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SYMMETRIC LINEARIZATIONS FOR MATRIX POLYNOMIALS*

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Abstract. A standard way of treating the polynomial eigenvalue problem $P(\lambda)x = 0$ is to convert it into an equivalent matrix pencil—a process known as linearization. Two vector spaces of pencils $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$, and their intersection $\mathbb{DL}(P)$, have recently been defined and studied by Mackey, Mackey, Mehl, and Mehrmann. The aim of our work is to gain new insight into these spaces and the extent to which their constituent pencils inherit structure from P. For arbitrary polynomials we show that every pencil in $\mathbb{DL}(P)$ is block symmetric and we obtain a convenient basis for $\mathbb{DL}(P)$ built from block Hankel matrices. This basis is then exploited to prove that the first $\deg(P)$ pencils in a sequence constructed by Lancaster in the 1960s generate $\mathbb{DL}(P)$. When P is symmetric, we show that the symmetric pencils in $\mathbb{L}_1(P)$ comprise $\mathbb{DL}(P)$, while for Hermitian P the Hermitian pencils in $\mathbb{L}_1(P)$ form a proper subset of $\mathbb{DL}(P)$ that we explicitly characterize. Almost all pencils in each of these subsets are shown to be linearizations. In addition to obtaining new results, this work provides a self-contained treatment of some of the key properties of $\mathbb{DL}(P)$ together with some new, more concise proofs.

Key words. matrix polynomial, matrix pencil, linearization, companion form, quadratic eigenvalue problem, vector space, block symmetry

AMS subject classifications. 65F15, 15A18

1. Introduction.

The polynomial eigenvalue problem $P(\lambda)x = 0$, where

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \qquad A_i \in \mathbb{C}^{n\times n}, \quad A_k \neq 0, \tag{1.1}$$

arises in many applications and is an active topic of study. The quadratic case ($k = 2$) is the most important in practice [24], but higher degree polynomials also arise [5], [12], [18], [23]. We continue the practice stemming from Lancaster [14] of developing theory for general k where possible, in order to gain the most insight and understanding.

The standard way of solving the polynomial eigenvalue problem is to linearize $P(\lambda)$ into $L(\lambda) = \lambda X + Y \in \mathbb{C}^{kn\times kn}$, solve the generalized eigenproblem $L(\lambda)z = 0$, and recover eigenvectors of P from those of L. Formally, L is a linearization of P if there exist unimodular $E(\lambda)$ and $F(\lambda)$ (that is, $\det(E(\lambda))$ and $\det(F(\lambda))$ are nonzero constants) such that

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(k-1)n} \end{bmatrix}.$$

Hence $\det(L(\lambda))$ agrees with $\det(P(\lambda))$ up to a nonzero constant multiplier, so that L and P have the same eigenvalues. The linearizations used in practice are almost

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invariably one of $C_1(\lambda) = \lambda X_1 + Y_1$ and $C_2(\lambda) = \lambda X_2 + Y_2$, called the first and second companion forms [15, Sec. 14.1], respectively, where

$$X_1 = X_2 = \mathrm{diag}(A_k, I_n, \dots, I_n), \tag{1.2a}$$

$$Y_1 = \begin{bmatrix} A_{k-1} & A_{k-2} & \dots & A_0 \\ -I_n & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -I_n & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} A_{k-1} & -I_n & \dots & 0 \\ A_{k-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \dots & 0 \end{bmatrix}. \tag{1.2b}$$

Yet many linearizations exist, and other than the convenience of their construction, there is no apparent reason for preferring the companion forms. Indeed one obvious disadvantage of the companion forms is their lack of preservation of certain structural properties of P, most obviously symmetry.

Three recent papers have systematically addressed the task of broadening the menu of available linearizations and providing criteria to guide the choice. Mackey, Mackey, Mehl, and Mehrmann [17] construct two vector spaces of pencils generalizing the companion forms and prove many interesting properties, including that almost all of these pencils are linearizations. In [16], the same authors identify linearizations within these vector spaces that respect palindromic and odd-even structures. Higham, D. S. Mackey, and Tisseur [9] analyze the conditioning of some of the linearizations introduced in [17], looking for a best conditioned linearization and comparing its condition number with that of the original polynomial.

Before discussing our aims, we recall some definitions and results from [17]. Let $\mathbb{F}$ denote $\mathbb{C}$ or $\mathbb{R}$. With the notation

$$\Lambda = [\lambda^{k-1}, \lambda^{k-2}, \dots, 1]^T \in \mathbb{F}^k, \quad \text{where} \quad k = \deg(P),$$

define two vector spaces of $kn \times kn$ pencils $L(\lambda) = \lambda X + Y$:

$$\mathbb{L}_1(P) = \left\{ L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{F}^k \right\}, \tag{1.3}$$

$$\mathbb{L}_2(P) = \left\{ L(\lambda) : (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda),\ w \in \mathbb{F}^k \right\}. \tag{1.4}$$

The vectors v and w are referred to as "right ansatz" and "left ansatz" vectors, respectively. It is easily checked that for the companion forms in (1.2), $C_1(\lambda) \in \mathbb{L}_1(P)$ and $C_2(\lambda) \in \mathbb{L}_2(P)$ with $v = e_1$ and $w = e_1$, respectively, where e_i denotes the ith column of I_k. The dimensions of $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ are both $k(k-1)n^2 + k$ [17, Cor. 3.6]. For any regular P (that is, any P for which $\det(P(\lambda)) \not\equiv 0$), almost all pencils in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ are linearizations of P [17, Thm. 4.7].

A crucial property of $\mathbb{L}_1$ and $\mathbb{L}_2$ is that eigenvectors of P can be directly recovered from eigenvectors of linearizations in $\mathbb{L}_1$ and $\mathbb{L}_2$. Specifically, for any pencil $L \in \mathbb{L}_1(P)$ with nonzero right ansatz vector v, x is a right eigenvector of P with eigenvalue λ if and only if $\Lambda \otimes x$ (if λ is finite) or $e_1 \otimes x$ (if $\lambda = \infty$) is a right eigenvector for L with eigenvalue λ. Moreover, if this $L \in \mathbb{L}_1(P)$ is a linearization for P, then *every* right eigenvector of L has one of these two Kronecker product forms; hence some right eigenvector of P can be recovered from every right eigenvector of L. A similar recovery property holds for left eigenvectors and pencils in $\mathbb{L}_2(P)$. For more details, see [17, Thms. 3.8, 3.14, and 4.4].

The subspace

$$\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P) \tag{1.5}$$

of "double ansatz" pencils is of particular interest, because there is a simultaneous correspondence via Kronecker products between left *and* right eigenvectors of P and those of pencils in $\mathbb{DL}(P)$. Two key facts are that $L \in \mathbb{DL}(P)$ if and only if L satisfies the conditions in (1.3) and (1.4) with $w = v$, and that every $v \in \mathbb{F}^k$ uniquely determines X and Y such that $L(\lambda) = \lambda X + Y$ is in $\mathbb{DL}(P)$ [17, Thm. 5.3]. Thus $\mathbb{DL}(P)$ is a k-dimensional space of pencils associated with P. Just as for $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$, almost all pencils in $\mathbb{DL}(P)$ are linearizations [17, Thm. 6.8].

Our contributions are now summarized. We show in Section 3 that the set of block symmetric pencils in $\mathbb{L}_1(P)$ is precisely $\mathbb{DL}(P)$. That $\mathbb{DL}(P)$ should comprise only block symmetric pencils is perhaps surprising, as P is arbitrary. We show that the pencils corresponding to $v = e_i$, $i = 1{:}\,k$, form a basis for $\mathbb{DL}(P)$ built from block diagonal matrices with block Hankel blocks. This basis is used in Section 4 to prove that the first $k = \deg(P)$ pencils in a sequence constructed by Lancaster [13], [14], generate $\mathbb{DL}(P)$. In Sections 5 and 6 we show that when P is symmetric the set of symmetric pencils in $\mathbb{L}_1(P)$ is the same as $\mathbb{DL}(P)$, while for Hermitian P the Hermitian pencils in $\mathbb{L}_1(P)$ form a proper subset of $\mathbb{DL}(P)$ corresponding to real ansatz vectors. In Section 7 we summarize the known "almost all pencils are linearizations" results and prove such a result for the Hermitian pencils in $\mathbb{L}_1(P)$.

Initially, our main motivation for this investigation was the problem of systematically generating symmetric linearizations for symmetric matrix polynomials. However, the analysis has led, via the study of *block symmetric* pencils, to new derivations of some of the general properties of $\mathbb{DL}(P)$. Therefore this paper should be useful as a self-contained introduction to $\mathbb{DL}(P)$ with proofs that are conceptually clearer and more concise than the original derivations in [17].

Finally, we motivate our interest in the preservation of symmetry. A matrix polynomial that is real symmetric or Hermitian has a spectrum that is symmetric with respect to the real axis, and the sets of left and right eigenvectors coincide. These properties are preserved in a symmetric (Hermitian) linearization by virtue of its structure—not just through the numerical entries of the pencil. A symmetry-preserving pencil has the practical advantages that storage and computational cost are reduced if a method that exploits symmetry is applied. The eigenvalues of a symmetric (Hermitian) pencil $L(\lambda) = \lambda X + Y$ can be computed, for small to medium size problems, by first reducing the matrix pair (Y, X) to tridiagonal-diagonal form [22] and then using the HR [4], [6] or LR [20] algorithms or the Ehrlich-Aberth iterations [3]. For large problems, a symmetry-preserving pseudo-Lanczos algorithm of Parlett and Chen [19], [2, Sec. 8.6], based on an indefinite inner product, can be used. For a quadratic polynomial $Q(\lambda)$ that is hyperbolic, or in particular overdamped, a linearization that is a symmetric definite pencil can be identified [10, Thm. 3.6]; this pencil is amenable to structure-preserving methods that exploit both the symmetry and the definiteness [26] and guarantee real computed eigenvalues for $Q(\lambda)$ not too close to being non-hyperbolic.

2. Block symmetry and shifted sum. We begin with some notation and results concerning block transpose and block symmetry. Our aim is to investigate the existence and uniqueness of solutions in block symmetric matrices of the equation $X \boxplus\!\!\rightarrow Y = Z$, where $\boxplus\!\!\rightarrow$ is a "shifted sum" operation and Z is a given arbitrary matrix. For the purposes of this paper we only consider block matrices in which all the blocks have the same size.

Definition 2.1 (Block transpose). *Let $A = (A_{ij})$ be a block $k \times \ell$ matrix with $m \times n$ blocks A_{ij}. The block transpose of A is the block $\ell \times k$ matrix $A^{\mathcal{B}}$ with $m \times n$*

blocks defined by $(A^{\mathcal{B}})_{ij} = A_{ji}$.

Recall that all pencils in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ are of size $kn \times kn$, where k is the degree of the $n \times n$ matrix polynomial $P(\lambda)$. Throughout this paper we regard these pencils as block $k \times k$ matrices with $n \times n$ blocks. The block transpose operation, performed relative to this partitioning, establishes an intimate link between $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$.

THEOREM 2.2. *For any matrix polynomial $P(\lambda)$, the block transpose map*

$$\begin{aligned} \mathbb{L}_1(P) &\longrightarrow \mathbb{L}_2(P) \\ L(\lambda) &\longmapsto L(\lambda)^{\mathcal{B}} \end{aligned}$$

is a linear isomorphism between $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$. In particular, if $L(\lambda) \in \mathbb{L}_1(P)$ has right ansatz vector v, then $L(\lambda)^{\mathcal{B}} \in \mathbb{L}_2(P)$ with left ansatz vector $w = v$.

Proof. It is straightforward to check that $\big(L(\lambda)(\Lambda \otimes I_n)\big)^{\mathcal{B}} = (\Lambda \otimes I_n)^{\mathcal{B}} L(\lambda)^{\mathcal{B}} = (\Lambda^T \otimes I_n) L(\lambda)^{\mathcal{B}}$ and $\big(v \otimes P(\lambda)\big)^{\mathcal{B}} = v^T \otimes P(\lambda)$. Hence if $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v, then block transposing the defining condition in (1.3) yields $(\Lambda^T \otimes I_n) L(\lambda)^{\mathcal{B}} = v^T \otimes P(\lambda)$. Thus $L(\lambda)^{\mathcal{B}} \in \mathbb{L}_2(P)$ with left ansatz vector v, and so block transpose gives a well-defined map from $\mathbb{L}_1(P)$ to $\mathbb{L}_2(P)$. Clearly this map is linear and the kernel is just the zero pencil, since $L(\lambda)^{\mathcal{B}} = 0 \Rightarrow L(\lambda) = 0$. Since $\dim \mathbb{L}_1(P) = \dim \mathbb{L}_2(P)$, the proof is complete. ∎

The companion forms give a nice illustration of Theorem 2.2. By inspection, $C_2(\lambda) = \big(C_1(\lambda)\big)^{\mathcal{B}}$ and, as noted earlier, $C_1(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector $v = e_1$ while $C_2(\lambda) \in \mathbb{L}_2(P)$ with left ansatz vector $w = v = e_1$.

Given the notion of block transpose, it is natural to consider block symmetric matrices, which will play a central role in our development. A block $k \times k$ matrix A with $m \times n$ blocks is *block symmetric* if $A^{\mathcal{B}} = A$. For example, a block 2×2 block symmetric matrix has the form $\left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{smallmatrix}\right]$. Note that if each block $A_{ij} \in \mathbb{F}^{n \times n}$ in a block symmetric matrix A is symmetric, then A is symmetric.

The *column-shifted sum* introduced in [17] is a simple operation on block matrices that enables us both to easily construct pencils in $\mathbb{L}_1(P)$ and to conveniently test when a given pencil is in $\mathbb{L}_1(P)$.

DEFINITION 2.3 (Column-shifted sum). *Let X and Y be block $k \times k$ matrices with $n \times n$ blocks X_{ij} and Y_{ij}. Then the column-shifted sum $X \boxplus\!\!\rightarrow Y$ of X and Y is*

$$X \boxplus\!\!\rightarrow Y := \begin{bmatrix} X_{11} & \dots & X_{1k} & 0 \\ \vdots & & \vdots & \vdots \\ X_{k1} & \dots & X_{kk} & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_{11} & \dots & Y_{1k} \\ \vdots & \vdots & & \vdots \\ 0 & Y_{k1} & \dots & Y_{kk} \end{bmatrix} \in \mathbb{F}^{kn \times k(n+1)},$$

where the zero blocks are $n \times n$.

The significance of this shifted sum operation is revealed by the following result [17, Lem. 3.4], which shows how membership in $\mathbb{L}_1(P)$ is equivalent to a specific Kronecker product form in the shifted sum.

LEMMA 2.4. *Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix polynomial of degree k, and let $L(\lambda) = \lambda X + Y$ be a $kn \times kn$ pencil. Then for $v \in \mathbb{F}^k$,*

$$L(\lambda) \in \mathbb{L}_1(P) \text{ with right ansatz vector } v \iff X \boxplus\!\!\rightarrow Y = v \otimes [A_k \; A_{k-1} \; \dots \; A_0].$$

We now prove two technical lemmas concerning solutions of the shifted sum equations $X \boxplus\!\!\rightarrow Y = Z$ and $X \boxplus\!\!\rightarrow Y = 0$. We show that the equation $X \boxplus\!\!\rightarrow Y = Z$ with an arbitrary Z may always be solved with block symmetric X and Y, and that the only block symmetric solution of $X \boxplus\!\!\!\rightarrow Y = 0$ is $X = Y = 0$.

First we define three special types of block symmetric matrix that play a central role in all that follows. Let

$$R_\ell = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{\ell\times\ell} \quad \text{and} \quad N_\ell = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{\ell\times\ell}. \quad \text{(Note that } N_1 = 0.\text{)} \tag{2.1}$$

For an arbitrary $n \times n$ block M, we define three block Hankel, block symmetric, block $\ell \times \ell$ matrices:

$$\mathcal{H}_\ell^{(0)}(M) := \begin{bmatrix} & & M \\ & \iddots & \\ M & & \end{bmatrix} = R_\ell \otimes M,$$

$$\mathcal{H}_\ell^{(1)}(M) := \begin{bmatrix} & & M & 0 \\ & \iddots & \iddots & \\ M & \iddots & & \\ 0 & & & \end{bmatrix} = (N_\ell R_\ell) \otimes M = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{bmatrix} \otimes M,$$

$$\mathcal{H}_\ell^{(-1)}(M) := \begin{bmatrix} & & & 0 \\ & & \iddots & M \\ & \iddots & \iddots & \\ 0 & M & & \end{bmatrix} = (R_\ell N_\ell) \otimes M = \begin{bmatrix} & & & 0 \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix} \otimes M.$$

The superscript (0), (1), or (-1) denotes that the blocks M are on, above, or below the anti-diagonal, respectively. Note that all three of these block Hankel matrices are symmetric if M is.

LEMMA 2.5. *Let Z be an arbitrary block $k \times (k+1)$ matrix with $n \times n$ blocks. Then there exist block symmetric block $k \times k$ matrices X and Y with $n \times n$ blocks such that $X \boxplus\!\!\!\rightarrow Y = Z$.*

Proof. Let $E_{ij}^\ell \in \mathbb{F}^{\ell\times(\ell+1)}$ denote the matrix that is everywhere zero except for a 1 in the (i,j) entry. Our proof is based on the observation that for arbitrary $M, P \in \mathbb{F}^{n\times n}$, the shifted sums

$$\mathcal{H}_\ell^{(0)}(M) \boxplus\!\!\!\rightarrow (-\mathcal{H}_\ell^{(1)}(M)) = \begin{bmatrix} 0 & \dots & \dots & 0 & 0 \\ \vdots & & \iddots & \iddots & \vdots \\ 0 & 0 & \iddots & & \vdots \\ M & 0 & \dots & \dots & 0 \end{bmatrix} = E_{\ell 1}^\ell \otimes M, \tag{2.2}$$

$$-\mathcal{H}_\ell^{(-1)}(P) \boxplus\!\!\!\rightarrow \mathcal{H}_\ell^{(0)}(P) = \begin{bmatrix} 0 & \dots & \dots & 0 & P \\ \vdots & & \iddots & 0 & 0 \\ \vdots & \iddots & \iddots & & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} = E_{1,\ell+1}^\ell \otimes P \tag{2.3}$$

place M and P at the bottom left corner and top right corner of a block $\ell \times (\ell+1)$ matrix, respectively.

The shifted sum $\boxplus\!\!\!\rightarrow$ is compatible with ordinary sums, i.e.,

$$\left(\sum X_i\right) \boxplus\!\!\!\rightarrow \left(\sum Y_i\right) = \sum (X_i \boxplus\!\!\!\rightarrow Y_i).$$

Hence if we can show how to construct block symmetric X and Y that place an arbitrary $n \times n$ block into an arbitrary (i, j) block-location in Z, then sums of such examples will achieve the desired result for an arbitrary Z.

For indices i, j such that $1 \le i \le j \le k$, let $\ell = j - i + 1$ and embed $\mathcal{H}_\ell^{(0)}(M)$ and $-\mathcal{H}_\ell^{(1)}(M)$ as principal submatrices in block rows and block columns i through j of block $k \times k$ zero matrices to get

$$
\begin{aligned}
\widehat{X}_{ij} \boxplus\!\!\rightarrow \widehat{Y}_{ij} &:= \begin{array}{cc} & \begin{array}{ccc} \; & i \quad\quad j & \; \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} & & \\ & \boxed{\mathcal{H}_\ell^{(0)}(M)} & \\ & & \end{bmatrix} \end{array} \boxplus\!\!\rightarrow \begin{array}{cc} & \begin{array}{ccc} \; & i \quad\quad j & \; \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} & & \\ & \boxed{-\mathcal{H}_\ell^{(1)}(M)} & \\ & & \end{bmatrix} \end{array} \\
&= \begin{array}{cc} & \begin{array}{ccc} \; & i \quad\quad\quad\quad j+1 & \; \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} & & \\ & \boxed{\mathcal{H}_\ell^{(0)}(M) \boxplus\!\!\rightarrow (-\mathcal{H}_\ell^{(1)}(M))} & \\ & & \end{bmatrix} \end{array} \\
&= E_{ji} \otimes M \quad (i \le j).
\end{aligned} \tag{2.4}
$$

Note that embedding $\mathcal{H}_\ell^{(0)}(M)$ and $-\mathcal{H}_\ell^{(1)}(M)$ as *principal* block-submatrices guarantees that $\widetilde{X}_{ij}$ and $\widetilde{Y}_{ij}$ are block symmetric. Similarly, defining the block symmetric matrices

$$
\widetilde{X}_{ij} = \begin{array}{cc} & \begin{array}{ccc} \; & i \quad\quad j & \; \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} & & \\ & \boxed{-\mathcal{H}_\ell^{(-1)}(P)} & \\ & & \end{bmatrix} \end{array}, \quad \widetilde{Y}_{ij} = \begin{array}{cc} & \begin{array}{ccc} \; & i \quad\quad j & \; \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} & & \\ & \boxed{\mathcal{H}_\ell^{(0)}(P)} & \\ & & \end{bmatrix} \end{array}, \tag{2.5}
$$

we have

$$
\widetilde{X}_{ij} \boxplus\!\!\rightarrow \widetilde{Y}_{ij} = E_{i,j+1} \otimes P \qquad (i \le j). \tag{2.6}
$$

Thus sums of these principally embedded versions of (2.2) and (2.3) can produce an arbitrary block $k \times (k+1)$ matrix Z as the column-shifted sum of block symmetric X and Y. □

LEMMA 2.6. *Suppose X and Y are both block symmetric block $k \times k$ matrices with $n \times n$ blocks. Then $X \boxplus\!\!\rightarrow Y = 0 \iff X = Y = 0$.*

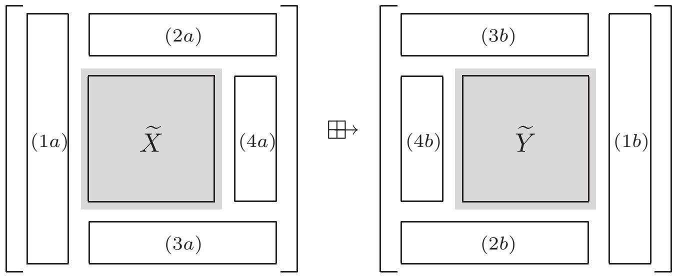
Proof. The proof is by induction on k. We focus on the nontrivial direction ($\Rightarrow$). There are two base cases to be checked, $k = 1$ and $k = 2$. The $k = 1$ case is immediate. Because X and Y are block symmetric, for $k = 2$ we have

$$
X \boxplus\!\!\rightarrow Y = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{12} & X_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_{11} & Y_{12} \\ 0 & Y_{12} & Y_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} + Y_{11} & Y_{12} \\ X_{12} & X_{22} + Y_{12} & Y_{22} \end{bmatrix}.
$$

Then $X \boxplus\!\!\rightarrow Y = 0$ clearly implies that $X = Y = 0$.

Now consider $k > 2$ and X and Y with their blocks "around the edges" grouped together as indicated in the diagram:

$X \boxplus\!\!\rightarrow Y =$



The only contribution to the first block column of $X \boxplus\!\!\rightarrow Y$ comes from $(1a)$, and the only contribution to the last block column of $X \boxplus\!\!\rightarrow Y$ comes from $(1b)$. Thus $X \boxplus\!\!\rightarrow Y = 0$ implies $(1a)$ and $(1b)$ are all zeros. (Note that this would be true for general X and Y.) The block-symmetry of X and Y now implies that the blocks in $(2a)$ and $(2b)$ are zero. The blocks of $(2a)$ interact in the shifted sum with those in $(3b)$; the $(2a)$ blocks being zero imply that all the $(3b)$ blocks are zero. Similarly the $(2b)$ blocks all zero imply that all the $(3a)$ blocks are zero. Finally, the block-symmetry of X and Y can be invoked once again to see that all the $(4a)$ and $(4b)$ blocks are zero. At this point we have that $X \boxplus\!\!\rightarrow Y = 0$ implies

$$X \boxplus\!\!\rightarrow Y = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \widetilde{X} & \vdots \\ 0 & \dots & 0 \end{bmatrix} \boxplus\!\!\rightarrow \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \widetilde{Y} & \vdots \\ 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \widetilde{X} \boxplus\!\!\rightarrow \widetilde{Y} & \vdots \\ 0 & \dots & 0 \end{bmatrix} = 0.$$

Since $\widetilde{X} \boxplus\!\!\rightarrow \widetilde{Y} = 0$, the induction hypothesis implies $\widetilde{X} = \widetilde{Y} = 0$, and consequently that $X = Y = 0$. ▯

3. Block symmetric pencils and $\mathbb{DL}(P)$ for general P. We now study the subspace of block symmetric pencils in $\mathbb{L}_1(P)$, which turns out to be the same as the space $\mathbb{DL}(P)$. This way of characterizing $\mathbb{DL}(P)$ leads to short proofs of some of its properties, as well as the identification of a useful basis.

3.1. Block symmetric pencils in $\mathbb{L}_1(P)$. For a general polynomial P we can use the results of Section 2 to analyze the subspace

$$\mathbb{B}(P) := \left\{ \lambda X + Y \in \mathbb{L}_1(P) : X^{\mathcal{B}} = X,\ Y^{\mathcal{B}} = Y \right\} \tag{3.1}$$

of all block symmetric pencils in $\mathbb{L}_1(P)$. We will see in Section 7 that almost all of these pencils are indeed linearizations for P.

Theorem 3.1. *For any matrix polynomial $P(\lambda)$ of degree k, $\dim \mathbb{B}(P) = k$, and for each vector $v \in \mathbb{F}^k$ there is a uniquely determined block symmetric pencil in $\mathbb{B}(P)$.*

Proof. Recalling that $\mathbb{L}_1$ is defined by (1.3), the theorem is proved if we can show that the map

$$\begin{aligned} \mathbb{B}(P) &\xrightarrow{\mathcal{M}} \mathcal{V}_P := \{ v \otimes P(\lambda) : v \in \mathbb{F}^k \} \\ L(\lambda) &\longmapsto L(\lambda)\,(\Lambda \otimes I_n) \end{aligned} \tag{3.2}$$

is a linear isomorphism.

First, recall from Lemma 2.4 that for any pencil $\lambda X + Y \in \mathbb{L}_1(P)$,

$$(\lambda X + Y)(\Lambda \otimes I_n) = v \otimes P(\lambda) \quad \Longleftrightarrow \quad X \boxplus\!\!\rightarrow Y = v \otimes [\, A_k \ A_{k-1} \ \dots \ A_0 \,]. \tag{3.3}$$

Thus $\lambda X + Y$ is in $\ker \mathcal{M}$ iff $X \boxplus\!\!\rightarrow Y = 0$. But X and Y are block symmetric, so by Lemma 2.6 we see that $\ker \mathcal{M} = \{0\}$, and hence $\mathcal{M}$ is 1-1.

To see that $\mathcal{M}$ is onto, let $v \otimes P(\lambda)$ with $v \in \mathbb{F}^k$ be an arbitrary element of $\mathcal{V}_P$. With $Z = v \otimes [A_k \; A_{k-1} \; \dots \; A_0]$, the construction of Lemma 2.5 shows that there exist block symmetric X and Y such that $X \boxplus\!\!\rightarrow Y = v \otimes [A_k \; A_{k-1} \; \dots \; A_0]$. Then by (3.3) we have $\mathcal{M}(\lambda X + Y) = v \otimes P(\lambda)$, showing that $\mathcal{M}$ is onto. □

3.2. Double ansatz pencils. Our goal is now to show that $\mathbb{DL}(P) := \mathbb{L}_1(P) \cap \mathbb{L}_2(P) = \mathbb{B}(P)$. The inclusion $\mathbb{DL}(P) \subseteq \mathbb{B}(P)$, which says that all pencils $\lambda X + Y$ in $\mathbb{DL}(P)$ are block symmetric, can be deduced immediately from the following formulae for the blocks of X and Y in terms of the ansatz vector v [17, Thm. 5.3]:

$$X_{ij} = v_{\max(i,j)} A_{k+1-\min(i,j)} + \sum_{\mu=1}^{\min(i-1,j-1)} (v_{j+i-\mu} A_{k+1-\mu} - v_\mu A_{k+1-j-i+\mu}),$$

$$Y_{ij} = \sum_{\mu=1}^{\min(i,j)} (v_\mu A_{k-j-i+\mu} - v_{j+i+1-\mu} A_{k+1-\mu}), \qquad i,j = 1\colon k.$$

However, the derivation of these formulas is long and tedious. We present a shorter proof, based on first principles, of the stronger result $\mathbb{DL}(P) = \mathbb{B}(P)$.

LEMMA 3.2. *For any matrix polynomial $P(\lambda)$, $\mathbb{B}(P) \subseteq \mathbb{DL}(P)$.*

Proof. Let $L(\lambda) \in \mathbb{B}(P) \subset \mathbb{L}_1(P)$. From Theorem 2.2 we know that $L(\lambda)^{\mathcal{B}} = L(\lambda)$ is in $\mathbb{L}_2(P)$, and so $L(\lambda) \in \mathbb{DL}(P)$. □

Now we consider the special case of $\mathbb{DL}(P)$-pencils with $v = 0$, showing that in this case $w = 0$ is forced and the pencil is unique. Note that the definition of $\mathbb{DL}(P)$ does not require that X and Y are block symmetric, so we cannot appeal to Lemma 2.6 here.

LEMMA 3.3. *Suppose $L(\lambda) = \lambda X + Y \in \mathbb{DL}(P)$ has right ansatz vector v and left ansatz vector w. Then $v = 0$ implies that w must also be 0, and that $X = Y = 0$.*

Proof. We first show that the ℓth block-column of X and the ℓth coordinate of w is zero for $\ell = 1\colon k$, by an induction on ℓ.

Suppose that $\ell = 1$. From Lemma 2.4 we know that $X \boxplus\!\!\rightarrow Y = v \otimes [A_k \; A_{k-1} \; \dots \; A_0]$. Since $v = 0$ we have $X \boxplus\!\!\rightarrow Y = 0$, and hence the first block-column of X is zero. Now from Theorem 2.2, $L(\lambda)$ being in $\mathbb{L}_2(P)$ with left ansatz vector w implies that $L(\lambda)^{\mathcal{B}} \in \mathbb{L}_1(P)$ with right ansatz vector w, which can be written in terms of the shifted sum as

$$X^{\mathcal{B}} \boxplus\!\!\rightarrow Y^{\mathcal{B}} = w \otimes [A_k \; A_{k-1} \; \dots \; A_0]. \tag{3.4}$$

The $(1,1)$-block of the right-hand side of (3.4) is $w_1 A_k$, while on the left-hand side the $(1,1)$-block of $X^{\mathcal{B}} \boxplus\!\!\rightarrow Y^{\mathcal{B}}$ is the same as the $(1,1)$-block of X. Hence $w_1 A_k = 0$. But the leading coefficient A_k of $P(\lambda)$ is nonzero by assumption, so $w_1 = 0$.

Now suppose that the ℓth block-column of X is zero and that $w_\ell = 0$. Then by (3.4) the ℓth block-row of $X^{\mathcal{B}} \boxplus\!\!\rightarrow Y^{\mathcal{B}}$ is zero. Since the ℓth block-row of $X^{\mathcal{B}}$ is zero, the ℓth block-row of $Y^{\mathcal{B}}$, or, equivalently, the ℓth block-column of Y, must also be zero. Combining this with $X \boxplus\!\!\rightarrow Y = 0$ implies that the $(\ell+1)$th block-column of X is zero. Now equating the $(\ell+1, 1)$-blocks of both sides of (3.4) gives $w_{\ell+1} A_k = 0$, and hence $w_{\ell+1} = 0$. This concludes the induction, and shows that $X = 0$ and $w = 0$.

Finally, $X = 0$ and $X \boxplus\!\!\rightarrow Y = 0$ implies $Y = 0$, completing the proof. □

We can now characterize $\mathbb{DL}(P)$ and give a precise description of all right/left ansatz vector pairs (v, w) that can be realized by some $\mathbb{DL}(P)$-pencil.

THEOREM 3.4. *For a matrix polynomial $P(\lambda)$ of degree k, suppose $L(\lambda) \in \mathbb{DL}(P)$ with right ansatz vector v and left ansatz vector w. Then $v = w$ and $L(\lambda) \in \mathbb{B}(P)$. Thus $\mathbb{DL}(P) = \mathbb{B}(P)$, $\dim \mathbb{DL}(P) = k$, and for each $v \in \mathbb{F}^k$ there is a uniquely determined pencil in $\mathbb{DL}(P)$.*

Proof. Let $\mathcal{L}(\lambda) \in \mathbb{B}(P)$ be the unique block symmetric pencil from Theorem 3.1 with v as its right ansatz vector. From Theorem 2.2 we know that $\mathcal{L}(\lambda)^{\mathcal{B}} = \mathcal{L}(\lambda)$ is in $\mathbb{L}_2(P)$ with left ansatz vector v, and so $\mathcal{L}(\lambda) \in \mathbb{DL}(P)$ with v as both its right and left ansatz vector. Thus the pencil $\widetilde{L}(\lambda) := L(\lambda) - \mathcal{L}(\lambda)$ is in $\mathbb{DL}(P)$ with right ansatz vector 0 and left ansatz vector $w - v$. Lemma 3.3 then implies that $v = w$ and $\widetilde{L}(\lambda) = \lambda \cdot 0 + 0$. Thus $L(\lambda) \equiv \mathcal{L}(\lambda) \in \mathbb{B}(P)$, so $\mathbb{DL}(P) \subseteq \mathbb{B}(P)$. In view of Lemma 3.2 we can conclude that $\mathbb{DL}(P) = \mathbb{B}(P)$. The rest of the theorem follows immediately from Theorem 3.1. ∎

The equality $\mathbb{DL}(P) = \mathbb{B}(P)$ can be thought of as saying that the pencils in $\mathbb{DL}(P)$ are *doubly* structured: they have block symmetry as well as the eigenvector recovery properties that were the original motivation for their definition.

3.3. The standard basis for $\mathbb{B}(P)$. The isomorphism established in the proof of Theorem 3.1 immediately suggests the possibility that the basis for $\mathbb{B}(P)$ corresponding (via the map $\mathcal{M}$ in (3.2)) to the standard basis $\{e_1, \dots, e_k\}$ for $\mathbb{F}^k$ may be especially simple and useful. In this section we derive a general formula for these "standard basis pencils" in $\mathbb{B}(P)$ as a corollary of the shifted sum construction used in the proof of Lemma 2.5. These pencils are of course also a basis for $\mathbb{DL}(P)$, since $\mathbb{DL}(P) = \mathbb{B}(P)$.

In light of Lemma 2.4, then, our goal is to construct for each $1 \le m \le k$ a block symmetric pencil $\lambda X_m + Y_m$ such that

$$X_m \boxplus\!\!\rightarrow Y_m \;=\; e_m \otimes [\, A_k \; A_{k-1} \; \dots \; A_0 \,]. \tag{3.5}$$

This is most easily done in two steps. First we show how to achieve the first m block-columns in the desired shifted sum, i.e., how to get $e_m \otimes [\, A_k \; \dots \; A_{k-m+1} \;\; 0 \; \dots \; 0\,]$. Then the last $k - m + 1$ block-columns $e_m \otimes [\, 0 \; \dots \; 0 \; A_{k-m} \; \dots \; A_1 \; A_0 \,]$ are produced by a related but slightly different construction. We use the following notation for principal block submatrices, adapted from [11]: for a block $k \times k$ matrix X and index set $\alpha \subseteq \{1, 2, \dots, k\}$, $X(\alpha)$ will denote the principal block submatrix lying in the block rows and block columns with indices in α.

To get the first m block-columns in the desired shifted sum we repeatedly use the construction in (2.4) to build block $k \times k$ matrices $\widehat{X}_m$ and $\widehat{Y}_m$, embedding once in each of the principal block submatrices $\widehat{X}_m(\alpha_i)$ and $\widehat{Y}_m(\alpha_i)$ for the index sets $\alpha_i = \{i, i+1, \dots, m\}$, $i = 1\!:\!m$. Accumulating these embedded submatrices, we obtain

$$\widehat{X}_m = \begin{array}{c} \qquad\qquad\quad m \\ \left[\begin{array}{cccc|c} & & & A_k & \\ & & \iddots & A_{k-1} & 0 \\ & \iddots & \iddots & \vdots & \\ A_k & A_{k-1} & \cdots & A_{k-m+1} & \\ \hline & & 0 & & 0 \end{array}\right] \end{array} m \quad ,$$

$$\widehat{Y}_m = - \begin{array}{c} \begin{array}{ccccc|c} & & & & m & \end{array} \\ \left[\begin{array}{ccccc|c} & & & A_k & 0 & \\ & & \iddots & A_{k-1} & 0 & \\ & \iddots & \iddots & \vdots & \vdots & 0 \\ A_k & A_{k-1} & \dots & A_{k-m+2} & 0 & \\ 0 & \dots & \dots & 0 & 0 & \\ \hline & & 0 & & & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ \\ m \\ \\ \end{array} ,$$

with the property that $\widehat{X}_m \boxplus\!\!\rightarrow \widehat{Y}_m = e_m \otimes [\, A_k \;\dots\; A_{k-m+1} \;\; 0 \;\dots\; 0\,]$.

To obtain the last $k-m+1$ columns we use the construction outlined in (2.5) and (2.6) $k-m+1$ times to build block $k \times k$ matrices $\widetilde{X}_m$ and $\widetilde{Y}_m$, embedding once in each of the principal block submatrices $\widetilde{X}_m(\beta_j)$ and $\widetilde{Y}_m(\beta_j)$ for the index sets $\beta_j = \{m, m+1, \dots, j\}$, $j = m\colon k$, which yields

$$\widetilde{X}_m = - \begin{array}{c} \begin{array}{c|ccccc} & m & & & & k \end{array} \\ \left[\begin{array}{c|ccccc} 0 & & & 0 & & \\ \hline & 0 & 0 & \dots\dots & & 0 \\ & 0 & A_{k-m-1} & \iddots & A_1 & A_0 \\ 0 & \vdots & \vdots & \iddots & \iddots & \\ & \vdots & A_1 & \iddots & & \\ & 0 & A_0 & & & \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ m \\ \\ \\ \\ k \end{array} ,$$

$$\widetilde{Y}_m = \begin{array}{c} \begin{array}{c|ccccc} & m & & & & k \end{array} \\ \left[\begin{array}{c|cccc} 0 & & & 0 & \\ \hline & A_{k-m} & \dots\dots & A_1 & A_0 \\ & \vdots & & \iddots & A_0 \\ 0 & \vdots & \iddots & \iddots & \\ & A_1 & \iddots & & \\ & A_0 & & & \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ m \\ \\ \\ \\ k \end{array} ,$$

satisfying $\widetilde{X}_m \boxplus\!\!\rightarrow \widetilde{Y}_m = e_m \otimes [\, 0 \;\dots\; 0 \; A_{k-m} \;\dots\; A_1 \; A_0\,]$. With $X_m := \widehat{X}_m + \widetilde{X}_m$ and $Y_m := \widehat{Y}_m + \widetilde{Y}_m$ we have $X_m \boxplus\!\!\rightarrow Y_m = e_m \otimes [\, A_k \; A_{k-1} \;\dots\; A_1 \; A_0\,]$, so $\lambda X_m + Y_m$ is the mth standard basis pencil for $\mathbb{B}(P)$.

A more concise way to express the mth standard basis pencil uses the following block Hankel matrices. Let $\mathcal{L}_j(P(\lambda))$ denote the lower block-anti-triangular, block Hankel, block $j \times j$ matrix

$$\mathcal{L}_j(P(\lambda)) := \begin{bmatrix} & & & A_k \\ & & \iddots & A_{k-1} \\ & \iddots & \iddots & \vdots \\ A_k & A_{k-1} & \dots & A_{k-j+1} \end{bmatrix} \tag{3.6}$$

formed from the first j matrix coefficients $A_k, A_{k-1}, \dots, A_{k-j+1}$ of $P(\lambda)$. Similarly,

TABLE 3.1
Block symmetric standard basis for the quadratic $P(\lambda) = \lambda^2 A + \lambda B + C$.

v	$L(\lambda) \in \mathbb{B}(P)$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\lambda \left[\begin{array}{c\|c} A & 0 \\ \hline 0 & -C \end{array}\right] + \begin{bmatrix} B & C \\ C & 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & A \\ A & B \end{bmatrix} + \left[\begin{array}{c\|c} -A & 0 \\ \hline 0 & C \end{array}\right]$

TABLE 3.2
Block symmetric standard basis for the cubic $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda C + D$.

v	$L(\lambda) \in \mathbb{DL}(P)$
$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\lambda \left[\begin{array}{c\|cc} A & 0 & 0 \\ \hline 0 & -C & -D \\ 0 & -D & 0 \end{array}\right] + \begin{bmatrix} B & C & D \\ C & D & 0 \\ D & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\lambda \left[\begin{array}{cc\|c} 0 & A & 0 \\ A & B & 0 \\ \hline 0 & 0 & -D \end{array}\right] + \left[\begin{array}{c\|cc} -A & 0 & 0 \\ \hline 0 & C & D \\ 0 & D & 0 \end{array}\right]$
$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & 0 & A \\ 0 & A & B \\ A & B & C \end{bmatrix} + \left[\begin{array}{cc\|c} 0 & -A & 0 \\ -A & -B & 0 \\ \hline 0 & 0 & D \end{array}\right]$

let $\mathcal{U}_j(P(\lambda))$ denote the upper block-anti-triangular, block Hankel, block $j \times j$ matrix

$$\mathcal{U}_j(P(\lambda)) := \begin{bmatrix} A_{j-1} \dots A_1\, A_0 & \\ \vdots \quad \cdot^{\cdot^{\cdot}} \cdot^{\cdot^{\cdot}} & \\ A_1 \; \cdot^{\cdot^{\cdot}} & \\ A_0 & \end{bmatrix} \tag{3.7}$$

formed from the last j matrix coefficients $A_{j-1}, A_{j-2}, \dots, A_1, A_0$ of $P(\lambda)$. Then the block symmetric matrices X_m and Y_m in the mth standard basis pencil $(m = 1\colon k)$ can be neatly expressed as a direct sum of block Hankel matrices:

$$X_m = X_m(P(\lambda)) = \begin{bmatrix} \mathcal{L}_m(P(\lambda)) & 0 \\ 0 & -\mathcal{U}_{k-m}(P(\lambda)) \end{bmatrix}, \tag{3.8a}$$

$$Y_m = Y_m(P(\lambda)) = \begin{bmatrix} -\mathcal{L}_{m-1}(P(\lambda)) & 0 \\ 0 & \mathcal{U}_{k-m+1}(P(\lambda)) \end{bmatrix}. \tag{3.8b}$$

($\mathcal{L}_j$ and $\mathcal{U}_j$ are taken to be void when $j = 0$.) From (3.8) it now becomes obvious that the coefficient matrices in successive standard basis pencils are closely related:

$$Y_m(P(\lambda)) \;=\; -X_{m-1}(P(\lambda)), \qquad m = 1\colon k. \tag{3.9}$$

Thus we have the following explicit formula for the standard basis pencils in $\mathbb{B}(P)$.

THEOREM 3.5. *Let $P(\lambda)$ be a matrix polynomial of degree k. Then for $m = 1\colon k$ the block symmetric pencil in $\mathbb{B}(P)$ with ansatz vector e_m is $\lambda X_m - X_{m-1}$, where X_m is given by* (3.8a).

The standard basis pencils in $\mathbb{B}(P)$ for general polynomials of degree 2 and 3 are listed in Tables 3.1 and 3.2, where the partitioning from (3.8) is shown in each case. As an immediate consequence we have, for the important case of quadratics

$P(\lambda) = \lambda^2 A + \lambda B + C$, the following description of all block symmetric pencils in $\mathbb{L}_1(P)$,

$$\mathbb{B}(P) = \left\{ L(\lambda) = \lambda \begin{bmatrix} v_1 A & v_2 A \\ v_2 A & v_2 B - v_1 C \end{bmatrix} + \begin{bmatrix} v_1 B - v_2 A & v_1 C \\ v_1 C & v_2 C \end{bmatrix} : \ v \in \mathbb{C}^2 \right\}.$$

4. Other constructions of block symmetric linearizations. Several other methods for constructing block symmetric linearizations of matrix polynomials have appeared previously in the literature.

Antoniou and Vologiannidis [1] have recently found new companion-like linearizations for general matrix polynomials P by generalizing Fiedler's results [7] on a factorization of the companion matrix of a scalar polynomial and certain of its permutations. From this finite family of $\frac{1}{6}(2 + \deg P)!$ pencils, all of which are linearizations, they identify one distinguished pencil that is Hermitian whenever P is Hermitian. But this example has structure even for general P: it is block symmetric. Indeed, it provides a simple example of a block symmetric linearization for $P(\lambda)$ that is not in $\mathbb{B}(P)$. In the case of a cubic polynomial $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda C + D$, the pencil is

$$L(\lambda) = \lambda \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & I \\ 0 & I & C \end{bmatrix} + \begin{bmatrix} B & -I & 0 \\ -I & 0 & 0 \\ 0 & 0 & D \end{bmatrix}. \tag{4.1}$$

Using the column-shifted sum it easy to see that $L(\lambda)$ is not in $\mathbb{L}_1(P)$, and hence not in $\mathbb{B}(P)$.

Contrasting with the "permuted factors" approach of [1], [7] and the additive construction used in this paper, is a third "multiplicative" method for generating block symmetric linearizations described by Lancaster in [13], [14]. In [13] only scalar polynomials $p(\lambda) = a_k\lambda^k + \cdots + a_1\lambda + a_0$ are considered; the starting point is the companion matrix of $p(\lambda)$,

$$C = \begin{bmatrix} -a_k^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a_{k-1} & a_{k-2} & \dots & a_0 \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \tag{4.2}$$

and the associated pencil $\lambda I - C$. Lancaster's idea is to seek a nonsingular symmetric matrix B such that BC is symmetric, thus providing a symmetric linearization $B(\lambda I - C) = \lambda B - BC$ for $p(\lambda)$. That such a B can always be found follows from a standard result in matrix theory [11, Cor. 4.4.11]. Lancaster shows further that B and BC symmetric implies BC^j is symmetric for all $j \geq 1$; thus $BC^{j-1}(\lambda I - C) = \lambda BC^{j-1} - BC^j$ is a symmetric pencil for any $j \geq 1$, and for $j \geq 2$ it is a linearization of $p(\lambda)$ if $a_0 \neq 0$. This strategy is realized in [13] with the particular choice of symmetric (Hankel) matrix

$$B = \begin{bmatrix} & & & a_k \\ & & \cdot^{\cdot^{\cdot}} & a_{k-1} \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ a_k & a_{k-1} & \dots & a_1 \end{bmatrix}, \tag{4.3}$$

which is nonsingular since $a_k \neq 0$, and it is observed that this particular B gives symmetric pencils $\lambda BC^{j-1} - BC^j$ with an especially simple form for $1 \leq j \leq k$, though apparently with a much more complicated form for $j > k$.

It is easy to see that these symmetric pencils, constructed for scalar polynomials $p(\lambda)$, can be immediately extended to block symmetric pencils for general matrix polynomials $P(\lambda)$ simply by formally replacing the scalar coefficients of $p(\lambda)$ in $B, BC, BC^2, \ldots$ by the matrix coefficients of $P(\lambda)$. This has been done in [14, Sect. 4.2] and [8]. Garvey et al. [8] go even further with these block symmetric pencils, using them as a foundation for defining a new class of isospectral transformations on matrix polynomials.

Since Lancaster's construction of pencils is so different from ours there is no reason to expect any connection between his pencils and the pencils in $\mathbb{DL}(P)$. The next result shows, rather surprisingly, that the first k pencils in Lancaster's sequence generate $\mathbb{DL}(P)$.

THEOREM 4.1. *For any matrix polynomial $P(\lambda)$ of degree k, the pencil $\lambda BC^{k-m} - BC^{k-m+1}$ from Lancaster's construction, with B and C defined by the block matrix analogue of* (4.2) *and* (4.3)*, is identical to $\lambda X_m - X_{m-1}$, the mth standard basis pencil for $\mathbb{DL}(P)$, for $m = 1{:}\,k$.*

Proof. We have to show that $X_m = BC^{k-m}$, $m = 0{:}\,k$, where X_m is given by (3.8a). For notational simplicity we will carry out the proof for a scalar polynomial; the same proof applies to a matrix polynomial with only minor changes in notation. The $m = k$ case, $X_k(p(\lambda)) = \mathcal{L}_k(p(\lambda)) = B$, is immediate from equations (3.6), (3.8), and (4.3). The rest follow inductively (downward) from the relation $X_{m-1}(p(\lambda)) = X_m(p(\lambda)) \cdot C$, which we now proceed to show holds for $m = 1{:}\,k$.

To see that $X_m C = X_{m-1}$, or equivalently that

$$\begin{bmatrix} \mathcal{L}_m(p(\lambda)) & 0 \\ 0 & -\mathcal{U}_{k-m}(p(\lambda)) \end{bmatrix} C = \begin{bmatrix} \mathcal{L}_{m-1}(p(\lambda)) & 0 \\ 0 & -\mathcal{U}_{k-m+1}(p(\lambda)) \end{bmatrix}$$

holds for $1 \le m \le k$, it will be convenient to rewrite the companion matrix (4.2) in the form

$$C = N_k^T - a_k^{-1} \begin{bmatrix} a_{k-1} & a_{k-2} & \ldots & a_0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = N_k^T - a_k^{-1} e_1 \,[\, a_{k-1} \quad a_{k-2} \quad \ldots \quad a_0 \,],$$

where N_k is defined in (2.1). Then

$$\begin{aligned} X_m(p(\lambda))C &= X_m(p(\lambda))N_k^T - a_k^{-1} X_m(p(\lambda))\, e_1 \,[\, a_{k-1} \quad a_{k-2} \quad \ldots \quad a_0 \,] \\ &= \begin{bmatrix} \mathcal{L}_m(p(\lambda)) & 0 \\ 0 & -\mathcal{U}_{k-m}(p(\lambda)) \end{bmatrix} N_k^T - e_m \,[\, a_{k-1} \quad a_{k-2} \quad \ldots \quad a_0 \,]. \end{aligned}$$

In the first term, postmultiplication by N_k^T has the effect of shifting the columns to the left by one (and losing the first column), thus giving

$$\begin{aligned} &X_m(p(\lambda))C = \\ &\quad \begin{bmatrix} \mathcal{L}_{m-1}(p(\lambda)) & 0 & 0 \\ a_{k-1} \ldots a_{k-m+1} & 0 & 0 \\ 0 & -\mathcal{U}_{k-m}(p(\lambda)) & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ a_{k-1} \ldots a_{k-m+1} & a_{k-m} \ldots a_1 & a_0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad = \left[\begin{array}{c|cc} \mathcal{L}_{m-1}(p(\lambda)) & 0 & 0 \\ \hline 0 & -a_{k-m} \ldots -a_1 & -a_0 \\ 0 & -\mathcal{U}_{k-m}(p(\lambda)) & 0 \end{array}\right] \end{aligned}$$

$$= \begin{bmatrix} \mathcal{L}_{m-1}(p(\lambda)) & 0 \\ 0 & -\mathcal{U}_{k-m+1}(p(\lambda)) \end{bmatrix} = X_{m-1}(p(\lambda)).$$

This completes the inductive step of the proof. ◻

5. Symmetric pencils in $\mathbb{L}_1(P)$ for symmetric P. We now return to the problem that originally motivated the investigation in this paper, that of systematically finding large sets of symmetric linearizations for symmetric polynomials, $P(\lambda) = P(\lambda)^T$. Our strategy is first to characterize the set

$$\mathbb{S}(P) := \left\{ \lambda X + Y \in \mathbb{L}_1(P) : X^T = X, Y^T = Y \right\} \tag{5.1}$$

of all symmetric pencils in $\mathbb{L}_1(P)$ when P is symmetric, and then later in Section 7 to show that almost all of these symmetric pencils are indeed linearizations for P.

We begin with a result for symmetric polynomials reminiscent of Theorem 2.2, but using transpose rather than block transpose.

LEMMA 5.1. *Suppose $P(\lambda)$ is a symmetric matrix polynomial and $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v. Then $L^T(\lambda) \in \mathbb{L}_2(P)$ with left ansatz vector $w = v$. Similarly, $L(\lambda) \in \mathbb{L}_2(P)$ with left ansatz vector v implies that $L^T(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v.*

Proof. Suppose $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v. Then

$$\big(L(\lambda)(\Lambda \otimes I)\big)^T = \big(v \otimes P(\lambda)\big)^T \implies (\Lambda^T \otimes I)L^T(\lambda) = v^T \otimes P(\lambda).$$

Thus $L^T(\lambda) \in \mathbb{L}_2(P)$ with left ansatz vector v. The proof of the second statement is analogous. ◻

We characterize the space $\mathbb{S}(P)$ in the next result by relating it to the previously developed space $\mathbb{DL}(P)$, which we already know equals $\mathbb{B}(P)$.

THEOREM 5.2. *For any symmetric polynomial $P(\lambda)$, $\mathbb{S}(P) = \mathbb{DL}(P)$.*

Proof. Suppose $L(\lambda) \in \mathbb{S}(P) \subseteq \mathbb{L}_1(P)$ with right ansatz vector v. Then by Lemma 5.1 we know that $L^T(\lambda) = L(\lambda)$ is in $\mathbb{L}_2(P)$ with left ansatz vector v, and so $L(\lambda) \in \mathbb{DL}(P)$. Thus $\mathbb{S}(P) \subseteq \mathbb{DL}(P)$.

By Lemma 5.1, $L(\lambda) \in \mathbb{DL}(P)$ with right/left ansatz vector v implies that $L^T(\lambda) \in \mathbb{DL}(P)$ with left/right ansatz vector v. But by Theorem 3.4 pencils in $\mathbb{DL}(P)$ are uniquely determined by their ansatz vector, so $L(\lambda) \equiv L^T(\lambda)$, and hence $\mathbb{DL}(P) \subseteq \mathbb{S}(P)$. Therefore $\mathbb{DL}(P) = \mathbb{S}(P)$. ◻

Once again one may refer to Tables 3.1 and 3.2 for examples of what are in effect *triply*-structured pencils whenever P is symmetric. Recall, however, that there are symmetric linearizations for P that are not in $\mathbb{S}(P)$: L in (4.1) is not in $\mathbb{S}(P)$, but is a symmetric linearization for any symmetric cubic P.

6. Hermitian pencils in $\mathbb{L}_1(P)$ for Hermitian P. For a Hermitian matrix polynomial $P(\lambda)$ of degree k, that is, $P(\lambda)^* = P(\overline{\lambda})$, let

$$\mathbb{H}(P) := \left\{ \lambda X + Y \in \mathbb{L}_1(P) : X^* = X,\ Y^* = Y \right\} \tag{6.1}$$

denote the set of all Hermitian pencils in $\mathbb{L}_1(P)$. A priori the right ansatz vector v of a pencil in $\mathbb{H}(P)$ might be any vector in $\mathbb{C}^k$, since P is a complex polynomial. However, the next result shows that any such v must in fact be real.

LEMMA 6.1. *Suppose $P(\lambda)$ is a Hermitian polynomial and $L(\lambda) \in \mathbb{H}(P)$ with right ansatz vector v. Then $v \in \mathbb{R}^k$ and $L(\lambda) \in \mathbb{DL}(P)$, so $\mathbb{H}(P) \subsetneq \mathbb{DL}(P)$.*

Proof. Since $L(\lambda) \in \mathbb{L}_1(P)$, we have $L(\lambda)(\Lambda \otimes I) = v \otimes P(\lambda)$. Then, since P and L are Hermitian,

$$\big(L(\lambda)(\Lambda \otimes I)\big)^* = \big(v \otimes P(\lambda)\big)^* \implies (\overline{\Lambda}^T \otimes I)L(\overline{\lambda}) = \overline{v}^T \otimes P(\overline{\lambda}).$$

This last equation holds for all λ, so we may replace $\overline{\lambda}$ by λ to get $(\Lambda^T \otimes I)L(\lambda) = \overline{v}^T \otimes P(\lambda)$, so that $L(\lambda) \in \mathbb{L}_2(P)$ with left ansatz vector $w = \overline{v}$. Thus $L(\lambda) \in \mathbb{DL}(P)$. But by Theorem 3.4 the right and left ansatz vectors of any $\mathbb{DL}(P)$-pencil must be equal. So $v = \overline{v}$, which means $v \in \mathbb{R}^k$. Since $\mathbb{DL}(P)$ includes pencils corresponding to nonreal v, $\mathbb{H}(P) \subsetneq \mathbb{DL}(P)$. ∎

Recall the map $\mathbb{DL}(P) \xrightarrow{\mathcal{M}} \mathcal{V}_P$ from (3.2), which we know from Theorems 3.1 and 3.4 to be an isomorphism. Lemma 6.1 implies that $\mathcal{M}$ can be restricted to the subspace $\mathbb{H}(P)$, giving a 1-1 map into the "real" part of $\mathcal{V}_P$, i.e. into the subspace $\mathcal{R}_P := \{\, v \otimes P(\lambda) : v \in \mathbb{R}^k \,\} \subsetneq \mathcal{V}_P$. The characterization of $\mathbb{H}(P)$ is then completed in the next result by showing that $\mathbb{H}(P) \xrightarrow{\mathcal{M}} \mathcal{R}_P$ is actually an isomorphism.

THEOREM 6.2. *For any Hermitian polynomial $P(\lambda)$, $\mathbb{H}(P)$ is the subset of all pencils in $\mathbb{DL}(P)$ with a real ansatz vector. In other words, for each vector $v \in \mathbb{R}^k$ there is a unique Hermitian pencil $H(\lambda) \in \mathbb{H}(P)$.*

Proof. We need to show that the map $\mathbb{H}(P) \xrightarrow{\mathcal{M}} \mathcal{R}_P$ is an isomorphism, and from the remarks preceding the theorem all that remains is to show that the map $\mathcal{M}$ is onto. By arguments analogous to the ones used in Lemma 5.1 and Theorem 5.2, it is straightforward to show that for Hermitian P, $L(\lambda) \in \mathbb{DL}(P)$ with right/left ansatz vector v implies that $L^*(\lambda) \in \mathbb{DL}(P)$ with left/right ansatz vector $\overline{v}$. Now if for an arbitrary $v \in \mathbb{R}^k$ we let $H(\lambda)$ be the unique pencil in $\mathbb{DL}(P)$ with right/left ansatz vector v, then $H^*(\lambda)$ is also in $\mathbb{DL}(P)$ with exactly the same ansatz vector v. The uniqueness of $\mathbb{DL}(P)$-pencils then implies that we must have $H(\lambda) \equiv H^*(\lambda)$, i.e., $H(\lambda) \in \mathbb{H}(P)$, thus showing that the map $\mathcal{M}$ is onto. ∎

7. Almost all pencils in $\mathbb{B}(P)$, $\mathbb{DL}(P)$, $\mathbb{S}(P)$, and $\mathbb{H}(P)$ are linearizations. The remaining fundamental issue is the question of which pencils in the subspaces $\mathbb{B}(P)$, $\mathbb{DL}(P)$, $\mathbb{S}(P)$, and $\mathbb{H}(P)$ are actually linearizations for P when P is regular. Some answers to this question are already known. First, a pencil L in $\mathbb{L}_1(P)$ or $\mathbb{L}_2(P)$ is a linearization precisely when L is a regular pencil [17, Thm. 4.3]. Second, for each of $\mathbb{L}_1(P)$, $\mathbb{L}_2(P)$, and $\mathbb{DL}(P)$ it is known that almost all pencils are linearizations, where "almost all" means all except for a closed, nowhere dense set of measure zero [17, Thms. 4.7, 6.8]. Because of Theorems 3.4 and 5.2, the same conclusion follows immediately for $\mathbb{B}(P)$, and for $\mathbb{S}(P)$ when P is symmetric. However, for $\mathbb{H}(P)$ the possible ansatz vectors lie in $\mathbb{R}^k$, a closed, nowhere dense set of measure zero in $\mathbb{C}^k$ (the ansatz vector set of $\mathbb{DL}(P)$ when P is Hermitian), so we cannot immediately deduce an "almost all" result for $\mathbb{H}(P)$. Some further analysis is therefore needed.

To a vector $v = [v_1, v_2, \ldots, v_k]^T \in \mathbb{F}^k$ associate the scalar polynomial

$$\mathsf{p}(x; v) = v_1 x^{k-1} + v_2 x^{k-2} + \cdots + v_{k-1} x + v_k,$$

referred to as the "v-polynomial" of the vector v. We adopt the convention that $\mathsf{p}(x; v)$ has a root at ∞ whenever $v_1 = 0$. The following result provides a condition that $L \in \mathbb{DL}(P)$ be a linearization of P.

THEOREM 7.1 (Eigenvalue Exclusion Theorem [17, Thm. 6.7]). *Suppose that $P(\lambda)$ is a regular matrix polynomial and $L(\lambda) \in \mathbb{DL}(P)$ with ansatz vector v. Then $L(\lambda)$ is a linearization for $P(\lambda)$ if and only if no root of the v-polynomial $\mathsf{p}(x; v)$ is an eigenvalue of $P(\lambda)$.*

With the aid of this result we can establish the desired genericity statement.

THEOREM 7.2 (Linearizations are generic in $\mathbb{H}(P)$). *Let $P(\lambda)$ be a regular Hermitian matrix polynomial. For almost all $v \in \mathbb{R}^k$ the corresponding pencil in $\mathbb{H}(P)$ is a linearization.*

Proof. Recall that the resultant $\operatorname{res}(f, g)$ of two polynomials $f(x)$ and $g(x)$ is a polynomial in the coefficients of f and g with the property that $\operatorname{res}(f, g) = 0$ if and only if $f(x)$ and $g(x)$ have a common (finite) root [21, p. 248], [25]. Now consider $r = \operatorname{res}\big(\mathsf{p}(x; v), \det P(x)\big)$, which, because P is Hermitian and fixed, can be viewed as a real polynomial $r(v_1, v_2, \ldots, v_k)$ in the components of $v \in \mathbb{R}^k$. The zero set $\mathcal{Z}(r) = \big\{ v \in \mathbb{R}^k : r(v_1, v_2, \ldots, v_k) = 0 \big\}$, then, is exactly the set of all $v \in \mathbb{R}^k$ for which some finite root of $\mathsf{p}(x; v)$ is an eigenvalue of $P(\lambda)$. Recall that by our convention the v-polynomial $\mathsf{p}(x; v)$ has ∞ as a root exactly for $v \in \mathbb{R}^k$ lying in the hyperplane $v_1 = 0$. Thus by Theorem 7.1 the set of vectors $v \in \mathbb{R}^k$ for which the corresponding pencil $L(\lambda) \in \mathbb{H}(P) \subset \mathbb{DL}(P)$ is *not* a linearization of $P(\lambda)$ is either the proper (real) algebraic set $\mathcal{Z}(r)$, or the union of two proper (real) algebraic sets, $\mathcal{Z}(r)$ and the hyperplane $v_1 = 0$. But the union of any finite number of proper (real) algebraic sets is always a closed, nowhere dense set of measure zero in $\mathbb{R}^k$. □

8. Conclusions. We have revisited $\mathbb{DL}(P)$, the space of double ansatz pencils introduced in [17], proving that it is the same as the set of block-symmetric pencils in the right ansatz space $\mathbb{L}_1(P)$. Our alternative characterization of $\mathbb{DL}(P)$ shows that even unstructured matrix polynomials admit linearizations that are symmetric at the block level, while simultaneously possessing the $\mathbb{DL}(P)$ property of revealing both left and right eigenvectors of P.

Our analysis shows how to find all the symmetric pencils in $\mathbb{L}_1(P)$ for a symmetric matrix polynomial P: these are precisely the pencils in $\mathbb{DL}(P)$. For Hermitian P, the Hermitian pencils in $\mathbb{L}_1(P)$ correspond to the double ansatz pencils that have a real ansatz vector. Almost all pencils in each of these vector spaces have been shown to be linearizations.

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