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Mathematical Abilities and Mathematical Skills

Alexandre V. Borovik and Tony Gardiner

1. Why do we need this discussion?

The concept of mathematical abilities is not something that is frequently discussed. At the personal level, however, almost every mathematician and mathematical educator has relatively firm views on the subject. In this document, we summarise less disputed aspects of the highly complex phenomenon and some of its immediate implications for educational policy.

The participants of this Conference run mathematics outreach programmes and mathematics competitions in countries with very diverse educational and cultural traditions. It would be futile to try to seek a single universal solution to educational and methodological problems that we encounter in our work. However, we have one thing in common: we work with mathematically able children. It is likely that some of us feel that their work goes against the grain of prevailing trends in educational policy and practice. It is likely that many of us are familiar with such phenomena as the dumbing down of the curriculum and "teaching to test". The task of explaining the importance of proper mathematical education to policymakers and to the wider public becomes ever more challenging. It is even more challenging to explain, to the lay person, the nature of mathematical abilities in children.

Promoting and developing our work, we have to reach a wider circle of teachers and parents. Given a widespread prejudice against mathematics, what can we say to them about mathematics and mathematical abilities?

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2. What can we say to a non-mathematician?

When talking to a non-mathematician, mathematical abilities can be usefully compared to musical abilities: in their developed form they appear highly specific, but are in fact quintessentially human, and so are widely spread in the population at large—in all social and ethnic groups. Like music, mathematics is a personality-building activity, it shapes the way the learner thinks and sees the world. As with music, mathematics has a profound educational impact even where someone no longer uses their mathematical training in later life. Like music, success in mathematics depends on systematic, cumulative learning, and each new skill needs to be built on a solid foundation laid at earlier stages.

Though mathematics is often thought to be a "cold" subject, this is a profound misunderstanding. As with music, mathematics requires a high level of motivation and emotional involvement on the part of the learner. Understanding is of course vital; but it is also essential for the learner to experience real difficulties: boredom and lack of challenge present far greater dangers when seeking to nurture mathematical talent. A degree of ***challenge and frustration*** are crucial.

Everyone has the ability to learn mathematics, although some children learn and make connections more quickly than others. Everyone has some mathematical ability, but some children have potential far beyond what most people are prepared to believe. Mathematical abilities in a child are often dormant and remain unnoticed both by the child and his or her teachers. This potential can be lost forever if it is not discovered and supported at the appropriate time. It may even be undermined by inappropriate experience: again, a comparison can be made with music, where a dissonant musical toy can seriously damage a child's perception of pitch.

Different mathematical traits appear at different ages. To develop a pupil's mathematical abilities effectively, one must tap into these abilities at appropriate times. A comparison can be made with language learning: almost every 7 year old child can master a foreign language with ease, though for an adult it could be an almost impossible task. Similarly, there are periods in a child's development when he or she is may respond to formal procedures and algorithms, and there times when they can be excited by the discovery of a new mathematical activity: generalisation. Matching a talented child's mathematical experience to their cognitive development is a challenge for every teacher since every child will be different!

3. Stratification

Abilities form a continuous spectrum; for the purpose of this discussion, we identify two groups of schoolchildren—the top 20% and top 1% in the school. The percentiles of 20% and 1% are chosen not for any intrinsic reasons but for purely practical purposes. The cohort of 20% is likely to include most of those who will end up working, in their adult life, in professions that require some mathematical background (i.e. beyond mere "numeracy"); these professions include engineering, information technology, the financial sector, etc. The group of 20% is sufficiently large to warrant the allocation of significant effort and resources to ensure that such a group can be realistically supported and nurtured within every school, via a sufficiently enriched curriculum, exciting problems and, of course, teachers (and appropriate professional development). Most importantly, we need some of the top 20% (and the best of them) to return to school as teachers of mathematics.

The top 1% group (in effect, 2–3 pupils per year in an average British school) should perhaps be more of concern to the professional academic community than to their school. Children in this group have the potential to join the next generation of mathematicians and computer scientists; they could become confident users of advanced "hardcore" mathematics in science, engineering, biotechnology and the financial sector. Everyone who works with gifted children in mathematical competitions and other outreach activities knows, that, as soon as such a child is encouraged in his or her interest in mathematics and is given some support and sufficiently challenging problems to solve, he or she rapidly outgrows the level of his or her school. We could not realistically expect that the school can provide sufficient intellectual nourishment for this group of children. But the health of the professional mathematical community and the needs of society demand that the top 1% group is not lost to mathematics and is nurtured via a network of outreach and enrichment activities run by universities and other professional agencies.

Experience in many countries shows that much of what we say concerning the select 1% group applies to the wider group of 20%.

4. Traits of mathematically able children

We by-pass a discussion of the deeper cognitive structures responsible for mathematical abilities, and offer a list of traits that might describe able mathematicians. We emphasise that these are possible indicators: those listed are not always present and the list is by no means exhaustive. Different traits start to manifest themselves at different stages of development; some appear only when a child is exposed to sufficiently rich and deep mathematics. The list has been carefully compiled, but is designed to be suggestive: it is neither complete, nor systematic.

- Ability to make and use generalisations—often quite quickly. One of the basic abilities, easily detectable even at the level of primary school: after solving a single example from a series, a child immediately knows how to solve *all* examples of the same kind.
- Rapid and sound memorisation of mathematical material.
- Ability to concentrate on mathematics for long periods without apparent signs of tiredness.
- Ability to offer and use multiple representations of the same mathematical object. (For example, a child switches easily between representations of the same function by tables, charts, graphs, and analytic expressions.)
- An instinctive tendency to approach a problem in different ways: even if a problem has been already solved, a child is keen to find an alternative solution.
- Ability to utilise analogies and make connections.
- Preparedness to link two (or more) elementary procedures to construct a solution to a multi-step problem.

- Ability to recognise what it means to "know for certain".
- Ability to detect unstated assumptions in a problem, and either to explicate and utilise them, or to reject the problem as ill-defined.
- A distinctive tendency for "economy of thought," striving to find the most economical ways to solve problems, for clarity and simplicity in a solution.
- Instinctive awareness of the presence and importance of an underlying structure.
- Lack of fear of "being lost" and having to struggle to find one's way through the problem.
- A tendency to rapid abbreviation, compression or a curtailment of reasoning in problem solving.
- An easy grasp of encapsulation and de-encapsulation of mathematical objects and procedures.

  These terms are less frequently used, and perhaps a few words of clarification may be useful; we quote Weller et al. [**3**]:

  "The encapsulation and de-encapsulation of processes in order to perform actions is a common experience in mathematical thinking. For example, one might wish to add two functions f and g to obtain a new function $f+g$. Thinking about doing this requires that the two original functions and the resulting function are conceived as objects. The transformation is imagined by de-encapsulating back to the two underlying processes and coordinating them by thinking about all of the elements x of the domain and all of the individual transformations $f(x)$ and $g(x)$ at one time so as to obtain, by adding, the new process, which consists of transforming each x to $f(x) + g(x)$. This new process is then encapsulated to obtain the new function $f + g$."

We use the term "traits" to emphasise that they are not just inborn; in a supportive learning environment, they may develop and flourish to spectacular effect, starting from the child's first shy attempts to try something unusual with a mathematical problem.

We emphasise again that attempts to assess mathematical ability are necessarily one-sided: a test or a child's performance may confirm the presence of significant mathematical ability; however, failure to perform does not necessarily mean that the child has less ability. Moreover, there are specific instances when mathematical abilities can be easily missed (and the list can be easily expanded):

- For example, able mathematicians are frequently quick thinkers. However, some able children appear to be slow—perhaps because it is natural for them to think about a problem in a form more general than the one given.
- Some children have a powerful ability for visualisation that allows them to instantly "see" the solution; after that they may have difficulty expressing their answer in the rigid form required by their teachers or textbooks.
- The ability to abbreviate, or to "compress" the process of thinking may lead to a child being misunderstood by a teacher or examiner because the child does not give all the details of the solution, skipping those which are too obvious and uninteresting for him/her.

A problem from Paper 6671 Edexcel GCE, Pure Mathematics P1 A/AS, 4.11.03:6

6. A container made from thin metal is in the shape of a right circular cylinder with height h cm and base radius r cm. The container has no lid. When full of water, the container holds 500 cm^3 of water.

(a) Show that the exterior surface area, A cm^2, of the container is given by

$$A = \pi r^2 + \frac{1000}{r}.$$

(4)

(b) Find the value of r for which A is a minimum. (4)

(c) Prove that this value of r gives a minimum value of A. (2)

(d) Calculate the minimum value of A, giving your answer to the nearest integer. (2)

***

A problem from a well known and widely available textbook:

Edexcel

12 The diagram shows a prism whose cross-section is a right-angled triangle with sides $3x$, $4x$ and $5x$ cm. The height is h cm.

(a) (i) Show that the volume, V cm^3, is given by $V = 6x^2h$.

(ii) Show that the surface area, S cm^2, is given by $S = 12x^2 + 12xh$.

(b) Given the the volume of the prism is 100 cm^3, show that

$$S = 12x^2 + \frac{200}{x}$$

(c) Find, correct to three significant figures, the value of x for which S is a minimum, showing that this value does give a minimum for S.

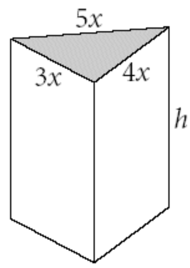

FIGURE 1. A typical British secondary school examination problem: a textbook mimics an exam paper. In Britain, the school examination system is privatised and run by several companies. Mathematics ceased to exist as a single discipline, we have instead textbooks specifically tailored for particular examination boards: Edexcel mathematics, OCR mathematics, etc. Teachers prefer to use in their classes examples that are as close as possible to problems appearing in actual examinations. However, the needs of convenient, easy to administer and uniform assessment impose specific restrictions on the format of the problem. The most important of them is that a problem has to be cut into smaller steps, each bearing certain number of points. When spread over all problems in the course, this requirement kills any possibility to initiate stronger students into the art of heuristic argument in search for a non-obvious multistep solution. There is a considerable concern that the examination bodies, some of which are owned by 'for profit' companies, benefit from publication of their own textbooks, creating a strong possibility for a conflict of interest. It should be noted that in Britain, apparently, no one controls the quality of these texts—for example, they are beyond the scope of responsibilities of Ofsted (Office for Standards in Education).

- Finally, an able child can show no apparent interest in mathematics whatsoever because he or she is bored by a dull curriculum or an uninspiring teacher.

Figure 1 illustrates the last point. Typical problems encountered by schoolchildren at British schools are routine and tend to be broken down into easily digestible bits. This is no way for able youngsters "to cut their mathematical teeth". But what is a good problem?

5. What makes a good problem?

We often fail to recognise mathematical ability because the nature of mathematics is misunderstood by many teachers and is misrepresented by the curriculum. Mathematical traits can only become visible if teaching routinely offers opportunities for pupils to express them. To offer such opportunities we need good problems. But what makes a good problem?

- A good problem has to be ***accessible*** and firmly rooted in the mathematical material familiar to children.
- A good problem is ***developmental***, it activates and trains new modes of thinking and stretches existing skills.
- A good problem is ***revealing***: it opens up new mathematical vistas or sheds new light on existing objects, structures and processes of mathematics.
- It helps if a problem is ***extendable*** and can be set at increasing levels of complexity, perhaps even leading to the development of a miniature "theory" (which can be fully explored and and justified on the level of the mathematics available to pupils).

Good problems frequently incorporate purely aesthetic criteria: simplicity, rhythm, naturalness, elegance and surprise—qualities which are often present in problems that help to develop an appetite for problem solving.

Good problems constitute perhaps one of the most precious resources of mathematics teaching. But they often exhibit features that make them inappropriate for large-scale formal assessment (e.g. pupils may find them hard). Since formal assessment dominates the British system of mathematical education, good developmental problems have been purged from the system. Look at Figure 1: could any of the mathematical traits discussed in this section be detected by run-of-the-mill problems like the one quoted there?

6. Attention span and delayed gratification

Work in mathematics, even at the school level, requires a significant attention span and a willingness to embrace delayed gratification. This is quite different from many other disciplines, where there is often a relatively low threshold to surmount before you can get into something: it might be a long time before you write anything on paper, but there is normally, fairly early on, a sense of the journey having been started. In mathematics you can spend much longer before "stepping into" a problem: solving mathematical problems frequently involves frustrating periods when the solver himself has no idea how to proceed. This is even a characteristic property of many really good mathematical problems: although accessible to the target audience, they betray no hint of where to begin.

When one tries to assess progress in mastering mathematics, one of the more telling quantitative characteristics is the "acceptable solution time" (for lack of a better term), the time learners are prepared to spend on a problem which they cannot immediately see how to tackle. This concept deservers to be at the centre of discourse on mathematics education, even if it challenges the present educational culture of many "western" countries. Most teachers think that their job is to "make mathematics easy", to provide a structured environment in which all problems are broken into small steps. In order to master mathematics, pupils need a regular diet of suitable but challenging problems, and the space and time to comprehend and to solve such problems. Too often teachers are afraid to provide that space; they "feel bad" about "dumping children in it".

To recognise and nurture the potential of able children will require considerable changes in the present culture of mathematics teaching.

7. Problem children?

We also have to touch upon sensitive issues arising from the interaction of an able student with his or her teacher and fellow students. Mathematics is empowering; when a young person starts to feel a grasp of mathematics, he or she discovers the wonderful feeling of knowing things for certain, the proverbial "mathematical certainty".

Locating and correcting errors in mathematical arguments and computations—either your own or others'—is an essential mathematical skill. It is a teacher's important duty to create in the class a safe environment for criticism, an atmosphere where being asked a question is not seen as a humiliation.

For an able student it is relatively easy to see that he or she is "better" at mathematics than others. Unfortunately, it could also work the other way: it is easy for able students to see when someone else is better than they are. The ease of superficial comparison and the element of competitiveness that is inherently present in problem solving can create a feeling of insecurity even in relatively strong students.

This feeling of insecurity can also affect teachers. In Britain, many observers get the impression that teachers systematically suppress able students—not out of vindictiveness, but either because of their own insecurity and fear of having their authority undermined, or in order to engage the rest of a group in a discussion that is not dominated by the able student. What is clear is that teachers need support and guidance on the best ways of interacting with able students.

8. "Acceleration"

The current British policy of encouraging the "acceleration" of able students is an example of educational policy based on a complete misunderstanding of the nature of mathematical ability.

"Acceleration" refers to the strategy of moving stronger students ahead to take tests, or to study material, from subsequent school years "when they are ready". As a policy "acceleration" is cheap, easy to administer, and does not require any additional professional development on the part of teachers—and so requires no additional effort from those responsible for administering the educational system! But in most hands, such "acceleration" offers simply "more of the same". "Acceleration" fails to ensure that earlier techniques become sufficiently robust, or that these techniques are linked together to provide a sufficiently strong foundation for subsequent work. Most important of all, it deprives students of those experiences, which are known to be most valuable in the long run: namely the daily reminder that, while school mathematics may be "elementary" (in the sense that the beginner can insist on understanding, rather than being obliged to trust the teacher's, or the textbook's authority), it is by no means always "easy".

"Acceleration", combined with overexamination and "teaching to test", is soul-destroying. To fully appreciate the ensuing horror, see Figure 1. This kind of teaching is the best way to suppress, or even destroy, mathematical ability in any self-respecting young person.

In contrast the series *Extension Mathematics* [**1**] illustrates how elementary mathematics can be developed—both for the 20% cohort and for the 1%—so as to provide a rich diet of good problems that lay a strong foundation for subsequent development.

9. Role of the University

The undergraduate experience of future mathematics teachers should hold the key to their understanding of the nature of mathematics. Unfortunately, the current approach to undergraduate mathematics teaching is based on a very narrow understanding of the nature of mathematical skills.

Even at university, teaching focuses narrowly on preparation for examinations. Students are not exposed to two important mathematical activities:

(i) finding flaws in other people's arguments, and
(ii) conveying to others their own mathematical thinking.

In Britain, we have a paradoxical situation when even the best mathematics graduates are ***mathematically*** unprepared for a fruitful career in teaching. Students start ***talking*** about mathematics only at the graduate level, and frequently feel the deficiency of their communicative and presentation skills. It is fairly common to find that PhD students organise their own semi-secret graduate seminars (from which more experienced "grown-ups" are excluded).

An effective mathematics teacher is a ***diagnostician*** and ***communicator***: he or she ***sees*** a student's error or difficulty, ***understands*** its underlying causes, and ***talks*** to the student using accessible mathematical language. The analytical and communicative skills required from a teacher should be part of a mathematics graduate's toolkit—but we do not give our students all the tools. Teachers have to give feedback to their students—but to appreciate the value of feedback, they have to experience efficient and supportive feedback from their own teachers. Sadly the level of feedback in undergraduate mathematics teaching is weak (for example, students are often prevented from seeing their marked examination scripts).

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http://www.maths.manchester.ac.uk/∼avb/pdf/WhereFromPreliminaryReport.pdf.

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Criticism, comments, corrections are welcome; please e-mail them at the addresses below.

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