A weak embedding property of probability measures on Lie groups

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The paper "A weak embedding property of probability measures on Lie groups" by S. G. Dani and M. McCrudden, was accepted for publication in final form by Math. Res. Letters in August 2007. Subsequently the paper was withdrawn by the authors when an error was discovered in the proof of Proposition 3.1, invalidating the proof of Corollary 3.2.

The proof of Theorem 1.1 in Section 4 is correct except for the last sentence where Corollary 3.2 is used. As a consequence the proofs of Theorem 1.1, Corollary 1.3 and Corollary 1.4 are all invalidated.

A weak embedding property of probability measures on Lie groups

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1   Introduction

Let G be a Lie group. We denote by $P(G)$ the space of all probability measures on G, equipped with the convolution product and the usual weak* topology. A $\mu \in P(G)$ is said to be *infinitely divisible* if for every $n \in \mathbb{N}$, there exists $\nu \in P(G)$ such that $\nu^n = \mu$, and it is said to be *embeddable* if there exists a continuous one-parameter convolution subsemigroup $\{\mu_t\}$ of $P(G)$ such that $\mu_1 = \mu$; in this case we say that μ is embeddable in $\{\mu_t\}$. Clearly every embeddable measure is infinitely divisible, and it is conjectured that for a connected Lie group G, conversely, every infinitely divisible measure is embeddable; when the latter holds we say that G has the embedding property. In [1] the conjecture was shown to hold for a large class of Lie groups, and an improved and transparent proof of the result was given in [2]; in the latter paper the groups in question are called class $\mathcal{C}$ groups.

In this note, from the results of [2] we deduce the following weak version of embeddability, for measures on any connected Lie group. Together with a result of Y. Guivarch and Riddhi Shah this enables us to deduce embeddability of a class of infinitely divisible measures, not covered by earlier results; see Corollaries 1.3 and 1.4.

We now begin with the details. Let G be a connected Lie group and R be the solvable radical of G. Then $\overline{[R,R]}$ is a connected nilpotent Lie group. Therefore every compact subgroup of $\overline{[R,R]}$ is contained in its center, and there is a unique maximal torus, say T, in $\overline{[R,R]}$; we call T the $\mathcal{C}$-kernel of G. The $\mathcal{C}$-kernel of G is contained in the center of G; firstly, being invariant under all automorphisms of R it is normal in G, and since

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G is connected every compact abelian normal subgroup is contained in the center. A connected Lie group G is of class $\mathcal{C}$ in the sense of [2] (for which the embedding property is established in [2]) if and only if the $\mathcal{C}$-kernel of G is trivial (see [2], Proposition 2.5). On the other hand, if G is a connected Lie group and T is its $\mathcal{C}$-kernel then G/T is of class $\mathcal{C}$; see Proposition 2.2 below. In this note we prove the following "weak embedding property" for all connected Lie groups.

Theorem 1.1. *Let G be a connected Lie group and $\mu \in P(G)$ be infinitely divisible. Let T be the $\mathcal{C}$-kernel of G. Let*

$$M = \{g \in G \mid gxg^{-1}x^{-1} \in T \text{ for all } x \in \operatorname{supp} \mu\}$$

and M^0 be the connected component of the identity in M. Then there exists an embeddable measure σ on G, and a sequence $\{g_i\}$ in M^0 such that μ is the limit of the sequence $\{g_i \sigma g_i^{-1}\}$ of embeddable measures.

A similar result was proved in [11] in the special case of a 4-dimensional solvable group, called the Walnut (see the part preceding the proof of Corollary 1.4 below, for a description of the group); also, the arguments as in [3] yield such a result for a somewhat larger class of groups (though the proof of the main theorem there, asserting the embedding property, turns out to be erroneous).

We next note the following, which is a special case of Theorem 1.3 of [6].

Theorem 1.2. *Let G be a Lie group (not necessarily connected) and $\mu \in P(G)$. Let $T(\mu) = \{g \in G \mid g\mu g^{-1} = \mu\}$ and $T^0(\mu)$ the connected component of the identity in $T(\mu)$. Let $Z^0(G)$ be the connected component of the identity in the center of G, and suppose that $T^0(\mu)/Z^0(G)$ is compact. Suppose also that there exists a sequence $\{\mu_i\}$ in $P(G)$ such that $\mu_i \to \delta_e$, the Dirac measure at the identity, and $\mu_i^{n_i} \to \mu$ for a sequence $\{n_i\}$ of natural numbers. Then μ is embeddable.*

Theorems 1.1 and 1.2 together imply the following.

Corollary 1.3. *Let the notation be as in Theorems 1.1 and 1.2. Suppose that $T^0(\mu)/Z^0(G)$ is compact, and that μ has no idempotent factor other than δ_e. Then μ is embeddable.*

In particular we get the following corollary for the "Walnut group" considered in [11] and [3] (described below).

Corollary 1.4. *The Walnut group has the embedding property.*

2 Preliminaries

Let G be a Lie group and $\mu \in P(G)$. For any $n \in \mathbb{N}$ we denote by $R_n(\mu, G)$ the set of all nth roots of μ on G, namely $R_n(\mu, G) = \{\rho \in P(G) \mid \rho^n = \mu\}$ and by $R(\mu, G)$ the set of all roots of μ on G, namely $R(\mu, G) = \cup_{n \in \mathbb{N}} R_n(\mu, G)$. We recall that μ is said to be *strongly root compact* if $R(\mu, G)$ is relatively compact in $P(G)$. Also, μ is said to be *rationally embeddable* on G if there exists a homomorphism ψ of $\mathbb{Q}^+$ (nonnegative rational numbers) into the semigroup $P(G)$ such that $\psi(1) = \mu$.

We denote by $\operatorname{supp} \mu$ the support of μ, and by $G(\mu)$ the smallest closed subgroup containing $\operatorname{supp} \mu$. For every subgroup H of G we denote by $Z(\mu, H)$ the centraliser of $G(\mu)$ in H, i.e. $Z(\mu, H) = \{h \in H \mid gh = hg \text{ for all } g \in G(\mu)\}$, and by $Z^0(\mu, H)$ the connected component of the identity in $Z(\mu, H)$.

A connected Lie group is said to be of class $\mathcal{C}$ if it admits a representation $\pi : G \to GL(V)$ over a finite-dimensional real vector space V such that the kernel of π is a discrete subgroup of G. In [2] we proved the following; see Theorem 1.2 and Remark 1.4 in [2].

Theorem 2.1. *Let G be a Lie group of class $\mathcal{C}$. Let $\mu \in P(G)$ be infinitely divisible, and let $r : \mathbb{N} \to P(G)$ be a map such that $r(m)^m = \mu$ for all $m \in \mathbb{N}$. Then there exist sequences $\{m_i\}$ in $\mathbb{N}$ and $\{z_i\}$ in $Z^0(\mu, [G, G])$, and $n \in \mathbb{N}$, such that n divides m_i for all i and the sequence $\{z_i r(m_i)^{m_i/n} z_i^{-1}\}$ converges to a n-th root ν of μ which is strongly root compact and rationally embeddable on the subgroup*

$$\{g \in G \mid gx = xg \text{ for all } x \in Z^0(\nu, G)\},$$

the centraliser of $Z^0(\nu, G)$ in G. Furthermore, the sequence $\{m_i\}$ can be chosen such that for all i, $i!n$ divides m_i, and for every $k \in \mathbb{N}$ the sequence $\{z_i r(m_i)^{m_i/k!n} z_i^{-1}\}_{i=k}^{\infty}$ converges.

For the proof of Theorem 1.1 we need also the following.

Proposition 2.2. *Let G be a connected Lie group and T be the $\mathcal{C}$-kernel of G. Then G/T is of class $\mathcal{C}$.*

Proof. Let R be the radical of G. Then T is contained in R, and R/T is the radical of G/T, and we have $\overline{[R/T, R/T]} = \overline{[R, R]}/T$. Let T' be the subgroup of G containing T and such that T'/T is a the unique maximal torus in

$\overline{[R/T, R/T]}$. Then T' is a compact connected nilpotent Lie group, and hence it is a torus. Since T is the (unique) maximal torus of $\overline{[R, R]}$ it follows that $T' = T$ and hence $\overline{[R, R]}/T$ contains no nontrivial torus. Thus the $\mathcal{C}$-kernel of G/T is trivial. This shows that G/T is of class $\mathcal{C}$; see [2], Proposition 2.5). This proves the proposition. □

3 Some topological dynamics

We now note some observations involving topological dynamics.

Consider a compact metric space (X, d) with a continuous action of a locally compact group L. The action is said to be *distal* if for any distinct pair of points $x, x' \in X$, $\{d(gx, gx') \mid g \in L\}$ is bounded away from 0. We note that distality of the action depends only on the topology, and not the specific metric (though it is convenient to express it with respect to a metric); an L-action on (X, d) as above fails to be distal if and only if there exists, $x, x' \in X$, $x \neq x'$, a sequence $\{g_i\}$ in L, and $y \in X$ such that, as $i \to \infty$, $g_i x \to y$ as well as $g_i x' \to y$ (then $\rho(g_i x, g_i x') \to 0$ for any metric ρ equivalent to d).

Now let G be a connected Lie group, T be the $\mathcal{C}$-kernel of G and $p : G \to G/T$ be the quotient homomorphism. Let $\mu \in P(G)$ and $\theta = p(\mu) \in P(G/T)$. Let $X = \{\lambda \in P(G) \mid p(\lambda) = \theta\}$, equipped with a metric d compatible with the topology induced by the weak* topology on $P(G)$. We note that (X, d) is a compact metric space. Let $L = p^{-1}(Z(\theta, G/T))$ and consider the action of L on X, where $g \in L$ acts on X by $\lambda \mapsto g\lambda g^{-1}$ for all $\lambda \in X$. We note the following:

Proposition 3.1. *The L-action on X is distal.*

Proof. Let $\lambda, \lambda' \in X \subset P(G)$, and suppose that there exists a sequence $\{g_i\}$ in L such that $d(g_i \lambda g_i^{-1}, g_i \lambda' g_i^{-1}) \to 0$. Then for any bounded continuous function f on G, $\int_G f \, d(g_i \lambda g_i^{-1}) - \int_G f \, d(g_i \lambda' g_i^{-1}) \to 0$, as $i \to \infty$.

The measures λ and λ' can be expressed as $\int_{G/T} \lambda_x d\theta(x)$ and $\int_{G/T} \lambda'_x d\theta(x)$ respectively, where $\{\lambda_x\}_{x \in G/T}$ and $\{\lambda'_x\}_{x \in G/T}$ are systems of conditional measures for λ and λ' respectively, $\lambda_x, \lambda'_x \in P(G)$, $x \in G/T$, being measures supported on $p^{-1}(x)$; we note that such a decomposition is uniquely defined θ-a.e..

Now let $x_0 \in G/T$ be arbitrary. We can find a neighbourhood Ω of $p^{-1}(x_0)$

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and a submanifold Σ (of dimension same as that of G/T) transversal to T, such that the multiplication map $m : \Sigma \times T \to G$ is a homeomorphism onto Ω. We identify the open subset Ω/T of G/T with Σ, via the map $x \mapsto xT$ for all $x \in \Sigma$. We consider bounded continuous functions f on G which vanish outside Ω and on Ω have the form $\varphi(x)\psi(t)$, where φ is a continuous function with compact support on Σ and ψ is a continuous function on T. We note that the fiber $p^{-1}(x)$ over $x \in G/T$ is $\{xt \mid t \in T\}$. We write the conditional measures λ_x and λ'_x as $x\sigma_x$ and $x\sigma'_x$, where σ_x, σ'_x are probability measures on T.

Since $L = p^{-1}(Z(\theta, G/T))$ it follows that the action of each g_i leaves $p^{-1}(x)$ invariant for θ-almost all x and hence

$$\int_G f \, d(g_i \lambda g_i^{-1}) = \int_\Omega f \, d(g_i \lambda g_i^{-1}) = \int_\Sigma \int_{p^{-1}(x)} f(y) \, d(g_i \lambda_x g_i^{-1})(y) d\theta(x).$$

For each i let $\alpha_i : G/T \to T$ be the map defined by $\alpha_i(xT) = x^{-1} g_i x g_i^{-1}$ for all $x \in G$; the map is well-defined since T is contained in the center of G. Then for all $x \in \Sigma$ and $i = 1, 2, \dots$ we have $g_i \lambda_x g_i^{-1} = (g_i x g_i^{-1})(g_i \sigma_x g_i^{-1}) = x\alpha_i(x)\sigma_x$. It follows that for all $x \in \Sigma$ and $i = 1, 2, \dots$

$$\int_{p^{-1}(x)} f(y) \, d(g_i \lambda g_i^{-1})(y) = \int_{xT} f(y) \, d(x\alpha_i(x)\sigma_x)(y) = \int_T \varphi(x)\psi(t) \, d(\alpha_i(x)\sigma_x)(t).$$

Substituting from the displayed equations and their analogues for λ', the original condition as above yields

$$\int_\Sigma \varphi(x)[\int_T \psi(t) \, d(\alpha_i(x)\sigma_x)(t) - \psi(t) \, d(\alpha_i(x)\sigma'_x)(t)]d\theta(x) \to 0,$$

as $i \to \infty$. As this holds for every continuous function φ with compact support on Σ it follows that the sequence of functions appearing in the square brackets must converge to 0, θ a.e., on $\Sigma \approx \Omega/T$. Again, since the conclusion holds for each bounded continuous function ψ on T, we get that for almost all x in Ω/T (with respect to θ), $\rho(\alpha_i(x)\sigma_x, \alpha_i(x)\sigma'_x) \to 0$ as $i \to \infty$, where ρ is a metric on $P(T)$. Since ρ can be chosen to be translation invariant this implies that $\sigma_x = \sigma'_x$ for almost all $x \in \Omega/T$. Hence $\lambda_x = x\sigma_x = x\sigma'_x = \lambda'_x$, for θ-almost all x in Ω/T. Thus we have shown that every $x_0 \in G/T$ has a neighbourhood in G/T such that for θ-almost all x in the neighbourhood $\lambda_x = \lambda'_x$. Hence $\lambda_x = \lambda'_x$ a.e. on G/T, with respect to θ, and therefore $\lambda = \lambda'$. This shows that the L-action on X is distal. $\qquad\square$

We need the following corollary in the proof of Theorem 1.1.

Corollary 3.2. *Let M be a closed subgroup of L. Let $\nu \in P(G)$ and suppose there exists a sequence $\{g_i\}$ in M such that $g_i \mu g_i^{-1} \to \nu$ as $i \to \infty$. Then there exists a sequence $\{h_i\}$ in M such that $h_i \nu h_i^{-1} \to \mu$, as $i \to \infty$.*

Proof. Since, by Proposition 3.1, the action of L on X is distal, the action of M on X is also distal. Under a distal action the space decomposes as a disjoint union of minimal (nonempty) closed invariant sets; cf.[5], Theorem 3.2. In particular, the closure, say C, of the M-orbit of μ is a minimal closed M-invariant subset, containing ν as in the hypothesis. By minimality of C, the closure of the M-orbit of ν must coincide with C, and in particular contains μ. This proves the corollary. $\square$

4 Proof of Theorem 1.1

Let G and T be as in the hypothesis of the theorem. By Proposition 2.2 G/T is of class $\mathcal{C}$. Let $G' = G/T$ and let $p : G \to G'$ be the quotient homomorphism. Let $\mu \in P(G)$ be infinitely divisible.

For each m choose $\lambda_m \in R_m(\mu, G)$ and let $r : \mathbb{N} \to P(G')$ be a map defined by $r(m) = p(\lambda_m)$. Then clearly $r(m)^m = p(\mu)$ for all m. Therefore by Theorem 2.1 there exist sequences $\{m_i\}$ in $\mathbb{N}$, and $\{z_i\}$ in $Z^0(p(\mu), [G', G'])$, and $n \in \mathbb{N}$ such that for all i, $i!n$ divides m_i, for every $k \in \mathbb{N}$ the sequence $\{z_i r(m_i)^{m_i/k!n} z_i^{-1}\}_{i=k}^{\infty}$ converges, and the limit ν' of $\{z_i r(m_i)^{m_i/n} z_i^{-1}\}$ is strongly root compact and rationally embeddable on the centraliser of $Z^0(\nu', G')$ in G'. Let $H' = Z^0(\nu', G')$ and $H = p^{-1}(H')$.

Let M be the subgroup as in the statement of the theorem. We note that $p(M) = Z(p(\mu), G')$. Considering the connected components we see that $p(M^0) = Z^0(p(\mu), G')$; that is, M^0 maps onto $Z^0(p(\mu), G')$ under p. Since $z_i \in Z^0(p(\mu), [G', G'])$, for all i, we can pick $g_i \in M^0$ such that $p(g_i) = z_i$, for all i. For each i we have $\sigma_i \in R_{m_i}(\mu, G)$ such that $p(\sigma_i) = r(m_i)$. For each $k \in \mathbb{N}$ consider the sequence $\{g_i \sigma_i^{m_i/k!n} g_i^{-1}\}$ in $P(G)$. Its image under p is the sequence $\{z_i r(m_i)^{m_i/k!n} z_i^{-1}\}$, which is convergent. Since the kernel of p is compact it follows that $\{g_i \sigma_i^{m_i/k!n} g_i^{-1}\}$ is relatively compact for all k. Passing to subsequences successively we see that there exists a sequence $\{i_j\}$ of integers such that for each k, $\{g_{i_j} \sigma_{i_j}^{m_{i_j}/k!n} g_{i_j}^{-1}\}$ converges, as $j \to \infty$; let ψ_k denote the limit. Then $\psi_k \in P(H)$ for all k, and $\psi_k^k = \psi_{k-1}$ for all $k \geq 2$.

This shows that ψ_1 is infinitely divisible on H. Also, $p(\psi_1) = \nu'$, and since ν' is strongly root compact on $H' = p(H)$ we get that ψ_1 is strongly root compact on H. Therefore ψ_1 is embeddable on H (see [7], Chapter 3). Also, as $\{g_{i_j}\sigma_{i_j}^{m_{i_j}/n} g_{i_j}^{-1}\}$ converges to ψ_1 and $\sigma_{i_j}^{m_{i_j}} = \mu$ we have $g_{i_j}\mu g_{i_j}^{-1} \to \psi_1^n$ as $j \to \infty$. Hence by Corollary 3.2 there exists a sequence $\{h_i\}$ in M^0 such that $h_i\psi_1^n h_i^{-1} \to \mu$ as $j \to \infty$. As ψ_1^n is embeddable, this proves the theorem. $\square$

For possible future reference we note that the above proof shows in fact that $\{g_i\}$ as in Theorem 1.1 may be found in the Lie subgroup $\{g \in [G,G] \mid gxg^{-1}x^{-1} \in T$ for all $x \in \text{supp}\,\mu\}$, or even in the connected component of the identity in that subgroup. We caution the reader that in general $[G,G]$, and the subgroups as above may not be closed subgroups.

Proof of Corollary 1.3: By Theorem 1.1 there exist a one-parameter convolution semigroup $\{\sigma_t\}$ in $P(G)$ and a sequence $\{g_i\}$ in M^0 such that $g_i\sigma_1 g_i^{-1} \to \mu$. Then σ_0 is an idempotent and it is the normalised Haar measure of a compact subgroup, say C. Since $\sigma_0\sigma_1 = \sigma_1$ and $p(\sigma_1) = p(\mu)$ it follows that $p(C)$ is contained in $p(G(\mu))$. Therefore every $z \in Z(p(\mu), G')$ centralises $p(C)$. Let $K = CT$ and $\varphi : Z(p(\mu), G') \to \text{Hom}\,(K,T)$ be the map defined by $\varphi(z)(g) = xgx^{-1}g^{-1}$, for all $g \in K$ and $z \in Z(p(\mu), G')$, with $x \in G$ such that $p(x) = z$; since $\ker p = T$ is contained in the center of G the map is well-defined, independently of the choice of the representative x. It can be verified that φ is a homomorphism of the topological groups. Since $\text{Hom}\,(K,T)$ is a discrete group, it follows that for all $z \in Z^0(p(\mu), G')$, $\varphi(z)$ is the trivial homomorphism. We note that $p(M^0) \subset Z^0(p(\mu), G')$, and hence the preceding conclusion implies that M^0 centralises K. In particular every g_i centralises C. It follows that ω_C, the normalised Haar measure of C, is a factor of $g_i\sigma_1 g_i^{-1}$ for all i, and hence also of μ. Since by hypothesis μ has no idempotent factor other than δ_e, we get that C must be trivial. Hence $\sigma_t \to \delta_e$ as $t \to 0$, and in turn, for each i, $g_i\sigma_t g_i^{-1} \to \delta_e$. Therefore we can find a sequence $\{n_i\}$ of natural numbers, tending to infinity, such that $g_i\sigma_{1/n_i}g_i^{-1}$ converges to δ_e. Now for all i let $\mu_i = g_i\sigma_{1/n_i}g_i^{-1}$. Then $\mu_i \to \delta_e$, and $\mu_i^{n_i} = g_i\sigma_1 g_i^{-1} \to \mu$, as $i \to \infty$. Thus the conditions of Theorem 1.2 hold for the measure μ and so it is embeddable. This proves the corollary. $\square$

The condition in the hypothesis of Corollary 1.3 that $T^0(\mu)/Z^0(G)$ be compact is rather strong, in the context of the general embedding problem for connected Lie groups. Nevertheless, Corollary 1.3 does yield embedding

property for the "Walnut" group, which had earlier seemed a test case among Lie groups for which the embedding property was not known; though embedding property was asserted for the group in [3] the argument turned out to have a flaw, as noted in the erratum to the paper. The argument in [3] did however establish the weak embedding property, namely the assertion as in Theorem 1.1, for the class of groups considered there, including the Walnut group. We now deduce the embedding property for the Walnut group as an illustration of application of the ideas as above. Further study along the lines, especially through use of the underlying general ideas involved in the proof of Theorem 1.2 rather than the specific theorem itself, is expected to yield stronger results, and is being taken up in collaboration with Riddhi Shah.

We shall begin with a description of the Walnut group. The reader is referred to [9] for more details. Let N be the 3-dimensional Lie group consisting of $\{(v, \theta) \mid v \in \mathbb{R}^2, \theta \in S^1\}$, where S^1 is the unit circle consisting of complex numbers of modulus 1, with the multiplication defined by $(v_1, \theta_1) \cdot (v_2, \theta_2) = (v_1 + v_2, \theta_1 \theta_2 e^{i\alpha(v_1, v_2)})$, where $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an nonzero alternating bilinear form; such a form is unique up to scalar multiples, and the corresponding Lie groups are isomorphic. Then N is a nilpotent Lie group, with $Z = \{(0, \theta) \mid \theta \in S^1\}$ as its center. Let K be the group of rotations of $\mathbb{R}^2$. Then for $\kappa \in K$, $(v, \theta) \to (\kappa(v), \theta)$ for all $v \in \mathbb{R}^2$ and $\theta \in S^1$, defines a Lie automorphism of N, and thus K can be realised as a group of Lie automorphisms of N. The semidirect product of K and N with respect to the action of K on N as above is called the Walnut group. It does not belong to class $\mathcal{C}$ and is also not nilpotent; it is a Lie group of minimum possible dimension satisfying the twin conditions. Thus the earlier known results on the embedding property do not apply to this case. We now show that nevertheless the embedding property does hold for the Walnut group, in the light of Corollary 1.3 and earlier known results for the case (see below).

Proof of Corollary 1.4: Let G be the Walnut group, with N and K as above. The $T = \{(0, \theta) \mid \theta \in S^1\}$ is the $\mathcal{C}$-kernel of G. Let $p : G \to G/T$ be the quotient homomorphism. By an argument as in the proof of Corollary 1.2 in [3] (originally from [11], Proposition 82) to prove the embedding property for G it suffices to prove embeddability of every infinitely divisible probability measure μ such that $G(\mu) \subset N$. If μ is infinitely divisible on N then it is embeddable, since N is nilpotent. We may therefore assume that μ is such with supp μ contained in N but not infinitely divisible on N. We note

that in this case $p(\mu)$, which is a measure on $N/T \approx \mathbb{R}^2$ is invariant under all rotations, namely under the action of K; see [4], Lemma 2.2. Hence $G(p(\mu)) = N/T$ unless $p(\mu)$ is the point mass at $\{0\}$, in which case $G(\mu)$ is contained in T. In case $G(\mu)$ is contained in T, using the fact that every root is of the form $x\lambda$ with $\lambda \in P(T)$ and T is central we conclude that μ is infinitely divisible on T, and hence embeddable. Therefore we may assume that $p(\mu)$ is not the point mass at $\{0\}$. Hence $G(p(\mu)) = N/T$, and this in turn implies that $G(\mu) = N$.

If ω_C is a factor of μ for a compact subgroup C, then clearly C must be contained in T. As T is contained in the center of G it follows that there exists a unique maximal compact subgroup C of T such that ω_C is a factor of μ. Let $q : G \to G/C$ be the quotient homomorphism. Since ω_C is a factor of μ, to prove that μ is embeddable it suffices to prove that $q(\mu)$ is embeddable; see [7], Theorem 1.2.15. We note that if $C = T$ then G/C is the group of motions of the plane, which is a group of class $\mathcal{C}$ and has the embedding property (see [1]; in this special case the result also from an earlier paper of the authors, cited in [1]), it follows that the infinitely divisible measure $q(\mu)$ is indeed embeddable. We may therefore assume C to be a proper, and hence finite, subgroup of T. Then G/C is isomorphic, as a Lie group, to the Walnut group, and $q(\mu)$ is an infinitely divisible measure on G/C with no nontrivial idempotent factor. Therefore to prove the corollary, by modifying the notation, we may further assume that μ as above has no nontrivial idempotent factor.

By Corollary 1.3 it now suffices to prove that $T^0(\mu)$ is compact. Clearly $T^0(\mu)$ is a closed subgroup containing T. Suppose it is noncompact. Then $p(T^0(\mu))$ is a closed noncompact subgroup of G/T. The latter is the semidirect product of K and $\mathbb{R}^2$, with K acting as rotations. Therefore $p(T^0(\mu))$ contains a one-parameter subgroup of $\mathbb{R}^2$. Hence $T^0(\mu)$ contains a one-parameter subgroup of N, that is not contained in T, say $\{x_t\}_{t \in \mathbb{R}}$. This means that μ is invariant under the maps $g \mapsto x_t g x_t^{-1} = (x_t g x_t^{-1} g^{-1})g$, $g \in G(\mu) = N$. These maps leave invariant each fibre $p^{-1}(y)$, $y \in N/T$, and in the fibre containing $g \in N$ the action of x_t is by translation by the element $x_t g x_t^{-1} g^{-1}$. We note that for g other than those commuting with each x_t the action of the one-parameter subgroup is transitive on the fibre. We note also that g commutes with each x_t if and only if $p(g) \in L = \{p(x_t) \mid t \in \mathbb{R}\}$, which is a line in $\mathbb{R}^2 = N/T$. Let $\{\mu_y\}_{y \in N/T}$ be a system of conditional measures for μ with respect to the fibration $p : N \to N/T$. The preceding observations

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therefore imply that μ_y is T-invariant for $p(\mu)$-almost all $y \in \mathbb{R}^2 \backslash L$. Since L is a line and by our assumption $p(\mu)$ is invariant under all rotations, and is not the point measure at $\{0\}$, it follows that $p(\mu)(L) = 0$. Thus we get that μ_y is T-invariant for $p(\mu)$-almost all y. Therefore μ is T-invariant; that is, the normalised Haar measure ω_T is a factor of μ. This contradicts the assumption as above, that μ has no nontrivial idempotent factor. It follows therefore that $T^0(\mu)$ is compact, and this proves the corollary. $\square$

References

[1] S.G. Dani and M. McCrudden, Embeddability of infinitely divisible distributions on linear Lie groups, Invent. Math. 110 (1992), 237-261.

[2] S.G. Dani and M. McCrudden, Convolution roots and embeddings of probability measures on Lie groups, Adv. Math. 209 (2007), 198-211.

[3] S.G. Dani, M. McCrudden and S. Walker, On the embedding problem for infinitely divisible distributions on certain Lie groups with toral center, Math. Zeits. 245 (2003), 781-790; erratum: ibid. 252 (2006), 457–458.

[4] S.G. Dani and Klaus Schmidt, Affinely infinitely divisible distributions and the embedding problem, Math. Res. Lett. 9 (2002), 607-620.

[5] H. Furstenberg, The structure of distal flows, Amer. J. Math. 85 (1963), 477 - 515.

[6] Y. Guivarc'h and Riddhi Shah, Asymptotic properties of convolution operators and limits of triangular arrays on locally compact groups, Trans. Amer. Math. Soc. 357 (2005), 3683–3723.

[7] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer Verlag, 1977.

[8] G. Hochschild, *The Structure of Lie Groups*, Holden-Dey Inc, San Francisco, 1965.

[9] M. McCrudden, The embedding problem for probabilities on locally compact groups, *Probability Measures on Groups: Recent Directions and Trends*, pp. 331–363, Tata Inst. Fund. Res., Mumbai, 2006; international distribution by the American Mathematical Society.

[10] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, 1967.

[11] Seth Walker, Probabilities on Semigroups, Ph. D. Thesis, University of Manchester, 2001.

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