

Invariant Manifolds and Sets

Glendinning, Paul

2005

MIMS EPrint: **2006.103**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: <http://eprints.maths.manchester.ac.uk/>

And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

NL3154 Invariant manifolds and sets

An invariant set S of a dynamical system has the property that every trajectory (or orbit) which starts in S remains in S . Invariant sets are the basic building blocks of dynamical systems and almost every set which is dynamically interesting has some invariance property. Every trajectory is an invariant set, and so equilibria and periodic orbits are simple examples of such objects. Attractors are also examples of invariant sets.

An invariant set which is also a manifold (i.e. it looks like R^n locally) is called an invariant manifold. Invariant manifolds provide a natural description of the dynamics close to an equilibrium or periodic orbit (using linearization) and can also make it possible to work in lower dimensions than the phase space of the problem, since a smooth dynamical system restricted to an invariant manifold is itself a dynamical system. This reduction in dimension is at the heart of center manifold techniques and inertial manifold methods.

Invariance is a topological property, and does not depend upon stability. This is often useful, and the sudden appearance or disappearance of attractors as parameters are varied can sometimes be understood as a change in stability of an invariant set which exists throughout the parameter region of interest. Indeed, many invariant sets, and particularly invariant manifolds, have persistence properties under perturbations of the dynamical system which make them useful when working with families of systems.

The basic definitions will be given for differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in R^n$ and solutions $\mathbf{x}(t)$, and maps $\mathbf{v}_{n+1} = \mathbf{g}(\mathbf{v}_n)$, $\mathbf{v} \in R^n$. In both cases the complications that arise if solutions do not exist for all time will be ignored.

For differential equations a set S is a *forward invariant* set (resp. *backward invariant* set) if $\mathbf{x}(0) \in S$ implies that $\mathbf{x}(t) \in S$ for all $t \geq 0$ (resp. all $t \leq 0$). S is *invariant* if it is both forward invariant and backward invariant.

For maps a set S is a *forward invariant* set (resp. *backward invariant* set) if $\mathbf{v} \in S$ implies that $\mathbf{g}(\mathbf{v}) \in S$ (resp. $\mathbf{g}^{-1}(\mathbf{v}) \in S$). S is *invariant* if it is both forward invariant and backward invariant.

In the remainder of this entry, some different sorts of invariant sets which arise in dynamics will be considered, illustrated by the Lorenz equations

$$\dot{x} = \sigma(y - x), \quad \dot{y} = -y + rx - xz, \quad \dot{z} = -bz + xy \quad (1)$$

where σ , r , and b are positive constants.

Trapping regions

A set R is a trapping region if it is forward invariant and if all solutions are eventually contained in R . Trapping regions provide the first approximation to the location of any attractors in the system.

For the Lorenz equations, (1), with $\sigma = 10$ and $b = \frac{8}{3}$ Sparrow (1982) shows that the region

$$R = \{(x, y, z) \mid rx^2 + 10y^2 + 10(z - 2r)^2 \leq 5r^2\} \quad (2)$$

is a trapping region. This can be proved using the Lyapunov function $V(x, y, z) = rx^2 + 10y^2 + 10(z - 2r)^2$.

Stable and unstable manifolds of stationary points

A stationary point of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a solution which is constant for all time, so it must satisfy $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The three equations obtained by setting $\dot{x} = \dot{y} = \dot{z} = 0$ in (1) can be solved to show that the origin is always a stationary point of the Lorenz equations and provided $r > 1$ there are two other stationary points: $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. Near a stationary point the most important terms of the differential equation are the linear terms so it is natural to ask whether the solutions of the linearized equation at the origin

$$\dot{x} = \sigma(y - x), \quad \dot{y} = -y + rx, \quad \dot{z} = -bz \quad (3)$$

obtained by ignoring the nonlinear terms in (1) reflect properties of the full equations close to the stationary point at the origin. These results are usually stated in terms of the eigenvalues of the matrix which defines the linear equation (3):

$$M = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \quad (4)$$

If M has no eigenvalues with zero real parts then the manifold (vector space) spanned by the eigenvectors with eigenvalues having negative real parts contains all the stable directions, and hence is called the stable manifold of the linear system, (3), while the manifold spanned by the eigenvectors with eigenvalues having positive real parts contains the unstable directions. The former manifold is called the stable manifold, E^s , of the origin, and the latter is the unstable manifold, E^u . If $0 < r < 1$ then all the eigenvalues of M are negative, so the stable manifold of the linear system is the whole space and the unstable manifold is the empty set. If $r > 1$ then M has two negative eigenvalues and one positive eigenvalue, so the unstable manifold of the linear system is a line, and the stable manifold is a plane.

This provides definitions of stable and unstable manifolds for linear systems such as (3) with no eigenvalues with zero real parts. To extend the definition to nonlinear systems we begin by noting that in the definition used for linear systems, the stable manifold of the origin contains the solutions that tend to the origin in forward time, and the unstable manifold contains those that tend to the origin in backwards time. These properties can be used to define the stable and unstable manifolds of a stationary point \mathbf{x}^* for a nonlinear system: the stable manifold of \mathbf{x}^* , $W^s(\mathbf{x}^*)$ is the set of initial conditions for which the solution $\mathbf{x}(t)$ tends to \mathbf{x}^* as $t \rightarrow \infty$, while the unstable manifold, $W^u(\mathbf{x}^*)$, is the set of initial conditions for which the solution $\mathbf{x}(t)$ tends to \mathbf{x}^*

as $t \rightarrow -\infty$. The stable manifold theorem states that provided the matrix that defines the linearized equation near a stationary point has no eigenvalues with zero real parts then $W^s(\mathbf{x}^*)$ and $W^u(\mathbf{x}^*)$ exist, are invariant sets, and are of the same dimension as the corresponding linear stable and unstable manifolds. Moreover, they are each tangential to the corresponding linear manifold at the stationary point itself. See Guckenheimer & Holmes (1983) for more details.

Calculating stable and unstable manifolds

The stable manifold theorem suggests how approximations to the stable and unstable manifolds of a stationary point can be calculated close to a stationary point. First, it is tangential to the corresponding linear manifold, and secondly it is invariant. This is enough to obtain a power series solution, and this will be illustrated using the unstable manifold of the origin of the Lorenz equations with $\sigma = 1$ and $r = 4$.

Step 1: [Linear approximation] The eigenvalues of M in equation (4) are 1, -3 , and $-b$ if $\sigma = 1$ and $r = 4$. The unstable manifold is one-dimensional, and the unstable manifold of the linear approximation is the eigenvector corresponding to the positive eigenvalue, i.e. $(1, 2, 0)$ or $y = 2x, z = 0$.

Step 2: [Canonical form] It is good practice to bring the linear part into canonical form at this stage, see Guckenheimer & Holmes (1983). For brevity this step will be omitted.

Step 3: [Power series] Since the unstable manifold is tangential to the linear approximation of Step 1 at the stationary point, it can be represented *locally* as a function of one of the variables. From the form of the linear approximation either x or y can be used here. If the defining equations are smooth then the stable and unstable manifolds are smooth, so choosing x as the independent variable we write

$$y = 2x + \alpha x^2 + O(x^3) \quad z = \beta x^2 + O(x^3) \quad (5)$$

where the linear terms are obtained from the linear approximation, α and β are constants to be determined, and $O(x^3)$ denotes the terms of order x^3 and higher that are small (locally) compared to the linear and quadratic terms. To identify the constants α and β two different ways of calculating \dot{y} and \dot{z} are used and then compared in the next three steps.

Step 4: [Power series evolution] Differentiating (5) gives $\dot{y} \sim 2\dot{x} + 2\alpha x\dot{x}$ and $\dot{z} \sim 2\beta x\dot{x}$ (ignoring the higher order terms). Now, on the unstable manifold given by (5) $\dot{x} = -x + y \sim x + \alpha x^2$, so

$$\dot{y} \sim 2(x + \alpha ax^2) + 2\alpha x^2 \sim 2x + 4\alpha x^2, \quad \dot{z} \sim 2\beta x^2 \quad (6)$$

where higher order terms have been ignored again.

Step 5: [Differential equation evolution] From (1) and (5) with $r = 4$ and $\sigma = 1$

$$\dot{y} = 4x - y - xz \sim 2x - \alpha x^2, \quad \dot{z} = -bz + xy \sim -b\beta x^2 + 2x^2 \quad (7)$$

Step 6: [Equate coefficients] To determine the two constants α and β we now simply equate coefficients of powers of x in (6) and (7). From the \dot{y} equations $4\alpha = -\alpha$, so

$\alpha = 0$, and from the \dot{z} equations $2\beta = -b\beta + 2$, so $\beta = 2/(2 + b)$. This gives the second order approximation to the unstable manifold:

$$y = 2x + O(x^3) \quad z = \frac{2}{2+b}x^2 + O(x^3) \quad (8)$$

which is valid close to the stationary point at the origin. Higher order terms can be calculated by including more terms in the power series expansions.

Center manifolds

If the matrix that defines the linearized differential equation at a stationary point has eigenvalues with zero real parts then three invariant manifolds can be defined: a strong stable manifold corresponding to eigenvalues with negative real parts, a strong unstable manifold corresponding to eigenvalues with positive real parts, and a center manifold which is tangential at the stationary point to the space spanned by the eigenvectors corresponding to eigenvalues with zero real parts. The motion on the strong stable and unstable manifolds is defined by the dominant linear terms, i.e. towards and away from the stationary point respectively. The motion on the center manifold depends on the nonlinear terms of the differential equation and may be stable or unstable or neutral. Center manifolds can be approximated locally using the same ideas as the approximation of the unstable manifold above, and this leads to the technique of center manifold reduction which is central to the development of bifurcation theory. The origin of the Lorenz equations has a one-dimensional center manifold if $r = 1$ (M in (4) has an eigenvalue of zero) which signals the bifurcation creating the non-trivial pair of stationary points if $r > 1$.

Unstable chaotic sets

The chaotic sets created by horseshoes are not stable, but they can be important from two points of view. First, if an unstable chaotic set exists in a system then it can manifest itself in chaotic transients before a stable attractor is reached. Second, if there are parameters in the system then the chaotic set may gain stability and so the sudden appearance of strange attractors can be explained by understanding the change of stability of a chaotic invariant set. Grebogi, Ott & Yorke (1982) describe some of these mechanisms as *crises*. In the Lorenz equations with $b = \frac{8}{3}$ and $\sigma = 10$ a strange invariant set (an unstable chaotic set with a fractal structure) is created by a homoclinic bifurcation as r is increased through $r \approx 13.93$ but it does not become stable until $r \approx 24.06$, see Sparrow (1982) for details.

There are many other types of invariant sets and properties of these sets that could have been described here—the selection above is only the tip of an invariant iceberg.

See also Attractors; Bifurcations; Center manifold reduction; Chaotic dynamics; Horseshoes and hyperbolicity in dynamical systems; Linearization; Lorenz equations; One-dimensional maps; Routes to chaos

Further Reading

Grebowi, C., Ott, E. & Yorke, J.A. 1982. Chaotic attractors in crisis. *Physics Review Letters*, 48: 1507–1510

Guckenheimer, J. & Holmes, P. 1983. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, New York: Springer

Sparrow, C. 1982. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, New York: Springer