge_{i=1}^{n}(S(k_i)\to F(k_i))\wedge S(h)), \quad (8)$$

$$w(F(h)\,|\,\bigwedge_{i=1}^{n+1}(S(k_i)\leftrightarrow F(k_i))\wedge S(h)) > w(F(h)\,|\,\bigwedge_{i=1}^{n}(S(k_i)\leftrightarrow F(k_i))\wedge S(h)), \quad (9)$$

$$w(F(h)\,|\,\bigwedge_{i=1}^{n+1}(S(k_i)\wedge F(k_i))\wedge S(h)) > w(F(h)\,|\,\bigwedge_{i=1}^{n}(S(k_i)\wedge F(k_i))\wedge S(h)), \quad (10)$$

for $n \geq 0$? (These are indeed enhancements since under the assumption of Ex+Px+SN, $w(F(h)\,|\,S(h)) = 1/2$.)

Our plan now is to relate these particular 'Indian Schema Principles' (8), (9) and (10) to the established symmetry principles stated in the previous section. We start with Atom Exchangeability Ax.

Assuming Atom Exchangeability

It turns out that (8), (9) and (10) all essentially follow from Ex+Ax. Indeed the following stronger result holds:

Theorem 1. *For w a probability function on L_q satisfying Ex+Ax, $\theta(a_1), \phi(a_1) \in QFSL_q$,*

$$w(\theta(a_{n+2})\,|\,\phi(a_{n+2})\wedge \bigwedge_{i=1}^{n+1}\theta(a_i)) \geq w(\theta(a_{n+1})\,|\,\phi(a_{n+1})\wedge \bigwedge_{i=1}^{n}\theta(a_i)),$$

with equality just if $\theta(a_i)\wedge\phi(a_i)$ is inconsistent or $\neg\theta(a_i)\wedge\phi(a_i)$ is inconsistent or $w = c_\infty^{L_q}$ or $w = c_0^{L_q}$ and $n > 0$.

Proof. It is straightforward to check that if $\theta(a_i)\wedge\phi(a_i)$ is inconsistent or $\neg\theta(a_i)\wedge\phi(a_i)$ is inconsistent then the result holds with equality so assume that neither of these hold. Let the de Finetti representation of w satisfying Ax be

$$w = \int w_{\vec{x}}\, d\mu$$

where μ is invariant under permutations of the coordinates.

Without loss of generality let

$$\theta(a_1) \equiv \bigvee_{i=1}^{r} \alpha_i(a_1), \qquad \phi(a_1) \equiv \bigvee_{i=1}^{m} \alpha_i(a_1) \vee \bigvee_{i=r+1}^{k} \alpha_i(a_1)$$

where $m \leq r$ and by our earlier assumption $0 < m, r < k$. Then the required inequality becomes

$$\frac{\int (\sum_{i=1}^{m} x_i)(\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{m} x_i)(\sum_{i=1}^{r} x_i)^{n}\, d\mu} \geq \frac{\int (\sum_{i=1}^{m} x_i + \sum_{i=r+1}^{k} x_i)(\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{m} x_i + \sum_{i=r+1}^{k} x_i)(\sum_{i=1}^{r} x_i)^{n}\, d\mu},$$

equivalently

$$\frac{\int (\sum_{i=1}^{m} x_i)(\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{m} x_i)(\sum_{i=1}^{r} x_i)^{n}\, d\mu} \geq \frac{\int (\sum_{i=r+1}^{k} x_i)(\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=r+1}^{k} x_i)(\sum_{i=1}^{r} x_i)^{n}\, d\mu}.$$

By Ax, for $1 \leq j \leq r$ and $s \in \mathbb{N}$,

$$m \int x_j \left(\sum_{i=1}^{r} x_i \right)^{s} d\mu = \int \left(\sum_{i=1}^{m} x_i \right) \left(\sum_{i=1}^{r} x_i \right)^{s} d\mu$$

and for $r + 1 \leq j \leq 2^q$

$$(k - r) \int x_j \left(\sum_{i=1}^{r} x_i \right)^{s} d\mu = \int \left(\sum_{i=r+1}^{k} x_i \right) \left(\sum_{i=1}^{r} x_i \right)^{s} d\mu.$$

Hence it suffices to show that

$$\frac{\int (\sum_{i=1}^{r} x_i)(\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{r} x_i)(\sum_{i=1}^{r} x_i)^{n}\, d\mu} \geq \frac{\int (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^{n}\, d\mu},$$

equivalently

$$\frac{\int (\sum_{i=1}^{r} x_i)^{n+2}\, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu} \geq \frac{\int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu - \int (\sum_{i=1}^{r} x_i)^{n+2}\, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n}\, d\mu - \int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}.$$

This amounts to

$$\frac{\int (\sum_{i=1}^{r} x_i)^{n+2}\, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu} \geq \frac{\int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n}\, d\mu} \tag{11}$$

which holds by EPIR.

Finally if equality held in the theorem for some n then we would have equality in (11) for some n, equivalently

$$\int \left((\sum_{i=1}^{r} x_i) - \frac{\int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n}\, d\mu} \right)^2 (\sum_{i=1}^{r} x_i)^n \, d\mu = 0. \tag{12}$$

Let

$$a = \frac{\int (\sum_{i=1}^{r} x_i)^{n+1}\, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n}\, d\mu}.$$

First consider $n = 0$. For (12) to hold, $\sum_{i=1}^{r} x_i$ must be equal to a for μ-almost all $\vec{x}$. Since μ is invariant under permutations of coordinates, the same must be true for the sum of any r of the x_i. Consequently, for μ-almost all $\vec{x}$, the sum of any r of the x_i must be a. Any two coordinates of such an $\vec{x}$ must be equal so μ is the discrete measure giving all the weight to $\vec{x} = \langle 2^{-q}, 2^{-q}, \ldots, 2^{-q} \rangle$ and hence $w = c_{\infty}^{L_q}$.

When $n \neq 0$, for (12) to hold $\sum_{i=1}^{r} x_i$ must be equal to a or to 0 for μ-almost all $\vec{x}$. Again, since μ is invariant under permutations of coordinates, the same must be true for the sum of any r of the x_i. Consequently, for μ-almost all $\vec{x}$, the sum of any r of the x_i must be a or 0. Any two coordinates of such an $\vec{x}$ must be either equal or differ by a. This is only possible when as before, $a = r2^{-q}$ and $\vec{x} = \langle 2^{-q}, 2^{-q}, \ldots, 2^{-q} \rangle$, or when $a = 1$ and $\vec{x} = \langle 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0 \rangle$. Since w satisfies Ax, it follows that $w = c_{\infty}^{L}$ or $w = c_0^{L_q}$. □

We remark that all of (8), (9) and (10) do follow from the theorem: we take $S(x)$ for $\phi(x)$ in both cases and $S(x) \to F(x)$, $S(x) \leftrightarrow F(x)$ or $F(x) \wedge S(x)$ respectively for $\theta(x)$ and note that $\theta(x) \wedge \phi(x)$ is logically equivalent to $F(x) \wedge S(x)$.

Again any argument against the evidence in (8), (9) and (10) providing a justification for $F(h)$ would seem to require one to hold the view that $c_{\infty}^{L_2}$ or $c_0^{L_2}$ was an acceptable choice. With $c_{\infty}^{L_2}$ there is the same counter as there was with Theorem 2 and using (4) a similar one can clearly be formulated for $c_0^{L_2}$.

Assuming Ex+Px+SN

Despite Johnson and Carnap's acceptance of Ax it seems that this is quite a step beyond Ex+Px+SN as far as being self evidently rational. For this reason it would be good if (8), (9) and (10) followed from just Ex+Px+SN since it would strengthen any claim as to their 'rationality'.

Treating (8) first:

Theorem 2. *Let w satisfy Ex+Px+SN. Then (8) holds for $n = 0, 1$, indeed*

$$w(F(h) \mid \bigwedge_{i=1}^{2}(S(k_i) \to F(k_i)) \wedge S(h)) \geq w(F(h) \mid (S(k_1) \to F(k_1)) \wedge S(h)) \quad (13)$$

$$\geq w(F(h) \mid S(h)) = 1/2 \quad (14)$$

with equality in (13) *just if* $w = c_\infty^{L_2}$ *or* $w = c_0^{L_2}$ *and equality in* (14) *just if* $w = c_\infty^{L_2}$.

Proof. The second inequality above is (5), and the result was proved in [11].

Turning to (13), $\alpha_1(x) = S(x) \wedge F(x)$, $\alpha_2(x) = S(x) \wedge \neg F(x)$, $\alpha_3(x) = \neg S(x) \wedge F(x)$, $\alpha_4(x) = \neg S(x) \wedge \neg F(x)$. We need to show that

$$\frac{w(\alpha_1(h) \wedge (\alpha_1(k_1) \vee \alpha_3(k_1) \vee \alpha_4(k_1)) \wedge (\alpha_1(k_2) \vee \alpha_3(k_2) \vee \alpha_4(k_2))}{w((\alpha_1(h) \vee \alpha_2(h)) \wedge (\alpha_1(k_1) \vee \alpha_3(k_1) \vee \alpha_4(k_1)) \wedge (\alpha_1(k_2) \vee \alpha_3(k_2) \vee \alpha_4(k_2))}$$

is greater or equal to

$$\frac{w(\alpha_1(h) \wedge (\alpha_1(k_1) \vee \alpha_3(k_1) \vee \alpha_4(k_1)))}{w((\alpha_1(h) \vee \alpha_2(h)) \wedge (\alpha_1(k_1) \vee \alpha_3(k_1) \vee \alpha_4(k_1)))}.$$

We can record this economically as

$$\frac{w(\alpha_1(\alpha_1 + \alpha_3 + \alpha_4)^2)}{w((\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3 + \alpha_4)^2)} \geq \frac{w(\alpha_1(\alpha_1 + \alpha_3 + \alpha_4))}{w((\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3 + \alpha_4))};$$

an equivalent inequality is

$$\frac{w(\alpha_1(\alpha_1 + \alpha_3 + \alpha_4)^2)}{w(\alpha_2(\alpha_1 + \alpha_3 + \alpha_4)^2)} \geq \frac{w(\alpha_1(\alpha_1 + \alpha_3 + \alpha_4))}{w(\alpha_2(\alpha_1 + \alpha_3 + \alpha_4))}.$$

Noting that by Ex, for distinct constants $a_{i_1}, a_{i_2}, a_{i_3}$ and $s, v, r \in \{1, 2, 3, 4\}$

$$w(\alpha_s(a_{i_1}) \wedge \alpha_v(a_{i_2}) \wedge \alpha_r(a_{i_3}))$$

depends only on s, v, r, we record this value as $w(\alpha_s\alpha_v\alpha_r)$ or, when for example $s = r$ as $w(\alpha_s^2\alpha_v)$ etc. Let $p = w(\alpha_1^3)$, $y = w(\alpha_1^2\alpha_2)$, $t = w(\alpha_1^2\alpha_4)$, $z = w(\alpha_1\alpha_2\alpha_3)$. By Px+SN we have furthermore

$$p = w(\alpha_2^3) = w(\alpha_3^3) = w(\alpha_4^3),$$

$$y = w(\alpha_1^2\alpha_3) = w(\alpha_4^2\alpha_2) = w(\alpha_4^2\alpha_3) = w(\alpha_2^2\alpha_1) = w(\alpha_2^2\alpha_4) = w(\alpha_3^2\alpha_1) = w(\alpha_3^2\alpha_4),$$

$$t = w(\alpha_4^2\alpha_1) = w(\alpha_2^2\alpha_3) = w(\alpha_3^2\alpha_2),$$

$$z = w(\alpha_1\alpha_2\alpha_4) = w(\alpha_1\alpha_3\alpha_4) = w(\alpha_2\alpha_3\alpha_4)$$

so any $w(\alpha_i\alpha_j\alpha_k)$ is given by one of p, y, t, z.

Consequently for example

$$w(\alpha_1\alpha_2) = w(\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)) = 2y + 2z,$$

and

$$w(\alpha_1\alpha_4) = 2t + 2z, \qquad w(\alpha_1^2) = p + t + 2y,$$

so since we also have $w(\alpha_1\alpha_3) = w(\alpha_1\alpha_2) = w(\alpha_2\alpha_4)$ and $w(\alpha_1\alpha_4) = w(\alpha_2\alpha_3)$ the required inequality becomes

$$\frac{p + 3t + 3y + 2z}{t + 2y + 6z} \;\geq\; \frac{p + 3t + 4y + 4z}{2t + 4y + 6z},$$

which simplifies to

$$pt + 2py + 4y^2 + 3t^2 + 8yt \geq 12z^2 + 6yz. \tag{15}$$

We now make two claims which will be proved later:

Claim 1: $p \geq y$ with equality just when $w = c_\infty^{L_2}$.

Claim 2: $y + t \geq 2z$.

Returning to the inequality (15), from Claim 2 we have $3(y+t)^2 \geq 12z^2$ and $y(y+t) \geq 2yz$ so

$$pt + 2py + 4y^2 + 3t^2 + 8yt \geq pt + 2py + 12z^2 + 2yz + yt.$$

But by Claims 1 and 2 we also have

$$pt + 2py + yt \geq yt + 2y^2 + yt = 2y(y+t) \geq 4yz, \tag{16}$$

so (15) follows.

It remains to show the Claims 1 and 2 and to consider when equality occurs in (15). Fortunately as the Claims are purely linear we can use the representation theorem which tells us that any probability function on the language $\{S, F\}$ satisfying Ex+Px+SN can be expressed as an integral (see [6, Lemma 6], dropping the redundant AP) using the probability functions

$$\begin{aligned} 8^{-1}(&w_{\langle x_1,x_2,x_3,x_4\rangle} + w_{\langle x_1,x_3,x_2,x_4\rangle} + w_{\langle x_4,x_2,x_3,x_1\rangle} + w_{\langle x_4,x_3,x_2,x_1\rangle} \\ &+ w_{\langle x_2,x_1,x_4,x_3\rangle} + w_{\langle x_2,x_4,x_1,x_3\rangle} + w_{\langle x_3,x_1,x_4,x_2\rangle} + w_{\langle x_3,x_4,x_1,x_2\rangle}), \end{aligned} \tag{17}$$

where the x_i are nonnegative real numbers summing to 1 and

$$w_{\langle x_i,x_j,x_k,x_r\rangle}(\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4}) = x_i^{m_1}x_j^{m_2}x_k^{m_3}x_r^{m_4}$$

Hence it is enough to show that $p \geq y$ and $y + t \geq 2z$ hold for this probability function. But these amount, respectively, to

$$2(x_1^3+x_2^3+x_3^3+x_4^3) \;\geq\; x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_4+x_3^2x_1+x_3^2x_4+x_4^2x_2+x_4^2x_3,$$

$$\iff \quad \left.\begin{array}{l}(x_1-x_2)^2(x_1+x_2)+(x_1-x_3)^2(x_1+x_3)\\ +\ (x_4-x_2)^2(x_4+x_2)+(x_4-x_3)^2(x_4+x_3)\end{array}\right\} \geq 0,$$

with equality just when $x_1 = x_2 = x_3 = x_4$, and

$$\begin{array}{c}x_1^2(x_2+x_3+2x_4)+x_2^2(x_1+x_4+2x_3)+\ x_3^2(x_1+x_4+2x_2)+x_4^2(x_2+x_3+2x_1)\\ \geq\ 2(2x_1x_2x_3+2x_2x_1x_4+2x_3x_1x_4+2x_4x_2x_3)\end{array}$$

$$\iff \quad \left.\begin{array}{l}x_2(x_1-x_3)^2+x_2(x_4-x_3)^2+x_3(x_1-x_2)^2+x_3(x_4-x_2)^2\\ +\ x_1(x_2-x_4)^2+x_1(x_3-x_4)^2+x_4(x_2-x_1)^2+x_4(x_3-x_1)^2\end{array}\right\} \geq 0.$$

with equality just when $x_1 = x_2 = x_3 = x_4$ or one of the x_i is 1.

Hence both claims hold and equality in Claim 1 can occur only when $w = c_\infty^{L_2}$ whilst in Claim 2 equality occurs just when the mixing measure featuring in the above mentioned representation of w gives measure 1 to the set of functions (17) with $x_1 = x_2 = x_3 = x_4$ or with one of the x_i equal to 1.

It follows that the first inequality in (16) is strict unless $t = y = 0$ or $w = c_\infty^{L_2}$. The former happens just when $w = c_0^{L_2}$ and hence equality in (15) occurs just when w is one of Carnap's $c_0^{L_2}$ and $c_\infty^{L_2}$.[13]

□

Contrary to expectations however (8) can fail under Ex+Px+SN for $n > 2$. A counter-example is provided by a function of the form (17), with suitable x_1, x_2, x_3, x_4. For given x_1, x_2, x_3, x_4 (to be specified later) let $\tilde{w}$ be

$$8^{-1}(w_{\langle x_1,x_2,x_3,x_4\rangle} + w_{\langle x_1,x_3,x_2,x_4\rangle} + w_{\langle x_4,x_2,x_3,x_1\rangle} + w_{\langle x_4,x_3,x_2,x_1\rangle}$$

[13]see e.g. [12].

$$+ \, w_{\langle x_2,x_1,x_4,x_3\rangle} + w_{\langle x_2,x_4,x_1,x_3\rangle} + w_{\langle x_3,x_1,x_4,x_2\rangle} + w_{\langle x_3,x_4,x_1,x_2\rangle}).$$

and define

$$\begin{aligned} R(n) &= \frac{\tilde{w}(\alpha_1(\alpha_1+\alpha_3+\alpha_4)^n)}{(\tilde{w}(\alpha_1+\alpha_2)(\alpha_1+\alpha_3+\alpha_4)^n)} \\ &= \tilde{w}\left(F(h) \,|\, \bigwedge_{i=1}^{n}(S(k_i) \to F(k_i)) \wedge S(h)\right). \end{aligned} \tag{18}$$

Write $A = x_1 + x_4$ and $B = x_2 + x_3$. We have

$$w_{\langle x_1,x_2,x_3,x_4\rangle}(\alpha_1(\alpha_1+\alpha_3+\alpha_4)^n) = x_1(A+x_3)^n,$$

$$w_{\langle x_1,x_2,x_3,x_4\rangle}((\alpha_1+\alpha_2)(\alpha_1+\alpha_3+\alpha_4)^n) = (x_1+x_2)(A+x_3)^n$$

etc so collecting terms, $R(n)$ is

$$\frac{A(A+x_3)^n + A(A+x_2)^n + B(B+x_1)^n + B(B+x_4)^n}{(A+2x_2)(A+x_3)^n + (A+2x_3)(A+x_2)^n + (B+2x_4)(B+x_1)^n + (B+2x_1)(B+x_4)^n}.$$

Multiplying $R(n+1) \geq R(n)$ by the denominators and substracting the RHS from the LHS shows it equivalent to $C - D \geq 0$ where C is the sum of products of terms from these two columns:

$$\begin{array}{ll} A(A+x_3)^{n+1} & (A+2x_2)(A+x_3)^n \\ A(A+x_2)^{n+1} & (A+2x_3)(A+x_2)^n \\ B(B+x_1)^{n+1} & (B+2x_4)(B+x_1)^n \\ B(B+x_4)^{n+1} & (B+2x_1)(B+x_4)^n \end{array}$$

and D is the sum of products in these two columns:

$$\begin{array}{ll} A(A+x_3)^n & (A+2x_2)(A+x_3)^{n+1} \\ A(A+x_2)^n & (A+2x_3)(A+x_2)^{n+1} \\ B(B+x_1)^n & (B+2x_4)(B+x_1)^{n+1} \\ B(B+x_4)^n & (B+2x_1)(B+x_4)^{n+1} \end{array}$$

This amounts to the sum of the following 12 terms being non-negative:

$$
\begin{array}{l}
(A+x_3)^n\ (A+x_2)^n\ A\ (A+2x_3)\ (x_3-x_2)\\
(A+x_3)^n\ (B+x_4)^n\ A\ (B+2x_1)\ (A+x_3-B-x_4)\\
(A+x_3)^n\ (B+x_1)^n\ A\ (B+2x_4)\ (A+x_3-B-x_1)\\
(A+x_2)^n\ (A+x_3)^n\ A\ (A+2x_2)\ (x_2-x_3)\\
(A+x_2)^n\ (B+x_1)^n\ A\ (B+2x_4)\ (A+x_2-B-x_1)\\
(A+x_2)^n\ (B+x_4)^n\ A\ (B+2x_1)\ (A+x_2-B-x_4)\\
(B+x_4)^n\ (A+x_3)^n\ B\ (A+2x_2)\ (B+x_4-A-x_3)\\
(B+x_4)^n\ (A+x_2)^n\ B\ (A+2x_3)\ (B+x_4-A-x_2)\\
(B+x_4)^n\ (B+x_1)^n\ B\ (B+2x_4)\ (x_4-x_1)\\
(B+x_1)^n\ (A+x_3)^n\ B\ (A+2x_2)\ (B+x_1-A-x_3)\\
(B+x_1)^n\ (A+x_2)^n\ B\ (A+2x_3)\ (B+x_1-A-x_2)\\
(B+x_1)^n\ (B+x_4)^n\ B\ (B+2x_1)\ (x_1-x_4)
\end{array}
$$

Combining the obvious pairs yields the sum of the following six terms

$$
\begin{array}{l}
(A+x_3)^n\ (A+x_2)^n\ A\ 2(x_3-x_2)^2\\
(A+x_3)^n\ (B+x_4)^n\ 2(Ax_1-Bx_2)\ (A+x_3-B-x_4)\\
(A+x_3)^n\ (B+x_1)^n\ 2(Ax_4-Bx_2)\ (A+x_3-B-x_1)\\
(A+x_2)^n\ (B+x_1)^n\ 2(Ax_4-Bx_3)\ (A+x_2-B-x_1)\\
(A+x_2)^n\ (B+x_4)^n\ 2(Ax_1-Bx_3)\ (A+x_2-B-x_4)\\
(B+x_4)^n\ (B+x_1)^n\ B\ 2(x_4-x_1)^2
\end{array}
$$

Rewriting A and B back in terms of the x_i, this is the sum of

$$
\begin{array}{l}
(1-x_2)^n\ (1-x_3)^n\ (x_1+x_4)\ 2(x_3-x_2)^2\\
(1-x_2)^n\ (1-x_1)^n\ 2(x_1^2-x_2^2+x_1x_4-x_2x_3)\ (x_1-x_2)\\
(1-x_2)^n\ (1-x_4)^n\ 2(x_4^2-x_2^2+x_1x_4-x_2x_3)\ (x_4-x_2)\\
(1-x_3)^n\ (1-x_4)^n\ 2(x_4^2-x_3^2+x_1x_4-x_2x_3)\ (x_4-x_3)\\
(1-x_3)^n\ (1-x_1)^n\ 2(x_1^2-x_3^2+x_1x_4-x_2x_3)\ (x_1-x_3)\\
(1-x_1)^n\ (1-x_4)^n\ (x_2+x_3)\ 2(x_4-x_1)^2
\end{array}
$$

Let $\epsilon > 0$ and

$$x_1 = 1-6\epsilon, \quad x_2 = 3\epsilon, \quad x_3 = 2\epsilon, \quad x_4 = \epsilon\,.$$

Then $1-x_1$ and x_2, x_3, x_4 are of order ϵ, so the second, fifth and sixth products are of order ϵ^n whilst the first, third and fourth are respectively

$$(1-3\epsilon)^n\ (1-2\epsilon)^n\ (1-5\epsilon)\ 2\epsilon^2$$

$$(1-3\epsilon)^n\ (1-\epsilon)^n\ 2(\epsilon-20\epsilon^2)\ (-2\epsilon)$$

$$(1-2\epsilon)^n\ (1-\epsilon)^n\ 2(\epsilon-15\epsilon^2)\ (-\epsilon)$$

so their sum is negative of order ϵ^2. Hence for $n > 2$ and a sufficiently small ϵ this is a counterexample to $R(n+1) \geq R(n)$. We remark that the very same approach does not work with $n = 2$ (3 kitchens) because with these $x_1, \ldots, x_4$ all but the last product are of the order ϵ^2 (the last one being of order ϵ^3) and their sum for small ϵ is positive.

Also, looking again at $R(n)$ with general x_1, x_2, x_3, x_4, if x_4 is the strictly smallest of the x_i then $(B + x_1)^n$ dominates and the limit of $R(n)$ is

$$\frac{B}{B + 2x_4} = \frac{x_2 + x_3}{x_2 + x_3 + 2x_4} \tag{19}$$

so not 1 but greater than 1/2, and similarly when it is another x_i.

The status of the one remaining case, when $n = 2$, the '3 kitchens problem', is open. Given these counter-examples one might wonder if it was possible that Ex+Px+SN was not enough to even justify jumping to the conclusion $F(h)$ on the basis of more than 2 kitchen fires. Fortunately that recommendation is still good:

Theorem 3. *For w satisfying Ex+Px+SN, and any $n \geq 1$*

$$w(F(a_{n+1}) \,|\, S(a_{n+1}) \wedge \bigwedge_{i=1}^{n} (S(a_i) \to F(a_i))) \geq 1/2 = w(F(a_{n+1}) \,|\, S(a_{n+1})),$$

with equality just if $w = c_\infty^{L_2}$.

Proof. Using the usual de Finetti Representation Theorem for w a probability function on L_2 satisfying Ex, the required inequality becomes

$$\frac{\int x_1(x_1 + x_3 + x_4)^n \, d\mu(\vec{x})}{\int (x_1 + x_2)(x_1 + x_3 + x_4)^n \, d\mu(\vec{x})} \geq \frac{1}{2}.$$

Simplifying gives

$$\int x_1(x_1 + x_3 + x_4)^n \, d\mu(\vec{x}) \geq \int x_2(x_1 + x_3 + x_4)^n \, d\mu(\vec{x}),$$

equivalently

$$\int (x_1 - x_2)(1 - x_2)^n \, d\mu(\vec{x}) \geq 0.$$

By the trick in [12, page 90] we can assume that the measure μ is invariant under those permutations of coordinates which are 'licensed' by SN+Px, in particular the permutation transposing x_1, x_2 (and x_3, x_4). Hence

$$\int (x_1 - x_2)(1 - x_2)^n \, d\mu(\vec{x}) = \int (x_2 - x_1)(1 - x_1)^n \, d\mu(\vec{x})$$

and it is enough to show that

$$\int (x_1 - x_2)(1-x_2)^n + (x_2 - x_1)(1-x_1)^n \, d\mu(\vec{x}) \geq 0.$$

But the polynomial being integrated here is just

$$(x_1 - x_2)((1-x_2)^n - (1-x_1)^n)$$

which equals

$$(x_1 - x_2)^2 \sum_{i=0}^{n-1} (1-x_1)^{n-1-i}(1-x_2)^i \ \geq \ 0 \tag{20}$$

so the result clearly holds.

Finally we can only have equality in (20) for all support points of μ if $x_1 = x_2$ (and perforce $x_1 = x_3 = x_4$ by the assumed invariance of μ under permutations licensed by Px+SN) for all support points so the only possible support point is $\langle 4^{-1}, 4^{-1}, 4^{-1}, 4^{-1} \rangle$ and w on this sublanguage must be $c_\infty^{L_2}$.

□

On a more positive note however we can fully answer this question when it comes to (9):

Theorem 4. *For w a probability function on L_2 satisfying Ex+Px+SN and $n \geq 0$,*

$$w(F(h) \,|\, \bigwedge_{i=1}^{n+1} (S(k_i) \leftrightarrow F(k_i)) \wedge S(h)) \geq w(F(h) \,|\, \bigwedge_{i=1}^{n} (S(k_i) \leftrightarrow F(k_i)) \wedge S(h)). \tag{21}$$

We remark that equality does hold for some special probability functions but they can be dismissed on similar grounds as $c_\infty^{L_2}$ and $c_0^{L_2}$, see below.

Proof. Using the same notation as above, we need to show that for a probability function w satisfying Ex+Px+SN and $n \geq 0$,

$$\frac{w(\alpha_1(\alpha_1+\alpha_4)^{n+1})}{w((\alpha_1+\alpha_2)(\alpha_1+\alpha_4)^{n+1})} \geq \frac{w(\alpha_1(\alpha_1+\alpha_4)^n)}{w((\alpha_1+\alpha_2)(\alpha_1+\alpha_4)^n)}, \tag{22}$$

equivalently that

$$\frac{w(\alpha_1(\alpha_1+\alpha_4)^n)}{w(\alpha_2(\alpha_1+\alpha_4)^n)}$$

is non-decreasing. To this end, it suffices to show that

$$\frac{w(\alpha_1(\alpha_1+\alpha_4)^n)}{w((\alpha_1+\alpha_4)^n)}$$

is non decreasing and

$$\frac{w(\alpha_2(\alpha_1+\alpha_4)^n)}{w((\alpha_1+\alpha_4)^n)}$$

is non-increasing. This follows by EPIR since by SN+Px

$$w(\alpha_1(\alpha_1+\alpha_4)^n) = \frac{1}{2} w((\alpha_1+\alpha_4)(\alpha_1+\alpha_4)^n)$$

and

$$w(\alpha_2(\alpha_1+\alpha_4)^n) = \frac{1}{2} w((\alpha_2+\alpha_3)(\alpha_1+\alpha_4)^n) = \frac{1}{2}(w((\alpha_1+\alpha_4)^n) - w((\alpha_1+\alpha_4)(\alpha_1+\alpha_4)^n))$$

□

As in the corresponding proof[14] of Theorem 1 if we have equality in (23) for some n and $n+1$ then either every point in the support of μ must be of one of the form $\langle x_1, x_2, (1/2)-x_2, (1/2)-x_1\rangle$ or every point in the support of μ must be of the form $\langle 0, x_2, 1-x_2, 0\rangle$ or $\langle x_1, 0, 0, 1-x_1\rangle$ and $n>0$.

These conditions are not enough to force w to be one of $c_\infty^{L_2}$ or $c_0^{L_2}$. However a similar argument can be made to rebuke the probability functions w which do give equality here. Namely they would have to satisfy

$$w((\alpha_1+\alpha_4) \,|\, (\alpha_1+\alpha_4)^n) = w((\alpha_1+\alpha_4) \,|\, (\alpha_1+\alpha_4))$$

for all $n>0$.

For (10) we would need to show that for a probability function w satisfying Ex+Px+SN - albeit possibly with some dismissable exceptions - and for $n \geq 0$,

$$\frac{w(\,\alpha_1^{n+2}\,)}{w((\alpha_1+\alpha_2)\,\alpha_1^{n+1})} \;\geq\; \frac{w(\,\alpha_1^{n+1}\,)}{w((\alpha_1+\alpha_2)\,\alpha_1^{n})}, \tag{23}$$

equivalently that

$$\frac{w(\,\alpha_2\,\alpha_1^n\,)}{w(\,\alpha_1^{n+1}\,)} \geq \frac{w(\,\alpha_2\,\alpha_1^{n+1}\,)}{w(\,\alpha_1^{n+2}\,)}.$$

It is straightforward to show that this is the case for all the functions of the form (17) but it is currently not clear if it holds in general. Nevertheless, we do have

[14] Arguing about the probability function for the language with one predicate R which we obtain from w upon replacing $\alpha_1 \vee \alpha_4$ by R and $\alpha_2 \vee \alpha_3$ by $\neg R$.

Theorem 5. *For w a probability function on L_2 satisfying Ex+Px+SN, and $n \geq 1$*

$$w(F(a_{n+1}) \,|\, S(a_{n+1}) \wedge \bigwedge_{i=1}^{n} (S(a_i) \wedge F(a_i))) \geq 1/2 = w(F(a_{n+1}) \,|\, S(a_{n+1})),$$

with equality just if $w = c_\infty^{L_2}$.

Proof. Proceeding as in the proof of Theorem 3, we use the usual de Finetti Representation Theorem for w a probability function on L_2 satisfying Ex. The required inequality becomes

$$\frac{\int x_1^{n+1} \, d\mu(\vec{x})}{\int (x_1 + x_2) x_1^n \, d\mu(\vec{x})} \geq \frac{1}{2},$$

equivalently

$$\int x_1^{n+1} \, d\mu(\vec{x}) \geq \int x_2 \, x_1^n \, d\mu(\vec{x}),$$

that is,

$$\int (x_1 - x_2) \, x_1^n \, d\mu(\vec{x}) \geq 0.$$

Again since w satisfies also Px we can assume that the measure μ is invariant under the permutation transposing x_1, x_2 and x_3, x_4. Hence

$$\int (x_1 - x_2) \, x_1^n \, d\mu(\vec{x}) = \int (x_2 - x_1) \, x_2^n \, d\mu(\vec{x})$$

and it is enough to show that

$$\int (\, (x_1 - x_2) \, x_1^n + (x_2 - x_1) \, x_2^n \,) \, d\mu(\vec{x}) \geq 0.$$

But the polynomial being integrated here is just

$$(x_1 - x_2)(x_1^n - x_2^n)$$

which is clearly nonnegative, so the result follows. The last part about $c_\infty^{L_2}$ follows as in Theorem 3. □

Mill's Property

In [9, Vol.7,p186] the Scottish philosopher J.S.Mill suggested (as others have since) that when we use for example

All men are mortal

to conclude that the Duke of Wellington is mortal it is not that we already know all instances of this universal but that we know a sufficient number of them to feel justified in saving mental storage space by rounding up our knowledge to 'All men are mortal'. In other words we are transforming an argument by induction into a fully deductive argument. From this viewpoint then the reality of the Indian Schema for one reasoning to oneself might be read as:

(a) *In the many cases I have experienced of smoke there has invariably been fire.*

(b) *There is smoke on the hill.*

(c) *Therefore there is fire on the hill.*

If such reasoning can be taken to be in some sense 'rational' then it suggests we should investigate the status within PIL of probability functions w on L_q satisfying the somewhat more general principle:

Mill's Property, MP

For $\theta(a_1), \phi(a_1) \in QFSL_q$ with $w(\theta(a_1) \wedge \phi(a_1)) > 0$,

$$lim_{n\to\infty} w\left(\theta(a_{n+1}) \mid \phi(a_{n+1}) \wedge \bigwedge_{i=1}^{n} \theta(a_i)\right) = 1 \tag{24}$$

Theorem 6. *Let w be a probability function on L_q satisfying Ax and with de Finetti prior μ. Then w satisfies MP just if all the points $\langle 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0\rangle$ are in the support of μ.*

Proof. First suppose that $\vec{x} = \langle 1, 0, 0, \ldots, 0\rangle$ is not in the support of μ, say that $\mu(A_\delta) = 0$ where $\delta > 0$ and

$$A_\delta = \{\vec{y} \in \mathbb{D}_{2^q} \mid |\vec{y} - \vec{x}| < \delta\}.$$

Then

$$\begin{aligned} w(\alpha_1(a_{n+1}) \mid \bigwedge_{i=1}^{n} \alpha_1(a_i)) &= \frac{\int_{\mathbb{D}_{2^q}} x_1^{n+1}\, d\mu}{\int_{\mathbb{D}_{2^q}} x_1^{n}\, d\mu} \\ &= \frac{\int_{\mathbb{D}_{2^q}-A_\delta} x_1^{n+1}\, d\mu}{\int_{\mathbb{D}_{2^q}-A_\delta} x_1^{n}\, d\mu} \\ &\leq \frac{\int_{\mathbb{D}_{2^q}-A_\delta} (1-\delta) x_1^{n}\, d\mu}{\int_{\mathbb{D}_{2^q}-A_\delta} x_1^{n}\, d\mu} \quad < 1, \end{aligned}$$

so MP fails for this θ and $\phi = \top$.

In the other direction suppose that each of these points $\langle 0,0,\ldots,0,1,0,\ldots,0,0\rangle$ is in the support of μ and let

$$\theta(a_1) \equiv \bigvee_{i=1}^{r} \alpha_i(a_1), \qquad \phi(a_1) \equiv \bigvee_{i=1}^{m} \alpha_i(a_1) \vee \bigvee_{i=r+1}^{k} \alpha_i(a_1)$$

where $0 < m \leq r$. We may assume that $k \geq r+1$ otherwise the result is immediate.

We need to show that

$$\frac{\int (\sum_{i=r+1}^{k} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu}{\int (\sum_{i=1}^{m} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu}.$$

tends to zero as $n \to \infty$. Using Ax as in the proof of Theorem 1 it is enough to show that

$$\frac{\int (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu}{\int (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu} \tag{25}$$

tends to zero as $n \to \infty$.

Let $0 < \delta < \nu$ and

$$B_\delta = \{\vec{x} \in \mathbb{D}_{2^q} \mid \sum_{i=1}^{r} x_i \geq 1 - \delta\}$$

and similarly for ν. By the assumption of MP, $\mu(B_\delta) > 0$.

We can write (25) as

$$\frac{\int_{B_\nu} (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu + \int_{\mathbb{D}_{2^q} - B_\nu} (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu}{\int_{B_\delta} (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu + \int_{B_\nu - B_\delta} (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu + \int_{\mathbb{D}_{2^q} - B_\nu} (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu}. \tag{26}$$

Since

$$\frac{\int_{B_\nu} (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu}{\int_{B_\delta} (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu + \int_{B_\nu - B_\delta} (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu} \leq \frac{\nu}{1-\nu},$$

and

$$\frac{\int_{\mathbb{D}_{2^q} - B_\nu} (\sum_{i=r+1}^{2^q} x_i)(\sum_{i=1}^{r} x_i)^n \, d\mu}{\int_{B_\delta} (\sum_{i=1}^{r} x_i)^{n+1} \, d\mu} \leq \frac{(1-\nu)^n (1 - \mu(B_\nu))}{(1-\delta)^{n+1} \mu(B_\delta)}$$

it follows that by choosing δ, ν sufficiently small and then n sufficiently large we can make (26) arbitrarily small, as required. □

This proof has assumed Ax. If we only assume Ex+Px+SN then Mill's Property may not hold as is apparent from (18) and (19).

We remark that a corollary of Theorem 6 is that, assuming Ax and regularity (i.e. $w(\theta) > 0$ whenever $\theta \in QFSL_q$ is consistent), Reichenbach's Axiom, see [12] for a formulation in the notation of this paper, implies Mill's Property since by Theorem 15.1 of that monograph that axiom is equivalent to *every* point in $\mathbb{D}_{2^q}$ being a support point of μ.

The Lake

The version of the Indian Schema which we have considered here is based on Sūtra 36 which is commonly referred to as a 'homogeneous example'. We have variously formalized this as

$$S(k) \to F(k), \quad S(k) \leftrightarrow F(k) \tag{27}$$

or

$$S(k) \wedge F(k). \tag{28}$$

However in Sūtra 37[15] Gotama describes another sort of example, a heterogeneous example. According to S.C.Vidyabhusana's rendering of the Sūtra, [15, p12]:

> *A heterogeneous (or negative) example is a familiar instance which is known to be devoid of the property to be established and which implies that the absence of this property is invariably rejected in the reason given.*[16]

In our smoke-fire scenario a commonly stated such example is that of the lake [which is both fire and smoke free] which is combined with the homogeneous example of the kitchen to give the schema

(a) *Where there is smoke there is fire, like in the kitchen and (un)like on the lake.*

(b) *There is smoke on the hill.*

[15] *tad-viparyayād vā viparītam.*

[16] Translations here from the original Sanskrit are considered notoriously difficult. We are grateful to one of the referees for suggesting the more literal

> *Or in the case opposite to that [i.e. the above-mentioned positive example] it [the udāharaṇa] is contrary [to the case at issue].*

where the parenthetic comments have been added by him/her for clarification and are not part of the Sanskrit original.

(c) *Therefore there is fire on the hill.*

Exactly how we should formalize the lake (denoted l) example is not clear (to us) but it would seem that given the formalizations in (27) it should be, respectively:

$$\neg F(l) \to \neg S(l), \ \neg S(l) \leftrightarrow \neg F(l),$$

which are simply covered by the two kitchen version. For (28) it should be

$$\neg F(l) \wedge \neg S(l).$$

However there are probability functions satisfying Ex+Ax for which the inequality

$$w(F(h) \,|\, S(h) \wedge S(k) \wedge F(k) \wedge \neg S(l) \wedge \neg F(l)) \geq w(F(h) \,|\, S(h) \wedge S(k) \wedge F(k)) \quad (29)$$

does not hold.[17] On this evidence then it seems that the appropriateness of capturing the example by a conjunction rather than an implication or bi-implication is questionable.

Conclusion

In this paper we have limited ourselves to the most natural present day formulations of the Indian Schema, namely treating 'smoke', or 'smoky', as a predicate and 'hill' as a constant, etc. and have shown how most of these can claim to be justified as rational, at least if $c_0^{L_q}, c_\infty^{L_q}$ are excluded, on the basis of following from various symmetry principles in PIL.

Given the arcane complexities of Sanskrit however it is certainly not clear, even unlikely, that the formalization presented here was how Gotama and the subsequent commentators on the Nyāyasūtra would have seen it. For example it has been suggested that 'hill' might have been thought of as a predicate and smoke, or smokiness, as a constant etc. or that they are both constants and the connection between them is via a binary relation A of 'happens at', see [8]. We plan to investigate these alternatives in a future paper but for the present we should emphasize

[17] For w satisfying Ax (29) reduces to $2y^2 \geq xz + yz$ where $x = w(\alpha_1^3)$, $y = w(\alpha_1^2\alpha_2)$ and $z = w(\alpha_1\alpha_2\alpha_3)$. This fails in the case of the probability function

$$4^{-1}(w_{\langle 1-3\epsilon,\epsilon,\epsilon,\epsilon\rangle} + w_{\langle \epsilon,1-3\epsilon,\epsilon,\epsilon\rangle} + w_{\langle \epsilon,\epsilon,1-3\epsilon,\epsilon\rangle} + w_{\langle \epsilon,\epsilon,\epsilon,1-3\epsilon\rangle})$$

with $\epsilon > 0$ very small since it does satisfy Ax but gives, up to lowest powers of ϵ, $x = 1/4$, $y = \epsilon/4$, $z = 3\epsilon^2/4$ so $2y^2 < xz$.

that our primary purpose in this paper, and the earlier paper [11] which it extends, has not been to argue about what Gotama et al could have meant but rather to justify the rationality of the version of the schema as it seemed many Victorian (and later) readers dismissively understood it.

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