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On the ratio X/Y for some elliptically symmetric distributions

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Abstract

The distributions of the ratio X/Y are derived when (X, Y) has the elliptically symmetric Pearson-type II distribution, elliptically symmetric Pearson-type VII distribution and the elliptically symmetric Kotz-type distribution.

AMS 2000 subject classification: 33C90; 62E99

Keywords: Elliptically symmetric distributions; Ratios of random variables

1. Introduction

For a bivariate random vector (X, Y), the distribution of the ratio X/Y is of interest in problems in biological and physical sciences, econometrics, and ranking and selection. Examples include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. The distribution of X/Y has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see [8,12] for normal family, [13] for Student's t family, [1] for Weibull family, [16] for stable family, [4] for non-central chi-squared family, and [14] for gamma family. However, there is relatively little work of this kind when X and Y are correlated random variables. Some of the known

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work include [5] for bivariate normal family, [7] for bivariate t family, and [11] for bivariate gamma family.

In this paper, we study the distribution of X/Y when (X, Y) has the

1. Elliptically symmetric Pearson-type II distribution given by the joint pdf

$$f(x, y) = \frac{N+1}{\pi\sqrt{1-\rho^2}}\left(1 - \frac{(x-\alpha)^2 + (y-\beta)^2 - 2\rho(x-\alpha)(y-\beta)}{1-\rho^2}\right)^N \tag{1}$$

for $\{(x-\alpha)2+(y-\beta)2-2\rho(x-\alpha)(y-\beta)\}/(1-\rho^2) < 1, -\infty < x < \infty, -\infty < y < \infty, -\infty < \alpha < \infty, -\infty < \beta < \infty, N > -1$, and $-1 < \rho < 1$.

2. Elliptically symmetric Pearson-type VII distribution given by the joint pdf

$$f(x, y) = \frac{N-1}{\pi m\sqrt{1-\rho^2}}\left(1 + \frac{(x-\alpha)^2 + (y-\beta)^2 - 2\rho(x-\alpha)(y-\beta)}{m\left(1-\rho^2\right)}\right)^{-N} \tag{2}$$

for $-\infty < x < \infty, -\infty < y < \infty, -\infty < \alpha < \infty, -\infty < \beta < \infty, N > 1, m > 0$, and $-1 < \rho < 1$. The bivariate t-distribution and the bivariate Cauchy distribution are special cases of (2) for $N = (m+2)/2$ and $m = 1$, $N = \frac{3}{2}$, respectively.

3. Elliptically symmetric Kotz-type distribution given by the joint pdf

$$f(x, y) = \frac{sr^{N/s}\left\{(x-\alpha)^2 + (y-\beta)^2 - 2\rho(x-\alpha)y-\beta)\right\}^{N-1}}{\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \times \exp\left\{-r\left(\frac{(x-\alpha)^2 + (y-\beta)^2 - 2\rho(x-\alpha)(y-\beta)}{1-\rho^2}\right)^s\right\} \tag{3}$$

for $-\infty < x < \infty, -\infty < y < \infty, -\infty < \alpha < \infty, -\infty < \beta < \infty, N > 0, r > 0, s > 0$ and $-1 < \rho < 1$. When $s = 1$, this is the original Kotz distribution introduced in [9]. When $N = 1$, $s = 1$ and $r = \frac{1}{2}$, (3) reduces to a bivariate normal density.

The parameter ρ is the correlation coefficient between the x and y components. For details on properties of these distributions see [2,6,10].

The last two decades have seen a vigorous development of elliptically symmetric distributions as direct generalizations of the multivariate normal distribution which has dominated statistical theory and applications for almost a century. Elliptically symmetric distributions retain most of the attract properties of the multivariate normal distribution. The distributions mentioned above are three of the most popular elliptical symmetric distributions. For instance, since 1990, there has been a surge of activity relating to the elliptical symmetric Kotz-type distribution. It has attracted applications in areas such as Bayesian statistics, ecology, discriminant analysis, mathematical finance, repeated measurements, shape theory and signal processing. The elliptical symmetric Pearson-type VII is becoming increasingly important in classical as well as in Bayesian statistical modeling. Its application is a very promising approach in multivariate analysis. Classical multivariate analysis is soundly and

rigidly tilted toward the multivariate normal distribution while the Pearson-type VII distribution offers a more viable alternative with respect to real-world data, particularly because its tails are more realistic. We have seen recently some unexpected applications in novel areas such as cluster analysis, discriminant analysis, missing data imputation, multiple regression, portfolio optimization, robust projection indices, security returns, and speech recognition.

The aim of this paper is to calculate the distributions of the ratio X/Y when (X, Y) has the joint pdfs (1), (2) and (3). The calculations of this paper involve the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, dt$$

and the Gauss hypergeometric function defined by

$$F(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in [3,15].

2. Pearson-type II distribution

Theorem 1 derives an explicit elementary expression for the pdf of $Z = X/Y$. Note that the resulting pdf depends only on ρ. The proof of this theorem is presented in the appendix.

Theorem 1. *If X and Y are jointly distributed according to* (1) *then the pdf of* $Z = X/Y$ *is*

$$f(z) = \frac{2g(\arctan(z))}{1+z^2}, \tag{4}$$

where

$$g(\theta) = \frac{\sqrt{1-\rho^2}}{2\pi\{1-\rho\sin(2\theta)\}}.$$

3. Pearson-type VII distribution

Theorem 2 derives an explicit expression for the pdf of $Z = X/Y$ in terms of the Gauss hypergeometric function. Its proof is also presented in the appendix.

Theorem 2. *If X and Y are jointly distributed according to* (2) *and let*

$$A = 1 - \rho\sin(2\theta), \tag{5}$$

$$B = (\rho\beta - \alpha)\cos\theta + (\rho\alpha - \beta)\sin\theta \tag{6}$$

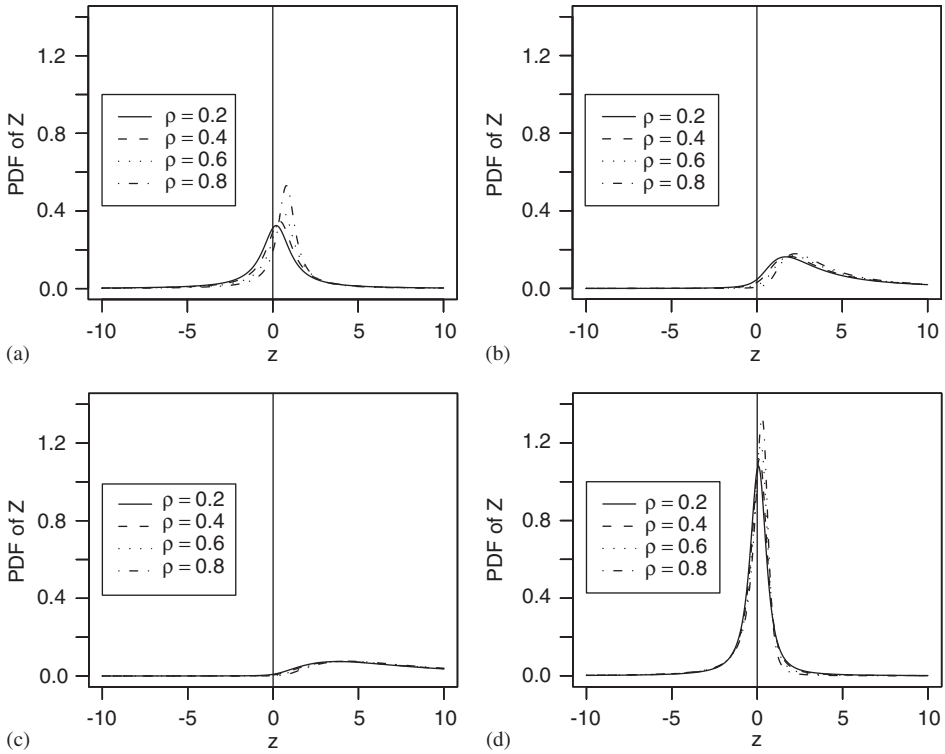

Fig. 1. Plots of the pdf (8) for $N = 2$, $m = 1$ and (a) $\alpha = 0$ and $\beta = 0$; (b) $\alpha = 0$ and $\beta = 2$; (c) $\alpha = 0$ and $\beta = 5$; and, (d) $\alpha = 1$ and $\beta = 0$.

and

$$C = \alpha^2 + \beta^2 - 2\rho\alpha\beta + m\left(1 - \rho^2\right). \tag{7}$$

Then, provided that $B^2 < AC$, *the pdf of* Z *can be expressed as*

$$f(z) = \frac{2g(\arctan(z))}{1 + z^2}, \tag{8}$$

where

$$g(\theta) = \frac{C^{1-N}\Gamma(2N-2)(N-1)m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{A\Gamma(2N)\pi} F\left(1, N-1; N+\frac{1}{2}; 1-\frac{B^2}{AC}\right). \tag{9}$$

Figs. 1 and 2 illustrate possible shapes of the pdf (8) for a range of values of α, β, ρ and N.

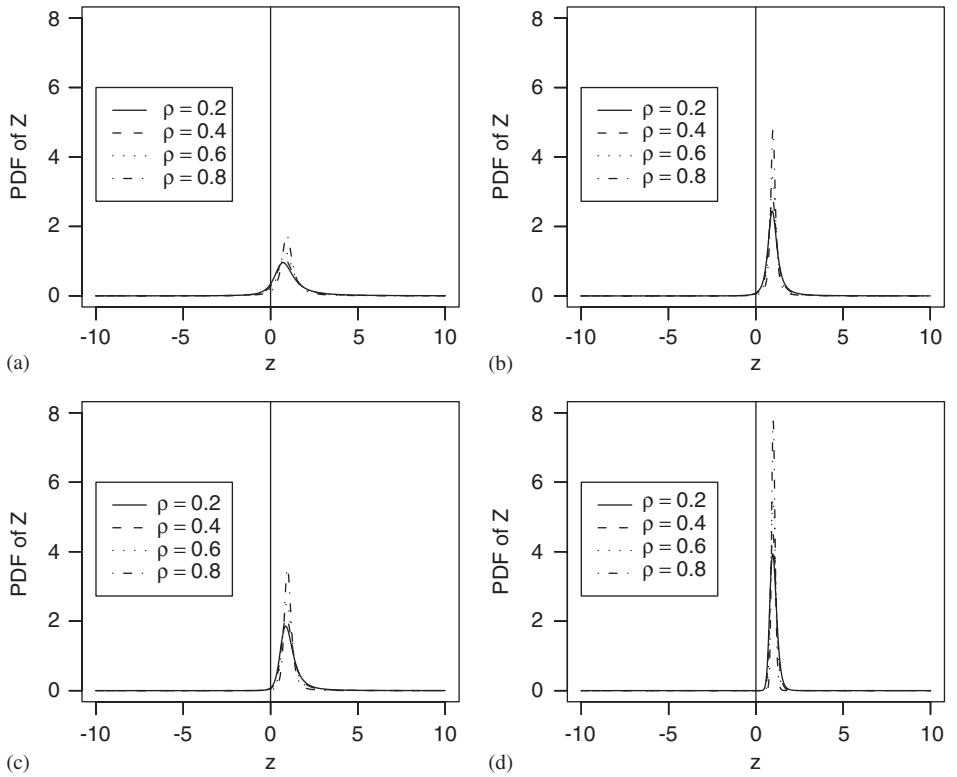

Fig. 2. Plots of the pdf (8) for $m = 1$ and (a) $\alpha = 1$, $\beta = 1$ and $N = 2$; (b) $\alpha = 3$, $\beta = 3$ and $N = 2$; (c) $\alpha = 1$, $\beta = 1$ and $N = 5$; and, (d) $\alpha = 1$, $\beta = 1$ and $N = 20$.

Theorems 3 and 4 and Corollaries 1–4 consider three particular forms for the pdf of Z involving only elementary forms. Corollary 1 considers the case $\alpha = \beta = 0$. Note that the resulting pdf depends only on ρ. Theorem 3 and Corollaries 2 and 3 consider integer values for N while Theorem 4 and Corollary 4 consider half-integer values for N.

Corollary 1. *If X and Y are jointly distributed according to* (2) *and if $\alpha = \beta = 0$ then* (9) *reduces to*

$$g(\theta) = \frac{\sqrt{1-\rho^2}}{2\pi\{1-\rho\sin(2\theta)\}}.$$

Theorem 3. *Suppose X and Y are jointly distributed according to* (2). *Assume $N \geqslant 2$ is an integer and $B > 0$, where A, B and C are given by* (5), (6) *and* (7), *respectively. Then, the pdf of Z can be expressed as*

$$f(z) = \frac{2g(\arctan(z))}{1+z^2}, \tag{10}$$

where

$$g(\theta)=\begin{cases}\dfrac{m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{2(-1)^{N-1}\pi(N-2)!}\dfrac{\partial^{N-2}}{\partial C^{N-2}} \\ \quad\times\left(\dfrac{B}{\left(AC-B^2\right)^{3/2}}\arctan\dfrac{B}{\sqrt{AC-B^2}}-\dfrac{1}{AC-B^2}\right), \\ \quad \textit{if } AC>B^2, \\ \dfrac{m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{2(-1)^{N-1}\pi(N-2)!}\dfrac{\partial^{N-2}}{\partial C^{N-2}} \\ \quad\times\left(\dfrac{B}{\left(B^2-AC\right)^{3/2}}\ln\dfrac{\sqrt{AC}}{B+\sqrt{B^2-AC}}-\dfrac{1}{AC-B^2}\right) \\ \quad \textit{if } AC<B^2, \\ \dfrac{A^{N-2}m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{2\pi(2N-1)B^{2N-2}} \quad \textit{if } AC=B^2.\end{cases} \tag{11}$$

Corollary 2. *Under the assumptions of Theorem* 3, *if* $AC>B^2$ *and* $N=2,3,4,5,6$ *then* (11) *reduces to*

$$g(\theta)=-\frac{m}{2\pi}\left(\frac{1-\rho^2}{AC-B^2}\right)^{3/2}\left\{\arctan\left(\frac{B}{\sqrt{AC-B^2}}\right)B-\sqrt{AC-B^2}\right\},$$

$$g(\theta)=-\frac{m^2}{4C\pi}\left(\frac{1-\rho^2}{AC-B^2}\right)^{5/2}\left\{B^2\sqrt{AC-B^2}+3B\arctan\left(\frac{B}{\sqrt{AC-B^2}}\right)AC\right.$$
$$\left.-2A\sqrt{AC-B^2}C\right\},$$

$$g(\theta)=\frac{m^3}{16C^2\pi}\left(\frac{1-\rho^2}{AC-B^2}\right)^{7/2}\left\{2B^4\sqrt{AC-B^2}-9AB^2\sqrt{AC-B^2}C\right.$$
$$\left.-15\arctan\left(\frac{B}{\sqrt{AC-B^2}}\right)A^2C^2B+8A^2\sqrt{AC-B^2}C^2\right\},$$

$$g(\theta)=-\frac{m^4}{96C^3\pi}\left(\frac{1-\rho^2}{AC-B^2}\right)^{9/2}\left\{8B^6\sqrt{AC-B^2}-38AB^4\sqrt{AC-B^2}C\right.$$
$$+87A^2B^2\sqrt{AC-B^2}C^2+105B\arctan\left(\frac{B}{\sqrt{AC-B^2}}\right)A^3C^3$$

$$-48A^3\sqrt{AC-B^2}C^3\Bigg\},$$

and

$$g(\theta) = \frac{m^5}{256C^4\pi}\left(\frac{1-\rho^2}{AC-B^2}\right)^{11/2}\Bigg\{16B^8\sqrt{AC-B^2}-88AB^6\sqrt{AC-B^2}C$$
$$+210A^2B^4\sqrt{AC-B^2}C^2-325\,A^3B^2\sqrt{AC-B^2}C^3$$
$$-315\arctan\left(\frac{B}{\sqrt{AC-B^2}}\right)A^4C^4B+128A^4\sqrt{AC-B^2}C^4\Bigg\},$$

respectively.

Corollary 3. *Under the assumptions of Theorem* 3, *if* $AC < B^2$ *and* $N = 2, 3, 4, 5, 6$ *then* (11) *reduces to*

$$g(\theta) = \frac{m}{2\pi}\left(\frac{1-\rho^2}{B^2-AC}\right)^{3/2}\left\{\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)B+\sqrt{B^2-AC}\right\},$$

$$g(\theta) = \frac{m^2}{4C\pi\left(B+\sqrt{B^2-AC}\right)}\left(\frac{1-\rho^2}{B^2-AC}\right)^{5/2}\Bigg\{B^4+\sqrt{B^2-AC}B^3$$
$$+3B^2\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)AC+B^2AC$$
$$+3B\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)AC\sqrt{B^2-AC}+2\sqrt{B^2-AC}ACB-2A^2C^2\Bigg\},$$

$$g(\theta) = \frac{m^3}{16C^2\pi\left(B+\sqrt{B^2-AC}\right)^2}\left(\frac{1-\rho^2}{B^2-AC}\right)^{7/2}$$
$$\times\Bigg\{4B^7+4\sqrt{B^2-AC}B^6-22ACB^5$$
$$-20B^4\sqrt{B^2-AC}AC-30B^3\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^2C^2+2B^3A^2C^2$$
$$-30B^2\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^2C^2\sqrt{B^2-AC}-7\sqrt{B^2-AC}A^2C^2B^2$$
$$+16BA^3C^3+15B\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^3C^3+8\sqrt{B^2-AC}A^3C^3\Bigg\},$$

$$g(\theta) = \frac{m^4}{96C^3\pi\left(B+\sqrt{B^2-AC}\right)^3}\left(\frac{1-\rho^2}{B^2-AC}\right)^{9/2}$$

$$
\begin{aligned}
&\times\Bigg\{32B^{10}+32\sqrt{B^2-AC}B^9-192B^8AC\\
&-176B^7\sqrt{B^2-AC}AC+546B^6A^2C^2+462B^5\sqrt{B^2-AC}A^2C^2\\
&+420B^4\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^3C^3-281B^4A^3C^3\\
&+420B^3\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^3C^3\sqrt{B^2-AC}-69B^3\sqrt{B^2-AC}A^3C^3\\
&-315B^2\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^4C^4-153B^2A^4C^4-144B\sqrt{B^2-AC}A^4C^4\\
&-105B\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^4C^4\sqrt{B^2-AC}+48A^5C^5\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
g(\theta)=&\frac{m^5}{256C^4\pi\left(B+\sqrt{B^2-AC}\right)^4}\left(\frac{1-\rho^2}{B^2-AC}\right)^{11/2}\Bigg\{128\,B^{13}+128\sqrt{B^2-AC}B^{12}\\
&-896B^{11}AC-832\sqrt{B^2-AC}B^{10}AC+2800\,B^9A^2C^2\\
&+2400B^8\sqrt{B^2-AC}A^2C^2-5472B^7A^3C^3-4368B^6\sqrt{B^2-AC}A^3C^3\\
&-2520B^5\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^4C^4+3716B^5A^4C^4\\
&-2520B^4\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^4C^4\sqrt{B^2-AC}+1786\sqrt{B^2-AC}A^4C^4B^4\\
&+2520B^3\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^5C^5+236B^3A^5C^5\\
&+1260B^2\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^5C^5\sqrt{B^2-AC}+699B^2\sqrt{B^2-AC}A^5C^5\\
&-512BA^6C^6-315B\ln\left(\frac{\sqrt{AC}}{B+\sqrt{B^2-AC}}\right)A^6C^6-128\sqrt{B^2-AC}A^6C^6\Bigg\},
\end{aligned}
$$

respectively.

Theorem 4. *Suppose X and Y are jointly distributed according to* (2). *Assume $N=M+1/2$, $M\geqslant 2$ is an integer and $B>0$, where A, B and C are given by* (5), (6) *and* (7), *respectively. Then, the pdf of Z can be expressed as*

$$
f(z)=\frac{2g(\arctan(z))}{1+z^2}, \tag{12}
$$

where

$$g(\theta) = \frac{(M-1/2)\, m^{M-1/2}(-2)^M \left(1-\rho^2\right)^M}{2\pi(2M-1)!!} \frac{\partial^{M-2}}{\partial C^{M-2}} \left(\frac{1}{\sqrt{C}\left(\sqrt{AC}+B\right)^2} \right). \tag{13}$$

Corollary 4. *Under the assumptions of Theorem* 4, *if* $M = 2, 3, 4, 5, 6$ *then* (13) *reduces to*

$$g(\theta) = \frac{m^{3/2}\left(1-\rho^2\right)^2}{\pi\sqrt{C}\left(\sqrt{AC}+B\right)^2},$$

$$g(\theta) = \frac{m^{5/2}\left(1-\rho^2\right)^3\left(3AC+\sqrt{AC}B\right)}{3\pi C^{3/2}\left(\sqrt{AC}+B\right)^3\sqrt{AC}},$$

$$g(\theta) = \frac{m^{7/2}\left(1-\rho^2\right)^4\left(5AC\sqrt{AC}+4ACB+\sqrt{AC}B^2\right)}{5\pi C^{5/2}\left(\sqrt{AC}+B\right)^4\sqrt{AC}},$$

$$g(\theta) = \frac{m^{9/2}\left(1-\rho^2\right)^5\left(35A^2C^2+47AC\sqrt{AC}B+25ACB^2+5\sqrt{AC}B^3\right)}{35\pi C^{7/2}\left(\sqrt{AC}+B\right)^5\sqrt{AC}}$$

and

$$g(\theta) = \frac{m^{11/2}\left(1-\rho^2\right)^6\left(63A^2C^2\sqrt{AC}+42ACB^3+122A^2C^2B+102AC\sqrt{AC}B^2+7\sqrt{AC}B^4\right)}{63\pi C^{9/2}\left(\sqrt{AC}+B\right)^6\sqrt{AC}},$$

respectively.

Corollary 1 follows from (9) by standard properties of the Gauss hypergeometric function (see Prudnikov et al. ([15], vol. 3, Section 7.3)). The result of Theorem 3 follows by repeating the proof of Theorem 2 and applying Lemma 2 to the integral in (25). Corollaries 2 and 3 follow by setting $N = 2, 3, 4, 5, 6$ into (11). The result of Theorem 4 follows by repeating the proof of Theorem 2 and applying Lemma 3 to the integral in (25). Corollary 4 follows by setting $M = 2, 3, 4, 5, 6$ into (13).

4. Kotz-type distribution

Theorem 5 derives an explicit expression for the pdf of $Z = X/Y$ in terms of the complementary incomplete gamma function. Its proof is presented in the appendix.

Theorem 5. *Suppose X and Y are jointly distributed according to* (3) *and let*

$$A = 1 - \rho \sin(2\theta), \tag{14}$$

$$B = (\rho\beta - \alpha)\cos\theta + (\rho\alpha - \beta)\sin\theta, \tag{15}$$

$$C = \alpha^2 + \beta^2 - 2\rho\alpha\beta, \tag{16}$$

and

$$D = \left(1 - \rho^2\right)(\alpha \sin\theta - \beta\cos\theta)^2 / A. \tag{17}$$

Furthermore, define

$$g_1(\theta) = \frac{\sqrt{1-\rho^2}}{2A\pi\Gamma(N/s)} \left\{ \Gamma\left(\frac{N}{s}, \frac{rC^s}{\left(1-\rho^2\right)^s}\right) + \frac{B}{\sqrt{A}} \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} \left(\frac{r^{1/s}}{1-\rho^2}\right)^{k+1/2} \Delta(C) \right\} \tag{18}$$

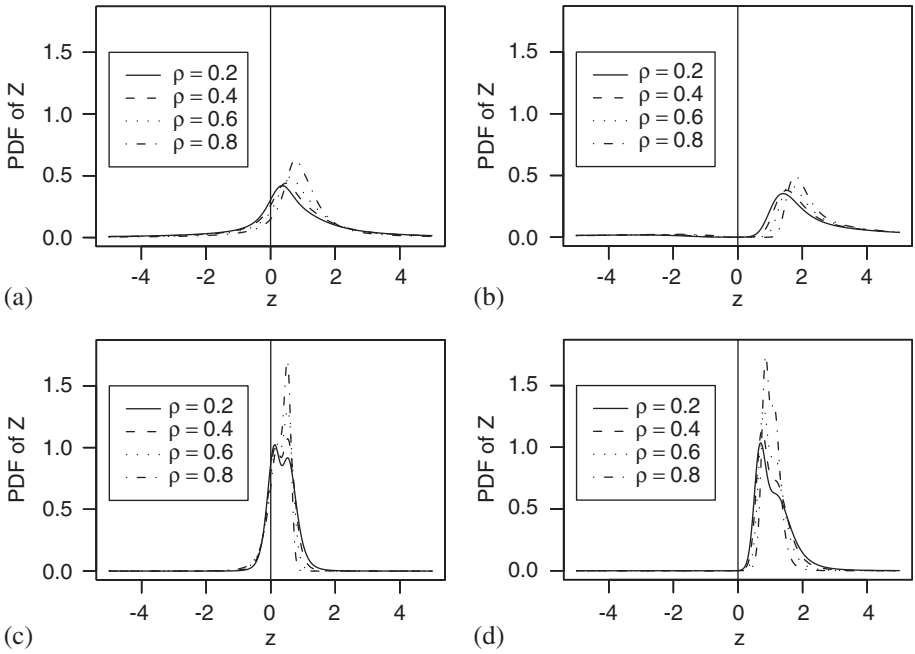

Fig. 3. Plots of the pdf (20)–(21) for $r = 1$, $s = 1$, $N = 2$ and (a) $\alpha = 1$ and $\beta = 1$; (b) $\alpha = 1$ and $\beta = 3$; (c) $\alpha = 3$ and $\beta = 1$; and, (d) $\alpha = 3$ and $\beta = 3$.

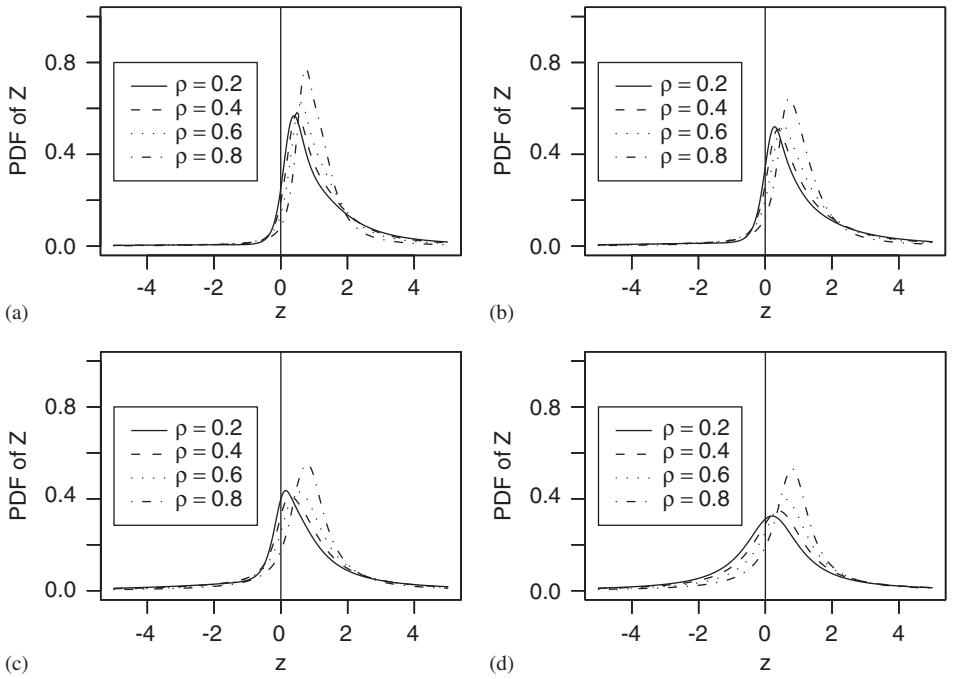

Fig. 4. Plots of the pdf (20)–(21) for $\alpha = 1$, $\beta = 1$, $r = 1$, $s = 2$ and (a) $N = 2$; (b) $N = 3$; (c) $N = 5$; and, (d) $N = 50$.

and

$$g_2(\theta) = \frac{B\sqrt{1-\rho^2}}{A^{3/2}\pi\Gamma(N/s)} \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} \left(\frac{r^{1/s}}{1-\rho^2}\right)^{k+1/2} \{\Delta(C) - \Delta(D)\}, \tag{19}$$

where

$$\Delta(a) = \left(a - \frac{B^2}{A}\right)^k \Gamma\left(\frac{N-k-1/2}{s}, \frac{ra^s}{\left(1-\rho^2\right)^s}\right).$$

If $B \geqslant 0$ *then the pdf of* $Z = X/Y$ *can be expressed as*

$$f(z) = \frac{g_1(\arctan(z)) + g_1(\pi + \arctan(z)) + g_2(\pi + \arctan(z))}{1+z^2}. \tag{20}$$

On the other hand, if $B < 0$ *then*

$$f(z) = \frac{g_1(\arctan(z)) + g_1(\pi + \arctan(z)) + g_2(\arctan(z))}{1+z^2}. \tag{21}$$

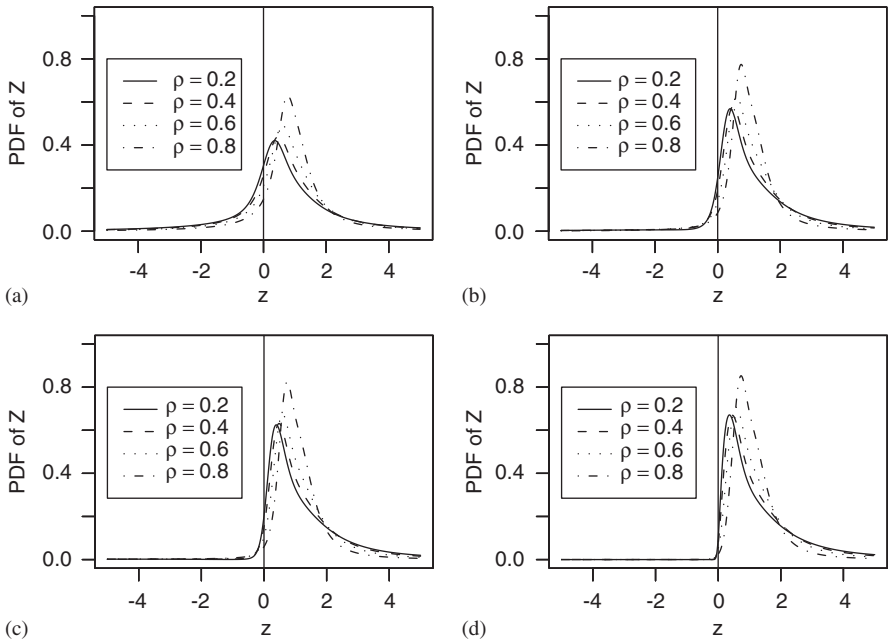

Fig. 5. Plots of the pdf (20)–(21) for $\alpha = 1$, $\beta = 1$, $r = 1$, $N = 2$ and (a) $s = 1$; (b) $s = 2$; (c) $s = 3$; and, (d) $s = 10$.

Figs. 3–5 illustrate possible shapes of the pdf (20)–(21) for a range of values of α, β, s, ρ and N. Note that several of the pdfs are multi-modal. This is interesting because the known ratio-distributions in the literature do not seem to have this feature.

Corollary 5 considers a particular form for the pdf of Z for the case $\alpha = \beta = 0$. Note that the resulting pdf is elementary and depends only on ρ. The proof of this corollary is also presented in the appendix.

Corollary 5. *If X and Y are jointly distributed according to* (3) *and if $\alpha = \beta = 0$ then the pdf of Z reduces to*

$$f(z) = \frac{2g(\arctan(z))}{1 + z^2}, \tag{22}$$

where

$$g(\theta) = \frac{\sqrt{1 - \rho^2}}{2\pi\{1 - \rho\sin(2\theta)\}}.$$

Appendix. Proofs

In this section, we outline proofs of the main results of the paper. We need the following technical lemmas.

Lemma 1 (*[15, vol. 1, Eq. (2.2.6.1)]*). *For $p > 0$ and $q > 0$,*

$$\int_a^b (x-a)^{p-1}(b-x)^{q-1}(cx+d)^r\,dx = (b-a)^{p+q-1}(ac+d)^r B(p,q) F\left(p,-r;p+q;\frac{c(a-b)}{ac+d}\right).$$

Lemma 2 (*[15, vol. 1, Eq. (2.2.9.14)]*). *For $a > 0$, $b > 0$, $c > 0$, and $n \geqslant 2$,*

$$\int_0^\infty \frac{x}{\left(ax^2+2bx+c\right)^n}\,dx = \begin{cases} \dfrac{(-1)^{n-1}}{2(n-1)!}\dfrac{\partial^{n-2}}{\partial c^{n-2}}\left(\dfrac{b}{\left(ac-b^2\right)^{3/2}}\arctan\dfrac{b}{\sqrt{ac-b^2}}-\dfrac{1}{ac-b^2}\right), \\ \qquad \text{if } ac > b^2, \\ \dfrac{(-1)^{n-1}}{2(n-1)!}\dfrac{\partial^{n-2}}{\partial c^{n-2}}\left(\dfrac{b}{\left(b^2-ac\right)^{3/2}}\ln\dfrac{\sqrt{ac}}{b+\sqrt{b^2-ac}}-\dfrac{1}{ac-b^2}\right), \\ \qquad \text{if } ac < b^2, \\ \dfrac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}}, \\ \qquad \text{if } ac = b^2. \end{cases}$$

Lemma 3 (*[15, vol. 1, Eq. (2.2.9.20)]*). *For $a > 0$, $b > 0$, $c > 0$, and $0 \leqslant m \leqslant n-1$,*

$$\int_0^\infty \frac{x^m}{\left(ax^2+2bx+c\right)^n}dx = \frac{(-1)^{n+m-1}2^{n-1}m!}{(2n-1)!!}\frac{\partial^{n-m-1}}{\partial c^{n-m-1}}\left(\frac{1}{\sqrt{c}\left(\sqrt{ac}+b\right)^{m+1}}\right).$$

Proof of Theorem 1. Set $(X,Y) = (\alpha + R\sin\Theta, \beta + R\cos\theta)$. Under this transformation, the Jacobian is R and so one can express the joint pdf of (R,Θ) as

$$g(r,\theta) = \frac{(N+1)\{1-\rho\sin(2\theta)\}^N}{\pi\left(1-\rho^2\right)^{N+1/2}} r\left(A-r^2\right)^N, \tag{23}$$

where $A = (1-\rho^2)/\{1-\rho\sin(2\theta)\}$. The region of integration $\{(x-\alpha)2 + (y-\beta)2 - 2\rho(x-\alpha)(y-\beta)\}/(1-\rho^2) < 1$ reduces to $r^2 \leqslant A$ or equivalently $r \leqslant \sqrt{A}$. Thus, the

marginal pdf of Θ can be obtained as

$$\begin{aligned} g(\theta) &= \frac{(N+1)\{1-\rho\sin(2\theta)\}^N}{\pi\left(1-\rho^2\right)^{N+1/2}} \int_0^{\sqrt{A}} r\left(A-r^2\right)^N dr \\ &= \frac{\{1-\rho\sin(2\theta)\}^N}{2\pi\left(1-\rho^2\right)^{N+1/2}} A^{N+1} \\ &= \frac{\sqrt{1-\rho^2}}{2\pi\{1-\rho\sin(2\theta)\}}. \end{aligned}$$

The result of the theorem follows by noting that the pdf of $Z = \tan\Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1+z^2} \tag{24}$$

and that $g(\arctan(\theta)) = g(\pi + \arctan(\theta))$ for the form for $g(\cdot)$. □

Proof of Theorem 2. Set $(X, Y) = (R\sin\Theta, R\cos\theta)$. Under this transformation, the Jacobian is R and so one can express the joint pdf of (R, Θ) as

$$g(r,\theta) = \frac{(N-1)m^{N-1}\left(1-\rho^2\right)^{N-1/2} r}{\pi\left(Ar^2+2Br+C\right)^N},$$

where A, B and C are given by (5), (6) and (7), respectively. Thus, the marginal pdf of Θ can be obtained as

$$g(\theta) = \frac{(N-1)m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{\pi} \int_0^\infty \frac{r}{\left(Ar^2+2Br+C\right)^N} dr \tag{25}$$

$$\begin{aligned} &= \frac{C^{1-N}\Gamma(2N-2)(N-1)m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{A\Gamma(2N)\pi} \\ &\quad \times F\left(1, N-1; N+\frac{1}{2}; 1-\frac{B^2}{AC}\right), \end{aligned} \tag{26}$$

which follows by a direct application of Lemma 1. The result of the theorem follows by noting that the pdf of $Z = \tan\Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1+z^2} \tag{27}$$

and that $g(\arctan(\theta)) = g(\pi + \arctan(\theta))$ for the form for $g(\cdot)$. □

Proof of Theorem 5. Set $(X, Y) = (T \sin\Theta, T\cos\theta)$. Under this transformation, the Jacobian is T and so one can express the joint pdf of (T, Θ) as

$$g(t,\theta) = \frac{sr^{N/s}\left(At^2+2Bt+C\right)^{N-1}}{\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \exp\left\{-\frac{r\left(At^2+2Bt+C\right)^s}{\left(1-\rho^2\right)^s}\right\}, \tag{28}$$

where A, B and C are given by (14), (15) and (16), respectively. Set $z(t) = At^2+2Bt+C$ and note that

$$\frac{dz(t)}{dt} = \pm 2\sqrt{B^2 - A(C-z)}.$$

Note further that $z(t)$ is an increasing function of t with $z(0) = C$ if $B \geqslant O$. On the other hand, if $B < 0$ then $z(t)$ decreases between $0 \leqslant t \leqslant z^{-1}(D)$ before increasing for all $t \geqslant z^{-1}(D)$, where D is given by (17). Thus, the marginal pdf of Θ can be expressed as $g(\theta) = g_1(\theta)$ if $B \geqslant 0$ and as $g(\theta) = g_l(\theta) + g_2(\theta)$ if $B < 0$, where

$$\begin{aligned} g_1(\theta) = {} & \frac{sr^{N/s}}{2A\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \int_C^\infty z^{N-1} \exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\} \\ & \times \left\{1 + \frac{B}{\sqrt{B^2 - A(C-z)}}\right\} dz \end{aligned} \tag{29}$$

and

$$g_2(\theta) = \frac{sr^{N/s}}{A\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \int_D^C \frac{Bz^{N-1}\exp\left\{-rz^s/\left(1-\rho^2\right)^s\right\}}{\sqrt{B^2 - A(C-z)}}\, dz \tag{30}$$

These two expressions actually reduce to those given by (18) and (19), respectively, as shown below. Consider (29). Note that $|\,(B^2 - AC)/(Az)\,| \leqslant 1$ for all $z \geqslant C$. Thus, using the series expansion

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^k,$$

one can expand (29) as

$$\begin{aligned} g_1(\theta) = {} & \frac{sr^{N/s}}{2A\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \int_C^\infty z^{N-1} \exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\} \\ & \times \left\{1 + \frac{B}{\sqrt{Az}} \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(\frac{B^2 - AC}{Az}\right)^k\right\} dz \end{aligned}$$

$$= \frac{sr^{N/s}}{2A\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}}\left[\int_C^\infty z^{N-1}\exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\}dz\right.$$

$$\left.+\frac{B}{\sqrt{A}}\sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{B\left(B^2-AC\right)^k}{A^{k+1/2}}\int_C^\infty z^{N-k-3/2}\exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\}dz\right]$$

$$= \frac{sr^{N/s}}{2A\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}}\left[\frac{\left(1-\rho^2\right)^N}{sr^{N/s}}\Gamma\left(\frac{N}{s},\frac{rC^s}{\left(1-\rho^2\right)^s}\right)\right.$$

$$+\frac{B}{\sqrt{A}}\sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{B\left(B^2-AC\right)^k}{A^{k+1/2}}\frac{\left(1-\rho^2\right)^{N-k-1/2}}{sr^{(N-k-1/2)/s}}$$

$$\left.\times\Gamma\left(\frac{N-k-1/2}{s},\frac{rC^s}{\left(1-\rho^2\right)^s}\right)\right],$$

where the last step follows from the definition of the complementary incomplete gamma function. One can similarly show that (30) reduces to the form given in (19). The result of the theorem follows by noting that the pdf of $Z = \tan\Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1+z^2} \tag{31}$$

and that $g(\theta) = g_1(\theta)$ if $B \geqslant 0$ and $g(\theta) = g_1(\theta) + g_2(\theta)$ if $B < 0$. □

Proof Corollary 5. Note that in this case (28) takes the form

$$g(t,\theta) = \frac{sr^{N/s}A^{N-1}}{\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}}\exp\left\{-\frac{rA^st^{2s}}{\left(1-\rho^2\right)^s}\right\},$$

where A is given by (14). Integrating this over $0 \leqslant t < \infty$ yields the form for $g(\cdot)$ given in the corollary. The result of the corollary follows by noting that the pdf of $Z = \tan\Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1+z^2} \tag{32}$$

and that $g(\arctan(\theta)) = g(\pi + \arctan(\theta))$ for the form for $g(\cdot)$. □

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