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ON THE PROBLEM OF STOCHASTIC INTEGRAL REPRESENTATIONS OF FUNCTIONALS OF THE BROWNIAN MOTION. II

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(Translated by A. A. Sergeev)

Abstract. In the first part of this paper [A. N. Shiryaev and M. Yor, Theory Probab. Appl. 48 (2004), pp. 304–313], a method of obtaining stochastic integral representations of functionals \( S(\omega) \) of Brownian motion \( B = (B_t)_{t \geq 0} \) was stated. Functionals \( \max_{t \leq T} B_t \) and \( \max_{t \leq T-a} B_t \), where \( T-a = \inf\{t : B_t = -a\} \), \( a > 0 \), were considered as an illustration. In the present paper we state another derivation of representations for these functionals and two proofs of representation for functional \( \max_{t \leq g_T} B_t \), where (non-Markov time) \( g_T = \sup\{0 \leq t \leq T : B_t = 0\} \) are given.

Key words. Brownian motion, Itô integral, max-functionals, stochastic integral representation

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2'. The second derivation of the representation for \( S_T = \max_{t \leq T} B_t \).

2.1. According to relation (4) of the first part of the paper,

\[
S_T = ES_T + 2 \int_0^T \left[ 1 - \Phi\left( \frac{S_t - B_t}{\sqrt{T - t}} \right) \right] dB_t
\]

or, equivalently,

\[
S_T = ES_T + \int_0^T \Psi\left( \frac{S_t - B_t}{\sqrt{T - t}} \right) dB_t,
\]

where \( S_t = \max_{u \leq t} B_u \), \( ES_t = \sqrt{2T/\pi} \), and \( \Psi(x) = 2[1 - \Phi(x)] \) (\( = 2P\{\mathcal{N}(0,1) > x\} \), \( \mathcal{N}(0,1) \) having the standard Gaussian distribution).

2.2. Let us demonstrate that for all \( t \geq 0 \) the following relation holds:

\[
E(S_T | \mathcal{F}_t) = \sqrt{\frac{2}{\pi}} T + \int_0^{T \wedge t} \Psi\left( \frac{S_u - B_u}{\sqrt{T - u}} \right) dB_u,
\]

which implies, obviously, formula (45) too. (Recall that \( \mathcal{F}_t = \sigma(B_s, s \leq t) \) is the \( \sigma \)-algebra, generated by the Brownian motion and completed with sets of \( P \)-probability zero from \( \sigma \)-algebra \( \mathcal{F} \) of the original complete probability space \( (\Omega, \mathcal{F}, P) \).)
Fix \( 0 \leq t < T \). Then, according to (5),

\[
E[S_T | F_t] = E\left( \int_0^\infty E[I(a < S_t) | F_t] \, da \right) = \int_0^\infty E[I(T_a < T) | F_t] \, da
\]

\[
= \int_0^\infty \left( I(T_a \leq t) + E[I(t < T_a < T) | F_t] \right) \, da
\]

\[
= \int_0^\infty I(T_a \leq t) \, da + \int_0^\infty P(t < T_a < T | F_t) \, da
\]

\[(47)\]

\[
= S_t + \int_0^\infty P(t < T_a < T | F_t) \, da.
\]

On the set \( \{ t < T_a \} \), according to the Markov property and relations (14), (15), we have

\[
P(t < T_a < T | F_t) = P(\exists s \in (t, T): B_s > a | F_t)
\]

\[
= P_B(\exists s \in (0, T-t): B_s > a | \{ T_a < T-t \})
\]

\[(48)\]

\[
= \int_0^{T-t} \frac{a - B_t}{\sqrt{2\pi s^3}} \exp\left\{ -\frac{(a-B_t)^2}{2s} \right\} \, ds,
\]

where \( P_x(\cdot) \) is the distribution of the Brownian motion starting at point \( x \).

From (47) and (48) we find that

\[
E[S_T | F_t] = S_t + \int_0^\infty \int_0^{T-t} \frac{a - B_t}{\sqrt{2\pi s^3}} \exp\left\{ -\frac{(a-B_t)^2}{2s} \right\} \, ds \, da
\]

\[
= S_t + \int_0^{T-t} \frac{1}{\sqrt{2\pi s}} \left( \int_s^\infty \frac{a - B_t}{\sqrt{2\pi s^2}} \exp\left\{ -\frac{(a-B_t)^2}{2s} \right\} \, da \right) \, ds
\]

\[(49)\]

\[
= S_t + \int_0^{T-t} \frac{1}{\sqrt{2\pi s}} \exp\left\{ -\frac{(S_t - B_t)^2}{2s} \right\} \, ds = S_t + H(S_t - B_t, t),
\]

where

\[
H(x, t) = \int_0^{T-t} \frac{1}{\sqrt{2\pi s}} e^{-x^2/(2s)} \, ds, \quad x \in \mathbb{R}, \quad 0 \leq t < T.
\]

It is obvious that

\[(50)\]

\[
H(0, 0) = \int_0^T \frac{1}{\sqrt{2\pi s}} \, ds = \sqrt{\frac{2T}{\pi}}
\]

and for \( x > 0 \) and \( 0 < t < T \)

\[(51)\]

\[
\frac{\partial}{\partial x} H(x, t) = -\int_0^{T-t} \frac{x}{\sqrt{2\pi s}} e^{-x^2/(2s)} \, ds = -P\{T_x < T - t\} = -\Psi\left( \frac{x}{\sqrt{T-t}} \right).
\]

Denoting \( X_t = S_t - B_t \) and applying Itô’s formula to \( H(X_t, t) \), we find that (in differential form)

\[
dH(X_t, t) = \frac{\partial}{\partial t} H(X_t, t) \, dt + \frac{\partial}{\partial x} H(X_t, t) \, dX_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} H(X_t, t) \, d[X]_t,
\]
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where \((X_t)_{t \leq T}\) is the square variation of the process \((X_t)_{t \leq T}\). It is obvious that in the case in question \(X_t = B_t = t\). So (in the integral form)

\[
H(S_t - B_t, t) = H(0, 0) + \left[ \int_0^t \frac{\partial}{\partial u} H(X_u, u) \, du + \int_0^t \frac{\partial^2}{\partial x^2} H(X_u, u) \, du \right] - \int_0^t \frac{\partial}{\partial x} H(X_u, u) \, dB_u.
\]

Denoting the expression in the square brackets \(A_t\), we find from (49) that

\[
\mathbb{E}(S_T \mid F_t) = H(0, 0) + (A_t + B_t) - \int_0^t \frac{\partial}{\partial x} H(X_u, u) \, dB_u.
\]

Since the processes \((\mathbb{E}(S_T \mid F_t))_{t \leq T}\) and \((\int_0^t \frac{\partial}{\partial x} H(X_u, u) \, dB_u)_{t \leq T}\) are continuous martingales, the process of bounded variation \((A_t + B_t)_{t \leq T}\) with \(A_0 + B_0 = 0\) is also a martingale. Therefore, this process is identically (to stochastic indistinguishability) equal to zero, and, hence,

\[
\mathbb{E}(S_T \mid F_t) = H(0, 0) - \int_0^t \frac{\partial}{\partial x} H(X_u, u) \, dB_u,
\]

which leads, taking formulas (50) and (51) into account, to the required representation (46).

3’. The second derivation of the representation for \(S_{T-a} = \max_{t \leq T-a} B_t\).

3.1. According to relation (44) from the first part of the paper,

\[
S_{T-a} = \int_0^{T-a} \log \frac{a}{a + S_u} \, dB_u.
\]

Let us demonstrate that for every \(M > 0\) and \(t \geq 0\) the following representation holds:

\[
\mathbb{E}[S_{T-a} \wedge M \mid F_t] = a \log \frac{a + M}{a} + \int_0^{t \wedge T-a} \log \frac{a + M}{a + M \wedge S_u} \, dB_u,
\]

whence (52) is derived by transition to the limit, as shown in what follows.

For fixed \(M > 0\)

\[
\mathbb{E}[S_{T-a} \wedge M \mid F_t] = \mathbb{E} \left[ \int_0^M I(\alpha < S_{T-a} \wedge M) \, d\alpha \bigg| F_t \right] = \int_0^M \mathbb{E}[I(\alpha < S_{T-a} \wedge M) \mid F_t] \, d\alpha = \int_0^M \mathbb{E}[I(T_\alpha < T-a) \mid F_t] \, d\alpha + \int_0^M \left( I(T_\alpha < T-a) + \mathbb{E}[I(t < T_\alpha < T-a) \mid F_t] \right) \, d\alpha.
\]
Due to the Markov property of the Brownian motion we find that on the set \( \{ t < T_{-a} \} \)

\[
\begin{align*}
\mathbb{E}[S_{T_{-a}} \wedge M \mid \mathcal{F}_t] &= \int_0^M I(T_{-a} \leq t) \, d\alpha + \int_0^M \mathbb{E}[I(t < T_{-a}) \mid \mathcal{F}_t] \, d\alpha \\
&= \int_0^M I(\alpha < S_t) \, d\alpha \\
&\quad + \int_0^M \mathbb{E}[I(t < T_{-a}) I(0 < T_{-a} \circ \theta_t < T_{-a} \circ \theta_t) \mid \mathcal{F}_t] \, d\alpha \\
&= S_t \wedge M + \int_0^M \mathbb{P}_t \{ T_{-a} < t \} I(t < T_{-a}) \, d\alpha,
\end{align*}
\]

(55)

where \( \theta_t \) is the shift operator. Yet, on the set \( \{ T_{-a} \leq t \} \)

\[
\begin{align*}
\mathbb{E}[S_{T_{-a}} \wedge M \mid \mathcal{F}_t] &= \int_0^M I(T_{-a} < T_{-a} < t) \, d\alpha = S_{T_{-a}} \wedge M.
\end{align*}
\]

(56)

Using the well-known relation

\[
\mathbb{P}_x \{ T_{-a} < t \} = \frac{x + a}{\alpha + a}, \quad -a < x < \alpha,
\]

we find from (55) that on the set \( \{ t < T_{-a} \} \)

\[
\begin{align*}
\mathbb{E}[S_{T_{-a}} \wedge M \mid \mathcal{F}_t] &= S_t \wedge M + \int_0^M \frac{B_t + a}{\alpha + a} I(t < T_{-a}) \, d\alpha \\
&= S_t \wedge M + B_t \int_0^M \frac{1}{\alpha + a} I(S_t < \alpha) \, d\alpha + \int_0^M \frac{a}{\alpha + a} I(S_t < \alpha) \, d\alpha \\
&= S_t \wedge M + (B_t + a) \log(M + a) - B_t \log(S_t \wedge M + a) \\
&\quad - a \log(S_t \wedge M + a) = A_t + \int_0^t \log \frac{M + a}{S_u \wedge M + a} \, dB_u,
\end{align*}
\]

(57)

where \( (A_t)_{t \geq 0} \) is the continuous process of bounded variation specified by the relation

\[
A_t = a \log(M + a) + S_t \wedge M - a \log(S_t \wedge M + a) - \int_0^t \frac{B_u}{S_u \wedge M + a} I(S_u \leq M) \, dS_u.
\]

(58)

(The last equality in (57) was obtained with the use of Itô’s formula applied to \( B_t \log(S_t \wedge M + a) \).)

As in the end of section 2’, we make use of the fact that a continuous martingale, which is at the same time a process of bounded variation, is constant. Then we find from (57) that the processes

\[
\left( \mathbb{E}[S_{T_{-a}} \wedge M \mid \mathcal{F}_{t \wedge T_{-a}}] \right)_{t \geq 0} \quad \text{and} \quad \left( A_0 + \int_0^{t \wedge T_{-a}} \log \frac{M + a}{S_u \wedge M + a} \, dB_u \right)_{t \geq 0}
\]

are indistinguishable.
From (58) we obtain equality $A_0 = a \log((M + a)/a)$, and, therefore, for every $t \geq 0$

$$E(S_{T-a} \land M | \mathcal{F}_t) = a \log \frac{M + a}{a} + \int_0^{t \land T-a} \log \frac{M + a}{S_u \land M + a} dB_u \quad (\mathbb{P}\text{-a.s.})$$

which is just the required relation (53).

Note that according to this formula

$$E(S_{T-a} \land M) = a \log \frac{M + a}{a}.$$  

This latter formula one can also find directly:

$$E(S_{T-a} \land M) = \int_0^M P\{S_{T-a} \land M > \alpha\} \, d\alpha = \int_0^M P\{T_a < T-a\} \, d\alpha = \int_0^M \frac{a}{\alpha + a} \, d\alpha = a \log \frac{M + a}{a}.$$ 

If one assumes $t = T-a$ in (53), then one will find that for every $M > 0$

$$S_{T-a} \land M = a \log \frac{M + a}{a} + \int_0^{T-a} \left[ \log(M + a) - \log(S_u \land M + a) \right] dB_u$$

$$= a \log \frac{M + a}{a} - a \log(M + a) - \int_0^{T-a} \log(S_u \land M + a) dB_u$$

$$= \int_0^{T-a} \left[ \log a - \log(S_u \land M + a) \right] dB_u = \int_0^{T-a} \log \frac{a}{S_u \land M + a} dB_u.$$ 

Assuming here $M \to \infty$ and using the continuity of the integral with respect to $M$, we get

$$S_{T-a} = \int_0^{T-a} \log \frac{a}{a + S_u} dB_u,$$

which is just the required relation (52).

3.2. Let $T_{b,a} = T_b \land T_{-a}, a, b > 0$. In other words, let $T_{b,a} = \inf\{t > 0: B_t \notin (-a, b)\}$. The reasoning, similar to the adduced one, allows us to validate the following representations:

(59) 

$$S_{T_{b,a}} = a \log \frac{a + b}{a} + \int_0^{T_{b,a}} \log \frac{b + a}{S_u + a} dB_u$$

and

(60) 

$$E(S_{T_{b,a}} | \mathcal{F}_t) = a \log \frac{a + b}{a} + \int_0^{t \land T_{b,a}} \log \frac{b + a}{S_u + a} dB_u.$$ 

Indeed, since $P\{S_{T_{b,a}} \leq b\} = 1$ and

$$P\{S_{T_{b,a}} > u\} = P\{T_u < T_{-a}\} = \frac{a}{u + a}, \quad 0 < u \leq b,$$
one can see that
\[ \mathbf{E}S_{T^b_a} = a \log \frac{b + a}{a}. \]

Fix \( t > 0 \). As in section 1 (see (54)), we have
\[ \mathbf{E}[S_{T^b_{-a}} \mid \mathcal{F}_t] = \int_0^\infty \mathbf{E}[I(u < S_{T^b_{-a}}) \mid \mathcal{F}_t] \, du = \int_0^\infty \mathbf{E}[I(T_u < T^b_{-a}) \mid \mathcal{F}_t] \, du \\
= \int_0^\infty (I(T_u < T^b_{-a} \leq t) + I(T_u \leq t < T^b_{-a})) \\
+ \mathbf{E}[I(t < T_u < T^b_{-a}) \mid \mathcal{F}_t] \, du. \]

Using the Markov property of the Brownian motion, we find that
\[ \mathbf{E}[I(t < T_u < T^b_{-a}) \mid \mathcal{F}_t] = \mathbf{P}\left( \exists s \in (t, T^b_{-a}): B_s > u \mid \mathcal{F}_t \right) I(t < T^b_{-a}) \\
= \mathbf{P}_{B_t}\{ \exists s \in (0, T^b_{-a}): B_s > u \} I(t < T_{-a}) \\
= \psi(B_t, u) I(t < T^b_{-a}), \]

where
\[ \psi(x, u) = \begin{cases} 
1, & x \geq u, \ u \in (0, b), \\
\frac{x + a}{u + a}, & -a < x < u, \ u \in (0, b), \\
0, & \text{otherwise}.
\end{cases} \]

Taking account of this designation, we find that on the set \( \{ t < T^b_{-a} \} \)
\[ \mathbf{E}[S_{T^b_{-a}} \mid \mathcal{F}_t] = S_t + \int_{S_t}^b \psi(B_t, u) \, du = S_t + \int_{S_t}^b \frac{B_t + a}{u + a} \, du = S_t + (B_t + a) \log \frac{b + a}{S_t + a}. \]

Applying Itô’s formula to the right-hand side of this relation and again, as in section 1, ignoring members with bounded variation, we arrive at the following relation:
\[ \mathbf{E}[S_{T^b_{-a}} \mid \mathcal{F}_t] = a \log \frac{a + b}{a} + \int_0^{t \wedge T^b_{-a}} \log \frac{b + a}{S_u + a} \, dB_u, \]

which is just the required relation (60), which obviously implies (59).

4. The case \( S_{g_T} = \max_{t \leq g_T} B_t \).

4.1. Let \( g_T = \sup\{0 < t \leq T: B_t = 0\} \) be the time of the last reaching of zero by the Brownian motion on \((0, T)\). If \( B_t \neq 0 \) for all \( 0 < t \leq T \), then assume \( g_T = 0 \).

THEOREM 3. For \( S_{g_T} \) the following stochastic integral representation is true:
\begin{equation}
S_{g_T} = \frac{1}{2} \mathbf{E}S_T + \int_0^T \left[ 1 - \Psi \left( \frac{2S_u - B_u}{\sqrt{T - u}} \right) - Z_u(B_u, S_u - S_{g_u}) \right] \, dB_u,
\end{equation}

where \( \mathbf{E}S_T = \sqrt{2T/\pi} \), \( \Psi(x) = 2[1 - \Phi(x)] \), and
\[ Z_u(B_u, S_u - S_{g_u}) = (S_u - S_{g_u}) \varphi_{T-u}(B_u) \quad \text{with} \quad g_u = \sup\{0 < t \leq u: B_t = 0\}, \]
or, equivalently,

\[
S_{gt} = \frac{1}{2} ES_T + \int_0^T \left[ \frac{1}{2} \psi \left( \frac{2S_u - B_u}{\sqrt{T-u}} \right) - Z_u(B_u, S_u - S_{gu}) \right] dB_u.
\]  

We give two different proofs, each of them of independent interest in view of the techniques used.

4.2. First proof. We have

\[
S_{gt} = \int_0^\infty I(a < S_{gt}) \, da = \int_0^\infty I(g_T > T_a) \, da = \int_0^\infty I(d_{T_a} < T) \, da,
\]

where for \( K > 0 \)

\[
d_K = \inf\{ t > K : B_t = 0 \}.
\]

By analogy with the scheme of the proof of Theorem 1 it is natural first to obtain a stochastic integral representation for \( I(d_{T_a} < T) \) (cf. Lemma 1, which provides the representation for \( I(T_a < T) \)).

**Lemma 4.** For any \( a > 0 \) and any \( T > 0 \)

\[
I(d_{T_a} < T) = P\{ T_{2a} < T \} + 2 \int_{T_a \wedge T} ^{d_{T_a} \wedge T} \varphi_{T-s} (B_s) dB_s
\]

\[
- 2 \int_0 ^{T_a \wedge T} \varphi_{T-s} (B_s - 2a) dB_s \quad (P\text{-a.s.}).
\]

**Proof.** It is obvious that

\[
d_{T_a} = \inf\{ t > T_a : B_t = 0 \} = T_a + \inf\{ u \geq 0 : B_{T_a + u} = 0 \}
\]

\[
= T_a + \inf\{ u \geq 0 : B_{T_a + u} - a = -a \} = T_a + \inf\{ u \geq 0 : \hat{B}_u = -a \},
\]

where \( \hat{B} = (\hat{B}_u)_{u \geq 0} \) with \( \hat{B}_u = B_{T_a + u} - a \) is the Brownian motion, independent of \( \sigma \)-algebra \( \mathcal{F}_{T_a} = \sigma\{ A \in \mathcal{F} : A \cap \{ T_a \leq t \} \in \mathcal{F}_t, t > 0 \} \) with \( \mathcal{F} = \bigvee_{t>0} \mathcal{F}_t \).

Denote \( \hat{T}_{-a} = \inf\{ u \geq 0 : \hat{B}_u = -a \} \). Then from (65) we have \( d_{T_a} = T_a + \hat{T}_{-a} \), and, hence,

\[
I(d_{T_a} < T) = I(\hat{T}_{-a} < T - T_a).
\]

Let us find a representation for \( I(\hat{T}_{-a} < b) \). If one denotes \( \hat{T}_{-a} = \hat{T}_{-a}(\hat{B}) \), then one can see that \( \hat{T}_{-a}(\hat{B}) = \hat{T}_a(-\hat{B}) \). According to Lemma 1,

\[
I(\hat{T}_a(-\hat{B}) < b) = P\{ \hat{T}_a(-\hat{B}) < b \} + 2 \int_0 ^{\hat{T}_a(-\hat{B}) \wedge b} \varphi_{\hat{B}_u - u} (\hat{B}_u - a) d(\hat{B}_u)
\]

\[
= P\{ T_a < b \} - 2 \int_0 ^{\hat{T}_{-a} \wedge b} \varphi_{\hat{B}_u - u} (\hat{B}_u + a) d\hat{B}_u.
\]

By (14) and (15)

\[
P\{ T_a < b \} = \int_0 ^\infty I(t < b) \gamma_a(t) \, dt \quad \text{with} \quad \gamma_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/(2t)}.
\]
Therefore,

\[(70) \quad \mathbb{E} \bigr|_{S_{\infty}} \leq \int_0^\infty \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \, d\mathbb{P}_{T_{a}} + \int_0^{T_{a}} \gamma(t) \, dt \, d\mathbb{P}_{T_{a}}.
\]

Using the independence of \( T_{a} \) from \( \sigma \)-algebra \( \mathcal{F}_{T_{a}} \), we find from (67) that

\[
\begin{align*}
\int_0^{T_{a}} \gamma(t) \, dt &= \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \\
&= \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \\
&= \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt.
\end{align*}
\]

From this, (67), (66), and (6) we obtain

\[
\begin{align*}
I(d_{T_{a}} < T) &= \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \\
&= \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt.
\end{align*}
\]

According to Lemma 5, mentioned below in section 3, for all \( s < T_{a} \) such that \( B_{s} < a \), the following equality holds:

\[
(69) \quad \int_0^{T_{a}} \gamma(t) \, dt = \int_0^{T_{a}} \gamma(t) \, dt = \int_0^{T_{a}} \gamma(t) \, dt.
\]

Hence, it follows from (68) that

\[
\begin{align*}
I(d_{T_{a}} < T) &= \mathbb{P}(T_{2a} < T) + 2 \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \\
&= \mathbb{P}(T_{2a} < T) + 2 \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt.
\end{align*}
\]

which is just the required relation (64). This proves Lemma 4.

Now we turn to proving representation (61).

By virtue of (63) and (64)

\[
\begin{align*}
S_{\infty} &= \int_0^{\infty} I(d_{T_{a}} < T) \, da \\
&= \int_0^{\infty} \mathbb{P}(T_{2a} < T) \, da + 2 \int_0^{\infty} \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \, d\mathbb{P}_{T_{a}} + \int_0^{T_{a}} \gamma(t) \, dt \, d\mathbb{P}_{T_{a}} \\
&= \int_0^{\infty} \mathbb{P}(T_{2a} < T) \, da + 2 \int_0^{\infty} \int_0^{T_{a}} \gamma(t) \, dt - \int_0^{T_{a}} \gamma(t) \, dt \, d\mathbb{P}_{T_{a}} + \int_0^{T_{a}} \gamma(t) \, dt \, d\mathbb{P}_{T_{a}}
\end{align*}
\]
Here

\[
\int_{0}^{\infty} P\{T_{2a} < T\} \, da = \frac{1}{2} \int_{0}^{\infty} P\{T_{b} < T\} \, db = \frac{1}{2} \int_{0}^{\infty} P\{S_{T} > b\} \, db = \frac{1}{2} ES_{T}
\]

and

\[
\int_{0}^{\infty} \left[ \int_{0}^{T_{a \wedge T}} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \int_{0}^{T} \left[ \int_{0}^{\infty} \varphi_{T-u}(B_{u} - 2a) I(S_{u} < a) \, da \right] \, dB_{u} = \int_{0}^{T} \left[ \int_{0}^{S_{u}} \varphi_{T-u}(B_{u} - 2a) \, da \right] \, dB_{u} = \frac{1}{2} \int_{0}^{T} \left[ \int_{2S_{u}}^{\infty} \varphi_{T-u}(B_{u} - b) \, db \right] \, dB_{u}
\]

\[
\int_{0}^{\infty} \left[ \int_{0}^{T_{a \wedge T}} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \int_{0}^{T_{0}} \left[ \int_{0}^{\infty} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \frac{1}{2} \int_{0}^{T_{0}} \left[ \int_{0}^{\infty} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \frac{1}{2} \int_{0}^{T_{0}} \left[ \int_{0}^{\infty} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da
\]

\[
\int_{0}^{\infty} \left[ \int_{0}^{T_{a \wedge T}} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \int_{0}^{T} \left[ \int_{0}^{\infty} \varphi_{T-u}(B_{u} - 2a) I(S_{u} < a) \, da \right] \, dB_{u} = \int_{0}^{T} \left[ \int_{0}^{S_{u}} \varphi_{T-u}(B_{u} - 2a) \, da \right] \, dB_{u} = \frac{1}{2} \int_{0}^{T} \left[ \int_{2S_{u}}^{\infty} \varphi_{T-u}(B_{u} - b) \, db \right] \, dB_{u}
\]

\[
(71)
\]

Finally, let us transform the last expression in the right-hand side of (70).

We have

\[
\int_{0}^{\infty} \left[ \int_{0}^{T_{a \wedge T}} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \int_{0}^{\infty} \left[ \int_{0}^{T_{a \wedge T}} \varphi_{T-s}(B_{s} - 2a) \, dB_{s} \right] \, da = \frac{1}{2} \int_{0}^{T} \left[ \int_{0}^{\infty} \varphi_{T-u}(B_{u} - 2a) I(S_{u} < a) \, da \right] \, dB_{u} = \frac{1}{2} \int_{0}^{T} \left[ \int_{0}^{S_{u}} \varphi_{T-u}(B_{u} - 2a) \, da \right] \, dB_{u} = \int_{0}^{T} \left[ \int_{0}^{\infty} \varphi_{T-u}(B_{u} - b) \, db \right] \, dB_{u}
\]

\[
(72)
\]

Thus, it follows from (70)–(73) that

\[
S_{gT} = \frac{1}{2} ES_{T} + \int_{0}^{T} \left[ 1 - \Phi\left( \frac{2S_{u} - B_{u}}{\sqrt{T - u}} \right) \right] \, dB_{u} + 2 \int_{0}^{T} (S_{u} - S_{g_{u}}) \varphi_{T-u}(B_{u}) \, dB_{u}
\]

Thereby (61) and (62) are proved.

4.3. In the above-mentioned proof, integral relation (69) linking the densities

\[
\varphi_{t}(a) = \frac{1}{\sqrt{2\pi t}} e^{-a^{2}/(2t)} \quad \text{and} \quad \gamma_{a}(t) = \frac{a}{\sqrt{2\pi t^{3}}} e^{-a^{2}/(2t)} \left( = -\frac{\partial}{\partial a} \varphi_{t}(a) \right)
\]

was used. It follows from the following lemma.

**Lemma 5.** For all \( a > 0 \) and \( \theta > 0 \)

\[
\int_{0}^{\theta} \gamma_{a}(t) \varphi_{\theta-t}(x - a) \, dt = \begin{cases} \varphi_{\theta}(x), & x > a, \\ \varphi_{\theta}(x - 2a), & x \leq a. \end{cases}
\]

**Proof.** Let

\[
I(a, x) = \frac{1}{\varphi_{\theta}(x)} \int_{0}^{\theta} \gamma_{a}(t) \varphi_{\theta-t}(x - a) \, dt.
\]
Using the above-mentioned explicit form of the functions \( \varphi_\theta(a) \) and \( \gamma_\alpha(t) \) and making the change of variables \( u = \sqrt{\theta}/t - 1 \), we find that

\[
I(a, x) = \frac{2a}{\sqrt{2\pi} t} e^{a(x-a)/\theta} \int_0^\infty e^{-a^2 - \beta/u^2} du
\]

with \( \alpha = a^2/(2\theta) \) and \( \beta = (x - a)^2/(2\theta) \).

By formula 3.325 of [2]

\[
\int_0^\infty e^{-\alpha u^2 - \beta/u^2} du = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} e^{-2\sqrt{\alpha\beta}} = \frac{\sqrt{2\pi\theta}}{2a} e^{-a|x-a|/\theta}.
\]

Therefore,

\[
I(a, x) = e^{a(x-a)/\theta} e^{-a|x-a|/\theta} = \begin{cases} 
1, & x > a, \\
e^{-2a(a-x)/\theta}, & x \leq a,
\end{cases}
\]

which proves (74).

Along with the adduced “analytic” proof of (74) the following “probabilistic” proof of this relation is not without interest.

Let \( f(x) \) be a measurable bounded function. Then

\[
(75) \quad E[f(\theta)] = E[f(\theta) I(T_a < \theta)] + E[f(\theta) I(T_a \geq \theta)].
\]

Here

\[
E[f(\theta) I(T_a < \theta)] = E[f(B_{T_a+(\theta-T_a)}) I(T_a < \theta)]
\]

\[
= \int_0^\theta \gamma_a(t) \left[ \int_{-\infty}^\infty f(x) \varphi_{\theta-t}(x-a) \, dx \right] \, dt
\]

and

\[
E[f(\theta) I(T_a \geq \theta)] = \int_\infty^\infty E[f(\theta) I(\theta \leq T_a) \mid B_\theta = x] \varphi_\theta(x) \, dx
\]

\[
= \int_\infty^\infty f(x) P(T_a \geq \theta \mid B_\theta = x) \varphi_\theta(x) \, dx.
\]

From (75)–(77) we obtain

\[
E[f(\theta)] = \int_{-\infty}^\infty f(x) \varphi_\theta(x) \, dx
\]

\[
= \int_{-\infty}^\infty f(x) \left[ \int_0^\theta \varphi_{\theta-t}(x-a) \gamma_a(t) \, dt + P(T_a \geq \theta \mid B_\theta = x) \varphi_\theta(x) \right] \, dx.
\]

From this in view of the arbitrariness of the function \( f(x) \) we have

\[
(78) \quad \int_0^\theta \gamma_a(t) \varphi_{\theta-t}(x-a) \, dt = \varphi_\theta(x)[1 - P(T_a \geq \theta \mid B_\theta = x)].
\]

If \( x > a \), then \( P(T_a \geq \theta \mid B_\theta = x) = 0 \), and (78) gives (74).

Now let \( x \leq a \); then

\[
1 - P(T_a \geq \theta \mid B_\theta = x) = P(T_a < \theta \mid B_\theta = x) = P\left( \max_{u \leq \theta} B_u > a \mid B_\theta = x \right)
\]

\[
= P(S_\theta > a \mid B_\theta = x)
\]

with \( S_\theta = \max_{u \leq \theta} B_u \).
Probability $\mathbb{P}(S_\theta > a \mid B_\theta = x)$ can be found, using, for example, the Seshadri result (see [3]) that random variables $S_\theta(S_\theta - B_\theta)$ and $B_\theta$ are independent and

$$S_\theta(S_\theta - B_\theta) \xrightarrow{law} \frac{\theta}{2} \mathcal{E},$$

where $\mathcal{E}$ is the standard exponentially distributed random variable ($\mathbb{P}\{\mathcal{E} > t\} = e^{-t}$, $t > 0$).

Indeed, from the stated assertions we find that for $x \leq a$, $a \geq 0$, and $b = a(a - x)$

$$\mathbb{P}(S_\theta > a \mid B_\theta = x) = \mathbb{P}(S_\theta(S_\theta - B_\theta) > b \mid B_\theta = x) = \mathbb{P}\left\{ \mathcal{E} > \frac{2b}{\theta} \right\} = e^{-2b/\theta} = e^{-2a(a-x)/\theta}.$$

Thereby, expression $1 - \mathbb{P}(T_a \geq \theta \mid B_\theta = x)$, being a part of (78), equals

$$\mathbb{P}(S_\theta > a \mid B_\theta = x) = e^{-2a(a-x)/\theta}$$

and, hence, for $x \leq a$,

$$\int_0^\theta \gamma_a(t) \varphi_{\theta-t}(x-a) \, dt = \varphi_\theta(x) e^{-2a(a-x)/\theta},$$

which is claimed in (74).

4.4. Second proof. By analogy with the distributions used in sections 2’ and 3’ let us demonstrate that for every $t > 0$ the following equality is true ($\mathbb{P}$-a.s.):

$$\mathbb{E}[S_\theta \mid \mathcal{F}_t] = \frac{1}{2} \mathbb{E}S_T + \int_0^{t\wedge T} \frac{1}{2} \Psi \left( \frac{2S_u - B_u}{\sqrt{T-u}} \right) - Z_u(B_u, S_u - S_{g_u}) \, dB_u. \tag{79}$$

(Formula (61), of course, follows from this representation.)

Fix $0 \leq t < T$. Then

$$\mathbb{E}[S_\theta \mid \mathcal{F}_t] = \int_0^\infty \mathbb{E}[I(a < S_{g_T}) \mid \mathcal{F}_t] \, da = \int_0^\infty \mathbb{E}[I(T_a < g_T) \mid \mathcal{F}_t] \, da \tag{80}$$

$$= \int_0^\infty \mathbb{E}[I(T_a < g_T \leq t) + I(T_a \leq t < g_T) + I(t < T_a < g_T) \mid \mathcal{F}_t] \, da.$$

Using the Markov property of the Brownian motion, we obtain the following relations:

(a) $\mathbb{E}[I(T_a < g_T \leq t) \mid \mathcal{F}_t] = \mathbb{E}[I(T_a < g_T < t) \mid \mathcal{F}_t]$

$$= \mathbb{E}[I(T_a < g_T < t) \mid \mathcal{F}_t] I(T_a < t)$$

$$= \mathbb{P}\{\exists s_1 < s_2 < t: B_{s_1} > a, B_{s_2} < 0; B_s \neq 0 \text{ for } s \in (t, T) \mid \mathcal{F}_t\} I(T_a < t)$$

$$= \mathbb{P}(T_a + T_0 \circ \theta_{T_a} < t \text{ and } B_s \neq 0 \text{ for } s \in (t, T) \mid \mathcal{F}_t) I(T_a < t)$$

$$= \mathbb{P}_{T_a \circ \theta_{T_a}}(B_\theta \neq 0 \text{ for } s \in (0, T-t) \mid \mathcal{F}_t) I(T_a + T_0 \circ \theta_{T_a} < t)$$

$$= \mathbb{P}(T_{|B_\theta|} > 1 - t \mid \mathcal{F}_t) I(T_a + T_0 \circ \theta_{T_a} < t);$$
(b) \( E[I(T_a \leq t \leq g_T) | F_t] = E[I(T_a < t < g_T) | F_t] \)
\[
= E[I(T_a < t < g_T) | F_t] I(t > T_a)
\]
\[
= P(\exists s < t: B_s > a) \quad \text{and} \quad P(\exists s_1 \in (t, T): B_{s_1} = 0) \quad I(t > T_a)
\]
\[
= P_{B_t}\{\exists s_1 \in (0, T - t): B_{s_1} = 0\} \quad I(t > T_a)
\]
\[
= P\{T_{|B| > T - t}\} \quad I(t > T_a);
\]
(c) \( E[I(t < T_a < g_T) | F_t] \)
\[
= P(\exists s_1, s_2: t < s_1 < s_2 < T, B_{s_1} > a, B_{s_2} = 0 \mid F_t) \quad I(t < T_a)
\]
\[
= P_{B_t}\{\exists s_1, s_2: 0 < s_1 < s_2 < T - t, B_{s_1} > a, B_{s_2} = 0\} \quad I(t < T_a)
\]
\[
= P_{B_t}\{\exists s \in (0, T - t): B_s = 2a\} \quad I(t < T_a)
\]
\[
= P\{T_{2a-B_t} < T - t\} \quad I(t < T_a).
\]

From (a), (b), and (c) we get
\[
(a^*) \quad \int_0^\infty E[I(T_a < g_T \leq t) \mid F_t] \, da
\]
\[
= P\{T_{|B|} > T - t\} \int_0^\infty I(T_a + T_0 < t) \, da
\]
\[
= P\{T_{|B|} > T - t\} S_{g_t} = [1 - P\{T_{|B|} < T - t\}] S_{g_t};
\]
(b*) \( \int_0^\infty E[I(T_a < t < g_t) \mid F_t] \, da = P\{T_{|B|} < T - t\} S_t; \)
(c*) \( \int_0^\infty E[I(t < T_a < g_T) \mid F_t] \, da \)
\[
= \int_0^\infty P\{T_{2a-B_t} < T - t\} \, I(t < T_a) \, da
\]
\[
= \int_{S_t} \int_0^{T-t} \frac{2a - B_t}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(2a - B_t)^2}{2s}\right\} \, ds \, da
\]
\[
= \int_0^{T-t} \int_{S_t} \frac{2a - B_t}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(2a - B_t)^2}{2s}\right\} \, ds \, da
\]
\[
= \frac{1}{2} \int_0^{T-t} \frac{1}{\sqrt{2\pi s}} \exp\left\{-\frac{(2S_t - B_t)^2}{2s}\right\} \, ds
\]
\[
= \frac{1}{2} \int_0^{T-t} \varphi_s(2S_t - B_t) \, ds = \frac{1}{2} H(2S_t - B_t, t),
\]
where \( H(x, t) = \int_0^{T-t} \varphi_s(x) \, ds \) and \( 0 \leq t < T. \)

Gathering relations \( (a^*), (b^*), \) and \( (c^*) \), we obtain
\[
E(S_{g_T} \mid F_t) = \frac{1}{2} H(2S_t - B_t, t) + P\{T_{|B|} < T - t\} (S_t - S_{g_t}) + S_{g_t},
\]
\[
(81) \quad = \frac{1}{2} H(2S_t - B_t, t) + \int_0^{T-t} \frac{|B_t|}{\sqrt{2\pi s^3}} \exp\left\{-\frac{|B_t|^2}{2s}\right\} \, ds \cdot (S_t - S_{g_t}) + S_{g_t}.
\]

Applying the Itô formula to \( H(X_t, t) \) with \( X_t = 2S_t - B_t \) (Bessel process of order 3),
we find that for $t < T$

$$H(2S_t - B_t, t) = H(0, 0) - \int_0^t \frac{\partial}{\partial x} H(X_u, u) dB_u + A_t,$$

where $(A_t)_{t < T}$ is a process of bounded variation. From this by virtue of (50) and (51) we have

$$H(2S_t - B_t, t) = \sqrt{\frac{2T}{\pi}} + \int_0^t \Psi\left(\frac{2S_u - B_u}{\sqrt{T}}\right) dB_u + A_t,$$

where $(A_t)_{t < T}$ is a continuous process of bounded variation.

Let

$$\tilde{H}(x, t) = \int_0^{T-t} \frac{x}{\sqrt{2\pi s}} e^{-x^2/(2s)} ds \left(= \Psi\left(\frac{x}{\sqrt{T-t}}\right)\right).$$

Applying the Itô–Tanaka formula to $\tilde{H}(|B_t|, t)$, we find that

$$d\tilde{H}(|B_t|, t) = \Psi\left(\frac{|B_t|}{\sqrt{T-t}}\right) + 2\varphi\left(\frac{|B_t|}{\sqrt{T-t}}\right) \frac{1}{\sqrt{T-t}} \text{sign} B_t \ dB_t + d\tilde{A}_t$$

where $(\tilde{A}_t)_{t < T}$ is a process of bounded variation.

Thus, from (81), (83)–(85) we find, neglecting members with bounded variation (cf. the reasoning in the end of section 2'), that for $t < T$,

$$\mathbb{E}[S_{gT} | \mathcal{F}_t] = \sqrt{\frac{T}{2\pi}} + \frac{1}{2} \int_0^{t\wedge T} \Psi\left(\frac{2S_u - B_u}{\sqrt{T-t}}\right) dB_u$$

$$- 2 \int_0^{t\wedge T} \varphi_{T-u}(B_u) \text{sign} B_u \cdot (S_u - S_{g_u}) dB_u.$$

Note that one can omit sign $B_u$ here, since if sign $B_u = -1$, then $S_u - S_{g_u} = 0$. Hence,

$$\int_0^{t\wedge T} \varphi_{T-u}(B_u) \text{sign} B_u \cdot (S_u - S_{g_u}) dB_u = \int_0^{t\wedge T} \varphi_{T-u}(B_u)(S_u - S_{g_u}) dB_u.$$

The required relation (79) follows for $t < T$ from (86) and (87). In the general case (when $t \geq 0$) it is sufficient to note that $\lim_{t \uparrow T} \mathbb{E}[S_{gT} | \mathcal{F}_t] = S_{gT}$ and the limits $\lim_{t \uparrow T}$ of the integrals $\int_0^{t\wedge T} (\cdot) dB_u$ in (86) are equal to the integrals $\int_0^T (\cdot) dB_u$.

**REFERENCES**

