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Krichever's formula**

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The universal equivariant genus and Krichever’s formula

V. M. Buchstaber and N. Ray

We consider smooth $2n$ -dimensional manifolds M , equipped with a smooth action θ of the circle S^1 . Such an M determines a *normally complex S^1 -manifold* (M, θ, c_ν) whenever an equivariant complex structure c_ν is chosen for the normal bundle $\nu(i)$ of some equivariant embedding $i: M \rightarrow \mathbb{C}^k$. In this situation, we use the standard action of S^1 on the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$ to define a $2(m+n)$ -dimensional smooth manifold $W_m = S^{2m+1} \times_{S^1} M$, and a k -dimensional complex vector bundle $q_m: E_m \rightarrow \mathbb{C}P^m$, where $E_m = S^{2m+1} \times_{S^1} \mathbb{C}^k$. Then i extends to an embedding $i': W_m \rightarrow E_m$, and the complex structure c_ν extends to a complex structure c' on the normal bundle $\nu(i')$.

The composition $p_m: W_m \xrightarrow{i'} E_m \xrightarrow{q_m} \mathbb{C}P^m$ is complex oriented in the sense of [1] and determines a complex cobordism class $\Phi_m(M, \theta, c_\nu) \in U^{-2n}(\mathbb{C}P^m)$. The standard embedding $\iota_m: \mathbb{C}P^m \rightarrow \mathbb{C}P^{m+1}$ acts by $\iota_m^* \Phi_{m+1} = \Phi_m$, hence the inverse sequence $(\Phi_m(M, \theta, c_\nu) : m \geq 0)$ defines an element of $\varprojlim U^{-2n}(\mathbb{C}P^m) \simeq U^{-2n}(\mathbb{C}P^\infty)$. We write this element as $\Phi(M, \theta, c_\nu)$ and call it the *universal S^1 -equivariant genus* of (M, θ, c_ν) . The genus Φ was used to deduce fundamental results on Hirzebruch genera of S^1 -manifolds in [2] and [3]. An analogue of Φ for a compact Lie group has been studied by several authors since the 1960s (see [4] and [1]).

The ring $U^*(\mathbb{C}P^\infty)$ is isomorphic to the algebra $\Omega^*[[u]]$ of formal power series over $\Omega^* = U^*(pt)$, where u is the cobordism class of the inverse sequence $(\iota_m : m \geq 0)$. So our initial aim is to construct stably complex manifolds $G_j(M)$ of dimension $2(n+j)$ such that $\Phi(M, \theta, c_\nu) = \sum_{j=0}^\infty [G_j(M)] u^j$. Clearly, $G_0 = M$ with the stably complex structure c_ν .

We consider $(S^3)^j = \{(y_1, z_1; \dots; y_j, z_j) \in \mathbb{C}^{2j} : |y_i|^2 + |z_i|^2 = 1, 1 \leq i \leq j\}$, on which the torus T^j acts freely by $t \cdot (y_1, z_1; \dots; y_j, z_j) = (t_1 y_1, t_1^{-1} z_1; \dots; t_j y_j, t_j^{-1} z_j)$ for all $t = (t_1, \dots, t_j)$. The quotient manifold $B_j = (S^3)^j / T^j$ is a j -fold iterated S^2 -sphere bundle over $B_0 = pt$ and admits complex line bundles η_1, \dots, η_j such that $E(\eta_i) = (S^3)^j \times_{T^j} \mathbb{C}$ via the action $t \cdot z = t_i^{-1} z$ for $z \in \mathbb{C}$. For $j > 0$, the isomorphism $\tau(B_j) \oplus \mathbb{C}^j \cong (\bar{\eta}_1 \oplus \eta_1) \oplus \dots \oplus (\bar{\eta}_j \oplus \eta_{j-1} \eta_j)$ defines a stably complex structure c_j^∂ such that $B_j = \partial D_j$, where D_j is the associated 3-disc bundle, hence $[B_j] = 0$ for $j > 0$.

Theorem 1. *For every $j \geq 0$ the manifold $G_j(M)$ is given by the quotient $(S^3)^j \times_{T^j} M$, where T^j acts on M by $t \cdot x = \theta(t_j^{-1}) \cdot x$. The stably complex structure $c(j)$ on $G_j(M)$ is induced from the fibration $M \rightarrow G_j(M) \rightarrow B_j$.*

The proof combines 1.7 in [1] with the fact (Proposition 2.2 in [5]) that the classifying maps for the bundles η_j over B_j represent a basis for $U_*(\mathbb{C}P^\infty)$ that is dual to the basis $\{u^k : k = 0, 1, 2, \dots\}$ in $U^*(\mathbb{C}P^\infty)$.

Remark 2. Our $G_j(M^{2n})$ are clearly complex bordant to the *non-connected* manifolds $\gamma^j(M^{2n})$ of Theorem 6.3 in [6]. Therefore, we deduce that the completion map of the homotopical bordism ring $MU_*^{S^1}$ at its augmentation ideal (see Ch. 27 in [7]) restricts to Φ on geometrical classes. Also, the $G_j(M^{2n})$ are *diffeomorphic* to the manifolds $M^{2(n+j)}$ of [8], so $c(j)$ induces a stably complex structure on the latter.

We interpret M as a *tangentially stably complex S^1 -manifold* (M, θ, c_τ) whenever a complex vector bundle $\xi \rightarrow M$ and an isomorphism

$$c_\tau: \tau(M) \oplus \mathbb{C}^{l-n} \rightarrow \xi \tag{1}$$

are chosen, where $\tau(M)$ is the stable tangent bundle, in such a way that the composition

$$\rho(t) : \xi \xrightarrow{c_\tau^{-1}} \tau(M) \oplus \mathbb{C}^{l-n} \xrightarrow{d\theta(t) \oplus I} \tau(M) \oplus \mathbb{C}^{l-n} \xrightarrow{c_\tau} \xi, \quad t \in S^1, \tag{2}$$

is a complex transformation for any $t \in S^1$, where $d\theta(t)$ is the differential of the action by $\theta(t)$. Hence, (2) corresponds to a representation $\rho : S^1 \rightarrow \text{Hom}_{\mathbb{C}}(\xi, \xi)$. Up to natural equivalence, the structure (M, θ, c_τ) determines (M, θ, c_ν) , and we may define $\Phi(M, \theta, c_\tau)$ to be the $\Phi(M, \theta, c_\nu)$ associated with the corresponding structure c_ν . On the other hand, c_ν may not determine an equivariant c_τ ([7], Ch. 28).

Let $x \in M$ be an isolated fixed point of the action θ . Then the associated representation $\rho_x : S^1 \rightarrow GL(l, \mathbb{C})$ decomposes the fibre $\xi_x \cong \mathbb{C}^l$ as $\mathbb{C}^n \oplus \mathbb{C}^{l-n}$, where ρ_x acts (without trivial summands) as $\rho_{x,1} \oplus \dots \oplus \rho_{x,n}$ on \mathbb{C}^n and trivially on \mathbb{C}^{l-n} . Here $\rho_{x,j}$ is the one-dimensional representation of weight $\omega_{x,j}$, that is, it acts by $\rho_{x,j}(t)v = t^{\omega_{x,j}}v$. To each isolated fixed point x we may therefore assign a sequence $\omega_x = (\omega_{x,1}, \dots, \omega_{x,n})$ of weights. Moreover, the isomorphism $c_{x,\tau}$ in (1) induces an orientation of the tangent space τ_x , which allows us to define $\sigma(x)$ as the sign of the determinant of the linear map

$$\tau_x(M) \xrightarrow{I \oplus 0} \tau_x(M) \oplus \mathbb{C}^{l-n} \xrightarrow{c_{\tau,x}} \xi_x \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{\pi} \mathbb{C}^n,$$

where π is the projection onto the first summand. The sign is important in toric topology [9].

Example 3. The S^1 -manifold $(B_1 = S^2, \theta, c_1^\partial)$, where $\theta(t)[z_1 : z_2] = [tz_1 : z_2]$, has fixed points $x_1 = [1, 0]$ and $x_2 = [0, 1]$ with weights $\omega_{x_1,1} = \omega_{x_2,1} = 1$ and signs $\sigma(x_1) = -\sigma(x_2) = 1$.

Now let $\{[q](u) \in \Omega^*[[u]] : q \in \mathbb{Z}\}$ denote the power system of the formal group $f(u, v)$ for complex cobordism (see [10]). The q -series $[q](u) = qu \pmod{u^2}$ is defined uniquely by $[0](u) = 0$ and $[q](u) = f(u, [q-1](u))$ for all $q \in \mathbb{Z}$.

Theorem 4. *If the action θ has a finite set X of isolated fixed points, then*

$$\Phi(M, \theta, c_\tau) = \sum_{x \in X} \sigma(x) \prod_{i=1}^n \frac{1}{[\omega_{x,i}](u)} \tag{3}$$

holds in $U^{-2n}(\mathbb{C}P^\infty)$.

The proof combines Proposition 3.8 in [1] with the construction of $\sigma(x)$ above.

Remark 5. If the S^1 -manifold M is almost complex, then we may take $l = n$ in (1). Then $\sigma(x) = 1$ for all $x \in X$, and (3) reduces to Krichever’s formula ((2.7) in [2]).

We may reformulate Theorems 1 and 4 for effective actions of the torus T^k on any $2n$ -dimensional manifold with $k \leq n$.

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