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Stability of Relative Equilibria of Point Vortices on the Sphere

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Abstract

We describe the linear and nonlinear stability and instability of certain configurations of point vortices on the sphere forming relative equilibria. These configurations consist of up to two rings, with and without polar vortices. Such configurations have dihedral symmetry, and the symmetry is used both to block diagonalize the relevant matrices and to distinguish the subspaces on which their eigenvalues need to be calculated.

1 Introduction

Since the work of Helmholtz [H] systems of point vortices on the plane have been studied as finite-dimensional approximations to vorticity dynamics in ideal fluids. For a general survey of patterns of point vortices see [ANSTV]. Point vortex systems on the sphere, introduced by Bogomolov [B77], provide simple models for the dynamics of concentrated regions of vorticity, such as cyclones and hurricanes, in planetary atmospheres. In this paper we consider a non-rotating sphere, since the rotation of the sphere induces a non-uniform background vorticity which makes the whole system infinite-dimensional.

As in the planar case, the equations governing the motion of N point vortices on a sphere are Hamiltonian [B77] and this property has been used to study them from a number of different viewpoints. Phase space reduction shows that the three vortex problem is completely integrable on both the plane and the sphere: the motion of three vortices of arbitrary vorticity on a sphere is studied in [KN98]. The stabilities of some of the relative equilibria described in [KN98] are computed in [PM98] and numerical simulations are presented in [MPS99]. The existence of relative equilibria of N vortices is treated in [LMR01], and the nonlinear stability of a latitudinal ring of N identical vorticities is computed in [BC03], and independently in the present paper. In fact the linear stability results of such as ring obtained by [PD93] coincide with the Lyapunov stability results. The stability of a ring of vortices on the sphere together with a central polar vortex is studied in [CMS03], and again independently in the present paper, though with different methods (and different results!). The existence and nonlinear stability of relative equilibria of N vortices of vorticity $+1$ together with N vortices of vorticity -1 are studied in [LP02]. It has also been proved in [LP] that relative equilibria formed of latitudinal rings of identical vortices for the non-rotating sphere persist to relative equilibria when the sphere rotates. However, the question of stability becomes much more delicate: for motions that are not relative equilibria, the vorticity of a point vortex is no longer preserved as it interacts with the background vorticity, and the problem becomes fundamentally infinite-dimensional. In [Ku04] Kurakin studies the stability of equilibrium configurations of identical vortices placed at the vertices of regular polyhedra; he finds that the tetrahedron, octahedron and icosahedron are stable, while the other two are unstable. Finally, studies of periodic orbits of point vortices on the sphere can be found in [ST, To01, LPth, LP04].

Our study of the stability of relative equilibria is based on the symmetries of the system, and especially the isotropy subgroups of the relative equilibria. The Hamiltonian is invariant under rotations and reflections of the sphere and permutations of identical vortices. However, some of these symmetries (eg reflections) are not symmetries of the equations of motion: they are *time-reversing* symmetries. From Noether's theorem, the rotational symmetry provides three conserved quantities, the components of the momentum map $\Phi : \mathcal{P} \to \mathbb{R}^3$ where $\mathcal{P}$ is the phase space.

Relative equilibria are dynamical trajectories that are generated by the action of a 1-parameter subgroup of the symmetry group. More intuitively, they correspond here to motions of the point vortices which are stationary in a steadily rotating frame. In other words, the motion of a relative equilibrium corresponds to a rigid rotation of N point vortices about some axis (which we always take to be the z-axis). In the same way as equilibria are critical points of the Hamiltonian H, relative equilibria are critical points of the restrictions of H to the level sets $\Phi^{-1}(\mu)$. Section 2 is devoted to a description of the system of point vortices on the sphere, and to an outline of stability theory for relative equilibria. The appropriate concept of stability for relative equilibria of Hamiltonian system is Lyapunov stability *modulo a subgroup*. The stability study is realized using on one hand the *energy-momentum method* [Pa92, Or98] which consists of computing the eigenvalues of a certain Hessian, and leads to *nonlinear* stability results, and on the other hand a linear study computing the eigenvalues of the linearization of the equations of motion. To both these ends, we block-diagonalize these matrices using a suitable basis, the *symmetry adapted basis* (Section 3), which makes use of the specific dihedral symmetry of the relative equilibrium. This is equivalent to noting that the matrices (or certain submatrices) are *circulant*, as noticed in [CMS03]. However, the symmetry is also used to apply the energy-momentum method as it helps distinguish on which subspaces computations are needed.

The remaining five sections each treat one of five different types of relative equilibria, consisting of rings of identical vortices together with possible vortices at the poles, whose existence were proved in [LMR01]. The notation for the different configurations is taken from the same source and is described at the end of the introduction. We now outline the main stability results.

We begin in Section 4 by computing the stability of the relative equilibria consisting of a single ring of identical vortices, a configuration denoted $\mathbf{C}_{nv}(R)$ (Figure 1.1(a)). We show in Theorem 4.2 that for $n \geq 7$, they are unstable for all co-latitudes of the ring, while for $n < 7$ there exist ranges of Lyapunov stability when the ring is near a pole. These results are not new [PD93, BC03], but serve to demonstrate the method used in later sections.

In Section 5, we study the stability of the relative equilibria $\mathbf{C}_{nv}(R,p)$ (Figure 1.1(b)) which are configurations formed of a ring of n identical vortices together with a polar vortex. For $n \geq 7$ they are all unstable if the vorticity κ of the polar vortex has opposite sign to that of the ring. However if the vorticities have the same sign then for each co-latitude of the ring there exists a range of κ for which the relative equilibrium is Lyapunov stable. Adding polar vortices can therefore *stabilize* the unstable pure ring relative equilibria. The detailed results are contained in Theorem 5.2, its corollary and the following discussion. Our results are consistent with those of [CMS03] (aside from an error in their Figure 7 where the wrong curves are plotted), though the present methods are stronger as they give more regions of stability than obtained in [CMS03]—see Remark 5.4.

In Section 6 we obtain analytic (in)stability criteria for the relative equilibria $\mathbf{C}_{nv}(R,2p)$ which are configurations formed of a ring of n identical vortices together with *two* polar vortices, but only with respect to certain modes ($\ell \geq 2$). As in the case of a single polar vortex, the two polar vortices play the role of control parameters for the stability. The details are contained in Theorem 6.2. A numerical investigation is needed for the remaining ($\ell = 1$) mode in order to provide stability criteria; this is being pursued separately.

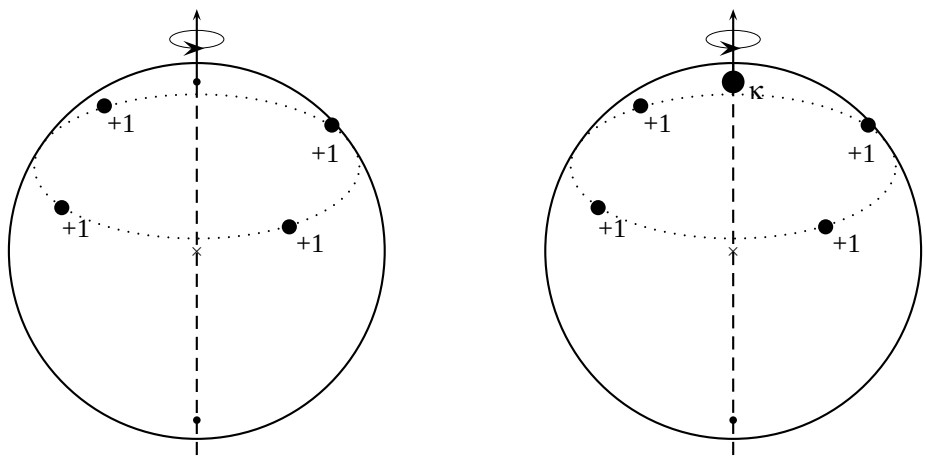

Figure 1.1: The $\mathbf{C}_{nv}(R)$ and $\mathbf{C}_{nv}(R,p)$ relative equilibria.
($\mathbf{C}_{nv}(R,2p)$ has a vortex at the South pole as well.)

Finally, in Sections 7 and 8 we investigate configurations formed of two rings of arbitrary vorticities (each ring, as always, consisting of identical vortices). In [LMR01] it was shown that two rings of n vortices can be relative equilibria if and only if they are either aligned or staggered. These two arrangements are denoted $\mathbf{C}_{nv}(2R)$ and $\mathbf{C}_{nv}(R,R')$ respectively (see Figure 1.2). Here we show that for almost all pairs of ring latitudes there is a unique ratio of the ring vorticities for which these configurations are relative equilibria. Numerical computations of their stabilities suggest that these relative equilibria can only be stable if $n \leq 6$, and in the aligned case the two rings must be close to opposite poles, and hence have opposite vorticities. In some cases, staggered rings may also be stable when in the same hemisphere.

In principle the method applies to larger numbers of rings but the algebraic problem of diagonalizing the matrices in general becomes intractable; however numerical studies for particular (numerical) values of the vorticities in the rings would be feasible.

Symmetry group notation All possible symmetry types of configurations of point vortices on the sphere were classified in [LMR01]. The symmetry group of the system is of the form $\mathrm{O}(3) \times S$, where S is a group of permutations, and a particular configuration with symmetry, or isotropy, subgroup $\Sigma < \mathrm{O}(3) \times S$ is denoted $\Gamma(A)$, where Γ is the projection of Σ to $\mathrm{O}(3)$ and A represents the way Σ permutes the point vortices. The classical *Schönflies-Eyring* notation for subgroups of $\mathrm{O}(3)$ is used.

In this paper we single out configurations consisting of concentric rings of identical vortices, with the same number of vortices in each ring, and with possible polar vortices. These configurations have cyclic symmetry (in the "horizontal plane"), and the Schönflies-Eyring notation for this subgroup of $\mathrm{O}(3)$ is $C_n < \mathrm{SO}(3)$. In fact we only consider the cases where the rings are either aligned (the vortices lie on the same longitudes) or staggered (they lie on intermediate longitudes, out of phase by π/n). In this case the symmetry group is the larger *dihedral* group C_{nv} (n being the number of vortices in each ring, and v denoting the fact that there are vertical planes of reflection). For such configurations, we write $C_{nv}(k_1R, k_2R', k_p p)$ to mean that there are $k_r = k_1 + k_2$ rings and k_p polar vortices. The difference between R and R' is that the k_1 rings R are aligned and the k_2 rings R' are staggered with respect to the first (and so aligned with each other). Of course $k_p = 0, 1$ or 2.

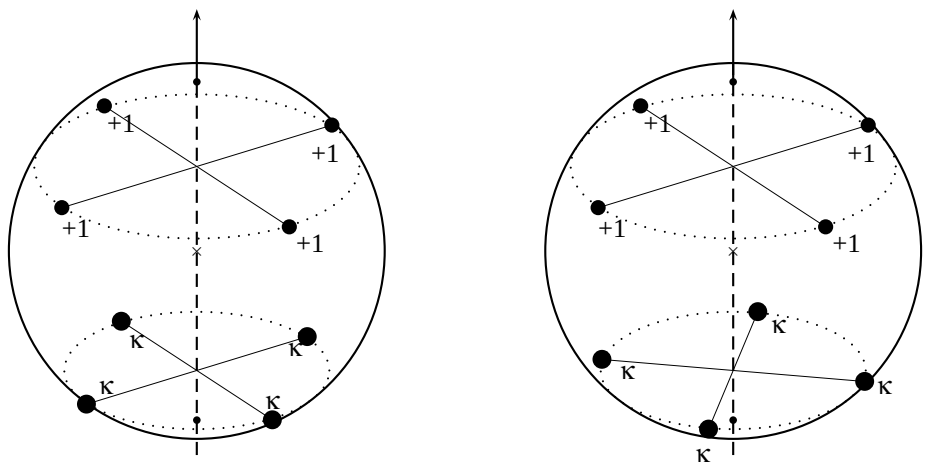

Figure 1.2: Configurations of types $\mathbf{C}_{nv}(2R)$ (2 aligned rings) and $\mathbf{C}_{nv}(R,R')$ (2 staggered rings). (Here $n=4$ of course.)

2 Point vortices on the sphere and stability theory

In this section we briefly recall that the system of point vortices on a sphere is a n-body Hamiltonian system with symmetry and we review the stability theory for relative equilibria.

2.1 Point vortices

Consider n point vortices $x_1,\dots,x_n \in S^2$ with vorticities $\kappa_1,\dots,\kappa_n \in \mathbb{R}$. Let θ_i,ϕ_i be respectively the co-latitude and the longitude of the vortex x_i. The dynamical system is Hamiltonian with Hamiltonian given by

$$H \;=\; -\sum_{i<j} \kappa_i\kappa_j \ln(1-\cos\theta_i\cos\theta_j - \sin\theta_i\sin\theta_j\cos(\phi_i-\phi_j))$$

and conjugate variables given by $q_i = \sqrt{|\kappa_i|}\cos\theta_i$ and $p_i = \mathrm{sign}(\kappa_i)\sqrt{|\kappa_i|}\phi_i$.

The phase space is $\mathcal{P} = \{(x_1,\dots,x_n) \in S^2\times\cdots\times S^2 \mid x_i \neq x_j \text{ if } i\neq j\}$ endowed with the symplectic form $\omega = \sum_i \kappa_i \sin\theta_i \, d\theta_i \wedge d\phi_i$. The Hamiltonian vector field X_H satisfies $\omega(\,\cdot\,, X_H(x)) = dH_x$. If we consider S^2 as a subset of $\mathbb{R}^3$, so the vortices $x_j \in \mathbb{R}^3$, then we obtain

$$\dot{x}_i \;=\; X_H(x)_i \;=\; \sum_{j,j\neq i} \kappa_j \frac{x_j\times x_i}{1-x_i\cdot x_j}\;,\; i=1,\dots,N, \tag{2.1}$$

$$H \;=\; -\sum_{i<j} \kappa_i\kappa_j \ln(\|x_i-x_j\|^2/2).$$

It follows that H is invariant under the action of $\mathrm{O}(3)$. The symplectic form is $\mathrm{SO}(3)$ invariant and so X_H is $\mathrm{SO}(3)$-equivariant. The reflections in $\mathrm{O}(3)$ reverse the sign of the symplectic form and so are time-reversing symmetries of X_H. Moreover H, ω and X_H are all invariant or equivariant with respect to permutations of vortices with equal vorticity.

The rotational symmetry implies the existence of a momentum map $\Phi : \mathcal{P} \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$:

$$\Phi(x) \;=\; \sum_{j=1}^{N} \kappa_j x_j \qquad (x_j \in S^2 \subset \mathbb{R}^3)$$

which is conserved under the dynamics. In other words each of the three components of $\Phi(x)$ is a conserved quantity.

2.2 Relative equilibria

A point $x_e \in \mathcal{P}$ is a relative equilibrium if and only if there exists $\xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3$ (the angular velocity) such that x_e is a critical point of the function $H_\xi(x) = H(x) - \langle \Phi(x), \xi \rangle$, where the pairing $\langle\ ,\ \rangle$ between $\mathbb{R}^3$ and its dual is identified with the canonical scalar product on $\mathbb{R}^3$. Equivalently, relative equilibria are critical points of the restriction of H to $\Phi^{-1}(\mu)$, since the level set $\Phi^{-1}(\mu)$ are always non-singular for point vortex systems of more than two vortices. The function H_ξ is called the *augmented Hamiltonian*.

Since the momentum is conserved, we can choose a frame for $\mathbb{R}^3$ such that Φ is parallel to the z-axis (provided the momentum is non-zero). It follows from the symmetry that the angular velocity $\xi \in \mathbb{R}^3$ is also parallel to the z-axis. We can therefore identify ξ and Φ with their z-components and the augmented Hamiltonian becomes simply $H_\xi(x) = H(x) - \xi\Phi(x)$.

Let $f : \mathcal{P} \to \mathbb{R}$ be a K-invariant function with K a compact group. Recall that $\mathrm{Fix}(K) = \{x \in \mathcal{P} \mid g \cdot x = x,\ \forall g \in K\}$. The *Principle of Symmetric Criticality* [P79] states that a critical point of the restriction of a K-invariant function f to $\mathrm{Fix}(K)$ is a critical point of f. As a corollary, if the Hamiltonian is invariant under K and x_e is an isolated point in $\mathrm{Fix}(K) \cap \Phi^{-1}(\mu)$, then x_e is a relative equilibrium. It follows in particular that all configurations of type $\mathbf{C}_{nv}(R)$, $\mathbf{C}_{nv}(R,p)$ (Figure 1.1), and $\mathbf{C}_{nv}(R,2p)$ are relative equilibria: take K such that $\pi(K) = \mathbf{C}_{nv}$ where $\pi : \mathrm{O}(3) \times S_n \to \mathrm{O}(3)$ is the cartesian projection.

Finally, one can show that if x_e is a relative equilibrium with angular velocity ξ, then H_ξ is a G_{x_e}-invariant function.

2.3 Stability theory

Stability is determined by the energy-momentum method together with an isotypic decomposition of the symplectic slice. We recall the main points of the method.

Let $x_e \in \mathcal{P}$ be a relative equilibrium, $\mu = \Phi(x_e)$, and ξ its angular velocity. The energy-momentum method consists of determining the symplectic slice

$$\mathcal{N} \ = \ (\mathfrak{so}(3)_\mu \cdot x_e)^\perp \cap \mathrm{Ker}\, D\Phi(x_e)$$

transversal to $\mathfrak{so}(3)_\mu \cdot x_e$, where

$$\mathrm{SO}(3)_\mu \ = \ \{g \in \mathrm{SO}(3) \mid \mathrm{Coad}_g \cdot \mu = \mu\}$$

and then examining the definiteness of the restriction $d^2H_\xi|_{\mathcal{N}}(x_e)$ of the Hessian $d^2H_\xi(x_e)$ to $\mathcal{N}$. (In practice we will represent μ as a vector, in which case $\mathrm{Coad}_g\, \mu = g\mu$ is just matrix multiplication.) If K is a group acting on the phase space a relative equilibrium x_e is said to be *Lyapunov stable modulo* K if for all K-invariant open neighbourhoods V of $K \cdot x_e$ there is an open neighbourhood $U \subseteq V$ of x_e which is invariant under the Hamiltonian dynamics. The *energy-momentum theorem* of Patrick [Pa92] holds since $\mathrm{SO}(3)$ is compact, and so we have:

If $d^2H_\xi|_{\mathcal{N}}(x_e)$ is definite, then x_e is Lyapunov stable modulo $\mathrm{SO}(3)_\mu$.

For $\mu \neq 0$, $\mathrm{SO}(3)_\mu$ is the set of rotations with axis $\langle \mu \rangle$, and so isomorphic to $\mathrm{SO}(2)$, while for $\mu = 0$, $\mathrm{SO}(3)_\mu = \mathrm{SO}(3)$. If $\mu \neq 0$ Lyapunov stability modulo $\mathrm{SO}(3)_\mu$ of a relative equilibrium with non-zero angular velocity corresponds to the ordinary stability of the corresponding periodic orbit.

The second step consists of performing an isotypic decomposition of the symplectic slice $\mathcal{N}$ in order to block diagonalize $d^2H_\xi|_{\mathcal{N}}(x_e)$. Let V be a finite dimensional representation of a compact Lie group K. Recall that a K invariant subspace $W \subset V$ of K is said to be *irreducible* if W has no proper K invariant subspaces. Since K is compact, V can be expressed as a direct sum of irreducible representations: $V = W_1 \oplus \cdots \oplus W_n$. In general this is not unique. There are a finite number of isomorphism classes of irreducible representations of K in V, say $U_1, \ldots, U_\ell$. Let V_k ($k = 1, \ldots, \ell$) be the sum of all irreducible representations $W_j \subset V$ such that W_j is isomorphic to U_k. Then $V = V_1 \oplus \cdots \oplus V_\ell$. This decomposition of V is unique and is called the *K-isotypic decomposition* of V [Se78]. By Schur's Lemma, the matrix of a K-equivariant linear map $f : V \to V$ block diagonalizes with respect to a basis $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_l\}$ where $\mathcal{B}_k$ is a basis of V_k, each block corresponding to a subspace V_k. The basis $\mathcal{B}$ is called a *symmetry adapted basis*.

Let G denote the group of all symmetries of H and X_H and G^χ the subgroup consisting of time-preserving symmetries. In the case of N identical vortices we have $G = \mathrm{O}(3) \times S_N$ and $G^\chi = \mathrm{SO}(3) \times S_N$. Since H_ξ is a G_{x_e}-invariant function, $d^2H_\xi|_{\mathcal{N}}(x_e)$ is G_{x_e}-equivariant as a matrix. Moreover the symplectic slice $\mathcal{N}$ is a G_{x_e}-invariant subspace and so we can implement a G_{x_e}-isotypic decomposition of $\mathcal{N}$ to block diagonalize $d^2H_\xi|_{\mathcal{N}}(x_e)$. This block diagonalization of $d^2H_\xi|_{\mathcal{N}}(x_e)$ simplifies the computation of its eigenvalues, and hence its definiteness. If it is definite then the relative equilibrium is Lyapunov stability modulo $\mathrm{SO}(3)_\mu$.

If $d^2H_\xi|_{\mathcal{N}}(x_e)$ is *not* definite then we study the spectral stability of x_e. In particular we examine the eigenvalues of $L_{\mathcal{N}}$, the matrix of the linearized system on the symplectic slice, that is $L_{\mathcal{N}} = \mathbf{J}_{\mathcal{N}} d^2H_\xi|_{\mathcal{N}}(x_e)$, where $\mathbf{J}_{\mathcal{N}}^{-1}$ is the matrix of $\omega|_{\mathcal{N}}$. The matrix $L_{\mathcal{N}}$ is $G^\chi_{x_e}$-equivariant and so we perform a $G^\chi_{x_e}$-isotypic decomposition of $\mathcal{N}$ to obtain a block diagonalization of $L_{\mathcal{N}}$, and so to determine the spectral stability of x_e. In particular, if $L_{\mathcal{N}}$ has eigenvalues with non-zero real part, then x_e is linearly unstable. Note that the block diagonalization of $d^2H_\xi|_{\mathcal{N}}(x_e)$ refines that of $L_{\mathcal{N}}$ since $G^\chi_{x_e} \subset G_{x_e}$.

Throughout this paper, *Lyapunov stable* will mean *Lyapunov stable modulo* $\mathrm{SO}(2)$ unless specified otherwise, and $\mu = \Phi(x_e)$ will denote the momentum of the configuration x_e. We will assume in most of the results that the relative equilibrium has a non-zero momentum. It is straightforward to check that almost all points of type $\mathbf{C}_{nv}(R, k_p p)$ have a non-zero momentum, so this is not a strong assumption for that case. This statement is also true for $\mathbf{C}_{nv}(2R)$ and $\mathbf{C}_{nv}(R, R')$ relative equilibria, as we shall see.

3 Symmetry adapted bases for rings and poles

In this section we give the ingredients needed to determine the symmetry adapted bases for the symplectic slice at the configurations described above, that is those of type $\mathbf{C}_{nv}(k_1R, k_2R', k_p p)$. In the first subsection we give a general symmetry adapted basis for the tangent space $T_{x_e}\mathcal{P}$ to the phase space at such a configuration, and express the derivative of the momentum map and tangent space to the group orbit in this basis. In the following two subsections we describe the isotypic decomposition of $T_{x_e}\mathcal{P}$, first for a single ring and then in general. Recall that the isotropy subgroup is always a dihedral group $\mathbf{C}_{nv}$ and that the irreducible representations of this group are of dimension 1 or 2. The actual symmetry adapted bases of the symplectic slices will be given case-by-case in the following sections. We do not give the proof of the results in the first subsection, since they can be easily deduced from the proofs of Propositions 4.1, 4.2, 4.3, 4.4 in [LP02].

3.1 Description of the symplectic slice

Let x_e be a $\mathbf{C}_{nv}(k_1R, k_2R', k_pp)$ configuration. Let $k_r = k_1 + k_2$ be the total number of rings. The total number of vortices is then $N = nk_r + k_p$. We suppose the vorticities in each of the k_r rings are κ_j for $j = 1, \ldots, k_r$ while the vorticities of the possible polar vortices are κ_N for the North pole and κ_S for the South pole. In this paper we only consider $k_r = 1$ or 2, but here we describe the more general case.

For each ring $j = 1, \ldots, k_r$ let $s = 1, \ldots, n$ label the vortices in the ring in cyclic order, and define tangent vectors in $T_{x_e}\mathcal{P}$ by

$$\begin{aligned} \alpha^{(\ell)}_{j,\theta} + \mathrm{i}\beta^{(\ell)}_{j,\theta} &= \textstyle\sum_{s=1}^{n} \exp(2\mathrm{i}\pi\ell s/n + \mathrm{i}\ell\phi^0_j)\,\delta\theta_{j,s} \\ \alpha^{(\ell)}_{j,\phi} + \mathrm{i}\beta^{(\ell)}_{j,\phi} &= \textstyle\sum_{s=1}^{n} \exp(2\mathrm{i}\pi\ell s/n + \mathrm{i}\ell\phi^0_j)\,\delta\phi_{j,s} \end{aligned} \tag{3.1}$$

where $\ell = 0, \ldots, n-1$, $\mathrm{i} = \sqrt{-1}$ and $\phi^0_j = 0$ or π/n depending on whether the jth ring of vortices is of type R or R'. Note that $\beta^{(\ell)}_{j,\theta}, \beta^{(\ell)}_{j,\phi}$ vanish for $\ell = 0$ and $n/2$ (for n even). For each pole $j = 1, \ldots, k_p$ we also have tangent vectors δx_j and δy_j.

The tangent vectors defined in the last paragraph are almost canonical, in the sense that

$$\begin{aligned} \omega\left(\alpha^{(\ell)}_{j,\theta}, \alpha^{(\ell)}_{j,\phi}\right) &= \begin{cases} n\sin\theta_j\,\kappa_j & \text{if } \ell = 0, n/2 \\ \frac{1}{2}n\sin\theta_j\,\kappa_j & \text{otherwise} \end{cases} \\ \omega\left(\beta^{(\ell)}_{j,\theta}, \beta^{(\ell)}_{j,\phi}\right) &= \tfrac{1}{2}n\sin\theta_j\,\kappa_j \\ \omega(\delta x_j, \delta y_j) &= \operatorname{sign}(z_j)\,\kappa_j, \end{aligned}$$

while the other pairings vanish.

In order to compute a specific basis for the symplectic slice it is necessary to have expressions for the derivative of the momentum map and the tangent space to the group orbit at a $\mathbf{C}_{nv}(k_1R, k_2R', k_pp)$ configuration. These expressions are given in the next two propositions. Since $\mathbf{C}_{nv}$ refers to fixed vertical reflection planes, the values of ϕ^0_j in (3.1) above can be taken to be:

$$\phi^0_j = \begin{cases} 0 & \text{if } j = 1 \ldots k_1 \\ \pi/n & \text{if } j = (k_1+1) \ldots k_r \end{cases}$$

where $k_r = k_1 + k_2$ is the total number of rings.

Proposition 3.1 *At a $\mathbf{C}_{nv}(k_1R, k_2R', k_pp)$ configuration, the differential of the momentum map is given by,*

$$\begin{aligned} & d\Phi(\delta x, \delta y, \alpha, \beta) \\ &= \sum_{j\,\text{polar}} \kappa_j(\delta x_j + \mathrm{i}\delta y_j) + \sum_{j=1}^{k_r} \kappa_j \cos\theta_j(\alpha^{(1)}_{j,\theta} + \mathrm{i}\beta^{(1)}_{j,\theta}) + \mathrm{i}\sum_{j=1}^{k_r} \kappa_j \sin\theta_j(\alpha^{(1)}_{j,\phi} + \mathrm{i}\beta^{(1)}_{j,\phi}) \\ &\qquad \bigoplus - \sum_{j=1}^{k_r} \kappa_j \sin\theta_j \alpha^{(0)}_{j,\theta} \end{aligned}$$

where the direct sum corresponds to the $\mathbf{C}_{nv}$-invariant decomposition of $\mathfrak{so}(3)^*$ as a direct sum of a plane and the line $\mathrm{Fix}(\mathbf{C}_{nv}, \mathfrak{so}(3)^*)$ *(the "z-axis").*

Proposition 3.2 *Let x_e be a $\mathbf{C}_{nv}(k_1R, k_2R', k_pp)$ configuration, and $\mu = \Phi(x_e)$. If $\mu \neq 0$, then the tangent space to the orbit $\mathfrak{so}(3)_\mu \cdot x_e$ is generated by the vector*

$$\sum_{j=1}^{k_r} \alpha_{j,\phi}^{(0)}.$$

If $\mu = 0$, then $\mathfrak{so}(3)_\mu \cdot x_e = \mathfrak{so}(3) \cdot x_e$ and is generated by the three vectors:

$$\sum_{j=1}^{k_r} \mathrm{i}\left(\alpha_{j,\theta}^{(1)} + \mathrm{i}\beta_{j,\theta}^{(1)}\right) - \cos\theta_j \sin\theta_j \left(\alpha_{j,\phi}^{(1)} + \mathrm{i}\beta_{j,\phi}^{(1)}\right) + \mathrm{i} \sum_{j\,polar} \mathrm{sign}(z_j)(\delta x_j + \mathrm{i}\delta y_j)$$
$$\sum_{j=1}^{k_r} \alpha_{j,\phi}^{(0)}$$

where one must take the real and imaginary parts of the first line.

3.2 A single ring

Let x_e be a $\mathbf{C}_{nv}(R, k_pp)$ configuration, that is a single ring together with k_p polar vortices where, of course, $k_p = 0, 1$ or 2. Since there is only one ring, we write $\alpha_\theta^{(\ell)}, \beta_\theta^{(\ell)}, \alpha_\phi^{(\ell)}, \beta_\phi^{(\ell)}$ instead of $\alpha_{1,\theta}^{(\ell)}, \beta_{1,\theta}^{(\ell)}, \alpha_{1,\phi}^{(\ell)}, \beta_{1,\phi}^{(\ell)}$ in order to lighten the formulae.

The irreducible representations (subspaces) of dimension 1 of the action of G_{x_e} on $T_{x_e}\mathcal{P}$ are:

$$\left\langle \alpha_\theta^{(0)} \right\rangle, \left\langle \alpha_\phi^{(0)} \right\rangle \quad \textit{if } n \textit{ is odd,}$$

while they are

$$\left\langle \alpha_\theta^{(0)} \right\rangle, \left\langle \alpha_\phi^{(0)} \right\rangle, \left\langle \alpha_\theta^{(n/2)} \right\rangle, \left\langle \alpha_\phi^{(n/2)} \right\rangle \quad \textit{if } n \textit{ is even.}$$

Of these, only $\left\langle \alpha_\theta^{(n/2)} \right\rangle$ and $\left\langle \alpha_\phi^{(n/2)} \right\rangle$ lie in the symplectic slice. Moreover, $\left\langle \alpha_\theta^{(n/2)} \right\rangle$ and $\left\langle \alpha_\phi^{(n/2)} \right\rangle$ are $G_{x_e}^\chi$-isomorphic representations, but not G_{x_e}-isomorphic representations.

The following spaces are irreducible representations of dimension 2 of the action of G_{x_e} on $T_{x_e}\mathcal{P}$:

$$\left\langle \alpha_\theta^{(\ell)}, \beta_\theta^{(\ell)} \right\rangle, \left\langle \alpha_\phi^{(\ell)}, \beta_\phi^{(\ell)} \right\rangle, 1 \leq \ell \leq n-1, \ell \neq n/2, \left\langle \delta x_j, \delta y_j \right\rangle, j = 1, \ldots, k_p.$$

The representations $\left\langle \alpha_\theta^{(1)}, \beta_\theta^{(1)} \right\rangle$, $\left\langle \alpha_\phi^{(1)}, \beta_\phi^{(1)} \right\rangle$, $\left\langle \delta x_j, \delta y_j \right\rangle$ do not lie in the symplectic slice, while the others do. Moreover, $\left\langle \alpha_\theta^{(1)}, \beta_\theta^{(1)} \right\rangle$, $\left\langle \alpha_\phi^{(1)}, \beta_\phi^{(1)} \right\rangle$, and $\left\langle \delta x_j, \delta y_j \right\rangle$ are $G_{x_e}^\chi$-isomorphic representations.

3.3 General case

In the general case where x_e is a $\mathbf{C}_{nv}(k_1R, k_2R', k_pp)$ configuration ($k_p = 0, 1$ or 2), we have the following decomposition.

The subspaces $\left\langle \alpha^{(0)}_{j,\theta} \right\rangle$, $\left\langle \alpha^{(0)}_{j,\phi} \right\rangle$, $j = 1,\ldots,k_r$ *are* G_{x_e}*-irreducible representations of dimension* 1 *and are* $G^{\chi}_{x_e}$*-isomorphic representations. If* n *is even, then we have* $2k_r$ *additional* G_{x_e}*-irreducible representations*

$$\left\langle \alpha^{(n/2)}_{j,\theta} \right\rangle, \left\langle \alpha^{(n/2)}_{j,\phi} \right\rangle, \; j = 1,\ldots,k_r$$

which are $G^{\chi}_{x_e}$*-isomorphic.*

The subspaces $\left\langle \alpha^{(\ell)}_{j,\theta}, \beta^{(\ell)}_{j,\theta} \right\rangle$, $\left\langle \alpha^{(\ell)}_{j,\phi}, \beta^{(\ell)}_{j,\phi} \right\rangle$, $j = 1,\ldots,k_r$, $1 \le \ell \le n-1, \ell \neq n/2$ *and* $\langle \delta x_r, \delta y_r \rangle$, $r = 1,\ldots,k_p$ *are* G_{x_e}*-irreducible representations of dimension* 2*. Moreover, the subspaces*

$$\left\langle \alpha^{(1)}_{j,\theta}, \beta^{(1)}_{j,\theta} \right\rangle, \left\langle \alpha^{(1)}_{j,\phi}, \beta^{(1)}_{j,\phi} \right\rangle, \langle \delta x_r, \delta y_r \rangle,$$

($j = 1,\ldots,k_r$, $r = 1,\ldots,k_p$*) are* $G^{\chi}_{x_e}$*-isomorphic representations.*

The difficulty in the case of several rings is that usually the subspaces listed above do not lie in the symplectic slice. One needs therefore to find linear combinations of the above vectors that do lie in the symplectic slice, and such that the *irreducible* and *isomorphism* properties of the representations are preserved. We will give the symmetry adapted bases for the cases of two rings without polar vortices, that is $\mathbf{C}_{nv}(2R)$ and $\mathbf{C}_{nv}(R,R')$ configurations, in the last two sections.

4 A ring of identical vortices: $\mathbf{C}_{nv}(R)$

The linear stability of $\mathbf{C}_{nv}(R)$ relative equilibria was determined by Polvani and Dritschel in [PD93]. Recently, Boatto and Cabral [BC03] studied their Lyapunov stability and found that the two types of stability coincide: whenever the relative equilibrium fails to be Lyapunov stable the linearization of X_H has real eigenvalues. In this section, we give another proof using the geometric method of this paper.

For n vortices of unit vorticity the Hamiltonian is

$$H(\theta_j,\phi_j) \;=\; -\sum_{j<k} \ln\left(1 - \sin\theta_j \sin\theta_k \cos(\phi_j - \phi_k) - \cos\theta_j \cos\theta_k\right)$$

and the augmented Hamiltonian is $H_\xi = H - \xi \sum_j \cos\theta_j$.

Let x_e be a $\mathbf{C}_{nv}(R)$ relative equilibrium and θ_0 the co-latitude of the ring. The angular velocity of x_e is

$$\xi \;=\; \frac{(n-1)\cos\theta_0}{\sin^2\theta_0},$$

since H_ξ has a critical point there and $\frac{\partial H_\xi}{\partial \theta_j}(x_e) = \frac{(n-1)\cos\theta_0 - \xi \sin^2\theta_0}{\sin\theta_0}$.

The second derivatives of H at the relative equilibrium are:

$$\begin{aligned}
\frac{\partial^2 H}{\partial \theta_j^2} &= -\frac{(n-1)(n-5)}{6\sin^2\theta_0} & \qquad \frac{\partial^2 H}{\partial \theta_j \partial \theta_k} &= \frac{1}{2\sin^2\theta_0 \sin^2(\pi(j-k)/n)} \\
\frac{\partial^2 H}{\partial \theta_j \partial \phi_j} &= 0 & \qquad \frac{\partial^2 H}{\partial \theta_j \partial \phi_k} &= 0 \\
\frac{\partial^2 H}{\partial \phi_j^2} &= \textstyle\sum_{r=1}^{n-1} \frac{1}{2\sin^2(r\pi/n)} & \qquad \frac{\partial^2 H}{\partial \phi_j \partial \phi_k} &= -\frac{1}{2\sin^2(\pi(j-k)/n)}.
\end{aligned}$$

We note that $\sum_{r=1}^{n-1} 1/\sin^2(\pi r/n) = \frac{1}{3}(n^2-1)$ [H75].

Notation. In order to harmonize the statements of the results between n even and n odd, we introduce the following notation: let $\eta_1^{(\ell)}, \eta_2^{(\ell)}, \eta_3^{(\ell)}, \eta_4^{(\ell)}$ be objects defined for all $2 \le \ell \le \left[\frac{n-1}{2}\right]$, where $[m]$ is the integer part of $m \in \mathbb{N}$, and only $\eta_1^{(\ell)}, \eta_2^{(\ell)}$ for $\ell = n/2$ when n is even. Then define

$$\left\{\eta_1^{(\ell)}, \eta_2^{(\ell)}, \eta_3^{(\ell)}, \eta_4^{(\ell)} \mid 2 \le \ell \le [n/2]\right\}^*$$

to be

$$\left\{\begin{array}{lcl} \text{for even } n & : & \left\{\eta_1^{(\ell)}, \eta_2^{(\ell)}, \eta_3^{(\ell)}, \eta_4^{(\ell)} \mid 2 \le \ell \le \frac{n}{2} - 1\right\} \cup \left\{\eta_1^{(n/2)}, \eta_2^{(n/2)}\right\} \\ \text{for odd } n & : & \{\eta_1^{(\ell)}, \eta_2^{(\ell)}, \eta_3^{(\ell)}, \eta_4^{(\ell)} \mid 2 \le \ell \le [n/2]\}. \end{array}\right. \tag{4.1}$$

Using this notation, the following proposition gives the symmetry adapted basis for a $\mathbf{C}_{nv}(R)$ configuration.

Proposition 4.1 *Assume $\mu \neq 0$. With respect to the following basis for the symplectic slice $\mathcal{N}$ at x_e,*

$$\left(e_1,\ e_2,\ \left\{\alpha_\theta^{(\ell)}, \alpha_\phi^{(\ell)}, \beta_\theta^{(\ell)}, \beta_\phi^{(\ell)} \mid 2 \le \ell \le [n/2]\right\}^*\right)$$

where

$$\begin{array}{rcl} e_1 & = & \sin\theta_0\ \alpha_\theta^{(1)} + \cos\theta_0\ \beta_\phi^{(1)} \\ e_2 & = & \sin\theta_0\ \beta_\theta^{(1)} - \cos\theta_0\ \alpha_\phi^{(1)} \end{array}$$

the Hessian $d^2H_\xi|_{\mathcal{N}}(x_e)$ block diagonalizes in 1×1 blocks, and $L_{\mathcal{N}}$ block diagonalizes in 2×2 blocks.

Proof. It is straightforward to check that the vectors above do form a basis for the symplectic slice at x_e thanks to Propositions 3.1 and 3.2.

The Hessian $d^2H_\xi|_{\mathcal{N}}(x_e)$ and the linearization $L_{\mathcal{N}}$ are both $G_{x_e}^\chi$-invariant. Assume n odd. It follows from Section 3.2 and Schur's Lemma (see the introduction) that $d^2H_\xi|_{\mathcal{N}}(x_e)$ and $L_{\mathcal{N}}$ both block diagonalize into 4×4 blocks and one 2×2 block corresponding to the subspaces $V_\ell = \left\langle \alpha_\theta^{(\ell)}, \beta_\theta^{(\ell)}, \alpha_\phi^{(\ell)}, \beta_\phi^{(\ell)} \right\rangle$ and $\langle e_1, e_2 \rangle$, respectively. See the proof of Theorem 4.5 of [LP02] for a detailed proof of a similar assertion.

Now fix ℓ and denote by s an anti-symplectic (time-reversing) element of G_{x_e}. For example s could be the reflection $y \mapsto -y$ together with an order two permutation of S_n. The restriction of H_ξ to V_ℓ is $\mathbb{Z}_2[s]$-invariant. Moreover $\left\langle \alpha_\theta^{(\ell)}, \beta_\phi^{(\ell)} \right\rangle$ and $\left\langle \beta_\theta^{(\ell)}, \alpha_\phi^{(\ell)} \right\rangle$ are non-isomorphic irreducible representation of $\mathbb{Z}_2[s]$ on V_ℓ. Hence $d^2H_\xi|_{\mathcal{N}}(x_e)$ block diagonalizes into 2×2 blocks which correspond to subspaces $\left\langle \alpha_\theta^{(\ell)}, \beta_\phi^{(\ell)} \right\rangle$, $\left\langle \beta_\theta^{(\ell)}, \alpha_\phi^{(\ell)} \right\rangle$, and $\langle e_1, e_2 \rangle$. This result does not depend on the details of the Hamiltonian, only its symmetries. However taking in account its particular form, one can improve the block diagonalization. Indeed one has

$$d^2H_\xi(x_e) \cdot (\alpha_\theta^{(\ell)}, \beta_\phi^{(\ell)}) \ = \ d^2H_\xi(x_e) \cdot (\beta_\theta^{(\ell)}, \alpha_\phi^{(\ell)}) \ = \ 0$$

and $d^2H_\xi(x_e) \cdot (e_1, e_2) = 0$ which gives the desired diagonalization of the Hessian.

The particular form of the symplectic form also enables us to improve the diagonalization of $L_{\mathcal{N}}$. Among the basis vectors of V_ℓ, only $\omega(\alpha_\theta^{(\ell)}, \alpha_\phi^{(\ell)})$ and $\omega(\beta_\theta^{(\ell)}, \beta_\phi^{(\ell)})$ do not vanish, and so the restriction of ω to V_ℓ block diagonalizes into two 2×2 blocks which correspond to the subspaces $\left\langle \alpha_\theta^{(\ell)}, \alpha_\phi^{(\ell)} \right\rangle$ and $\left\langle \beta_\theta^{(\ell)}, \beta_\phi^{(\ell)} \right\rangle$. The block diagonalization of $L_{\mathcal{N}}$ then follows from $L_{\mathcal{N}} = \mathbf{J}_{\mathcal{N}} d^2H_\xi|_{\mathcal{N}}(x_e)$.

The case n even is very similar, except that there is an additional 2×2 block in the $G^{\chi}_{x_e}$-isotypic decomposition, and leads to the same result. □

The block diagonalization of $d^2H_\xi|_{\mathcal{N}}(x_e)$ and $L_{\mathcal{N}}$ enable us to find formulae for their eigenvalues, and thus to conclude criteria for both Lyapunov and linear stability.

Theorem 4.2 *The stability of a ring of n identical vortices depends on n and the co-latitude* θ_0 *as follows:*

n=2 *is Lyapunov stable at all latitudes;*

n=3 *is Lyapunov stable at all latitudes;*

n=4 *is Lyapunov stable if* $\cos^2\theta_0 > 1/3$*, and linearly unstable if the inequality is reversed;*

n=5 *is Lyapunov stable if* $\cos^2\theta_0 > 1/2$*, and linearly unstable if the inequality is reversed;*

n=6 *is Lyapunov stable if* $\cos^2\theta_0 > 4/5$*, and linearly unstable if the inequality is reversed;*

n≥7 *is always (linearly) unstable.*

Proof. Any arrangement of two vortices is a relative equilibrium [KN98]. When perturbing such a relative equilibrium, we obtain a new relative equilibrium close to the first. Thus any relative equilibrium of two vortices is Lyapunov stable modulo $\mathrm{SO}(2)$ (modulo $\mathrm{SO}(3)$ if $\mu = 0$).

Hence assume $n \geq 3$. We first study Lyapunov stability. Suppose further that $\mu \neq 0$ and so the ring is not equatorial. A simple calculation shows that $d^2H_\xi(x_e)\cdot(\beta_\theta^{(\ell)},\beta_\theta^{(\ell)}) = d^2H_\xi(x_e)\cdot(\alpha_\theta^{(\ell)},\alpha_\theta^{(\ell)})$ and $d^2H_\xi(x_e)\cdot(\beta_\phi^{(\ell)},\beta_\phi^{(\ell)}) = d^2H_\xi(x_e)\cdot(\alpha_\phi^{(\ell)},\alpha_\phi^{(\ell)})$. Hence it follows from Proposition 4.1 that

$$d^2H_\xi|_{\mathcal{N}}(x_e) \;=\; \mathrm{diag}\left(\lambda_1,\lambda_1,\{\lambda_\theta^{(\ell)},\lambda_\phi^{(\ell)},\lambda_\theta^{(\ell)},\lambda_\phi^{(\ell)} \mid 2\leq \ell \leq [n/2]\}^*\right)$$

(recall notation from (4.1)) where

$$\begin{aligned}
\lambda_1 &= \sin^2\theta_0\lambda_\theta^{(1)} + \cos^2\theta_0\lambda_\phi^{(1)},\\
\lambda_\theta^{(\ell)} &= d^2H_\xi(x_e)\cdot(\alpha_\theta^{(\ell)},\alpha_\theta^{(\ell)}),\\
\lambda_\phi^{(\ell)} &= d^2H_\xi(x_e)\cdot(\alpha_\phi^{(\ell)},\alpha_\phi^{(\ell)}).
\end{aligned}$$

Thanks to the following formula [H75] (p.271)

$$\sum_{j=1}^{n-1}\frac{\cos(2\pi\ell j/n)}{\sin^2(\pi j/n)} \;=\; \frac{1}{3}(n^2-1) - 2\ell(n-\ell),$$

we find after some computations that $\lambda_\phi^{(\ell)} = n\ell(n-\ell)/2$ and

$$\lambda_\theta^{(\ell)} \;=\; \frac{n}{2\sin^2\theta_0}\left[-(\ell-1)(n-\ell-1) + (n-1)\cos^2\theta_0\right].$$

The eigenvalues $\lambda_\phi^{(\ell)}$ are all positive and $\lambda_1 = n(n-1)\cos^2\theta_0 > 0$, thus the relative equilibrium is Lyapunov stable (modulo $\mathrm{SO}(2)$) if $(n-1)\cos^2\theta_0 > (\ell-1)(n-\ell-1)$ for all $\ell = 2,\ldots,[n/2]$, that is if $\cos^2\theta_0 > ([n/2]-1)(n-[n/2]-1)/(n-1)$. This gives the desired values.

We now turn to linear stability. It follows from Proposition 4.1 and the block diagonalization of $d^2H_\xi|_{\mathcal{N}}(x_e)$ that

$$L_{\mathcal{N}} = \operatorname{diag}\left(\begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}, \left\{\begin{pmatrix} 0 & -\lambda_\phi^{(\ell)} \\ \lambda_\theta^{(\ell)} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\lambda_\phi^{(\ell)} \\ \lambda_\theta^{(\ell)} & 0 \end{pmatrix} \mid 2 \le \ell \le [n/2]\right\}^*\right)$$

where the blocks are given up to a strictly positive scalar factor. The eigenvalues of $L_{\mathcal{N}}$ are therefore

$$\pm i\lambda_1, \left\{\pm i\sqrt{\lambda_\theta^{(\ell)}\lambda_\phi^{(\ell)}} \mid 2 \le \ell \le [n/2]\right\},$$

(up to a positive factor) and so the relative equilibrium is linearly unstable if $\lambda_\theta^{(\ell)} > 0$ for some ℓ, that is if

$$\cos^2\theta_0 < \frac{1}{n-1}\left(\left[\frac{n}{2}\right]-1\right)\left(\left[\frac{n+1}{2}\right]-1\right).$$

In particular this inequality is satisfied if $\theta_0 = \pi/2$ and $n > 3$.

When the ring is equatorial, one has $\theta_0 = \pi/2$ and $\mu = 0$. In particular $\lambda_1 = 0$. This is because the symplectic slice is smaller ($G_{\mu=0} = \mathrm{SO}(3)$): it follows from Proposition 3.2 that we have to remove the vectors e_1, e_2 from the basis for $\mu \neq 0$ (that is to remove λ_1 from the previous eigenvalue study). It follows that the $\mathbf{C}_{nv}$ equatorial relative equilibria are linearly unstable for $n > 3$, and Lyapunov stable (modulo $\mathrm{SO}(3)$) for $n = 3$. □

The proof shows that the 'critical mode' for stability is $\ell = [n/2]$. For $n \ge 7$ a ring is always unstable to this mode, while for $4 \le n \le 6$ the ring first loses stability to this mode as it moves closer to the equator. This loss of stability is accompanied by a pitchfork bifurcation to a pair of staggered rings when $n = 4$ or 6 (i.e. to types $\mathbf{C}_{2v}(R,R')$ and $\mathbf{C}_{3v}(R,R')$ respectively, in the notation of [LMR01]). In the case $n = 5$ the bifurcation is transcritical to an "equatorial" vortex and two pairs that are reflections of each other in that equator (ie type $C_h(2R,E)$ in [LMR01]). The bifurcations for $n = 4$ and 5 are illustrated in Figures 7 and 8 of [LMR01] respectively.

5 A ring and a polar vortex: $\mathbf{C}_{nv}(R,p)$

We assume that the polar vortex lies at the North pole and its vorticity is κ, while the remaining n vortices are all identical with vortex strength 1 and lie in a ring. The relative equilibrium is of symmetry type $\mathbf{C}_{nv}(R,p)$ and denoted x_e. In this case, the Hamiltonian is given by

$$H = H_r + H_p$$

where H_r is the ring Hamiltonian given in the previous section and

$$H_p(x,y,\theta_i,\phi_i) = -\kappa\sum_{j=1}^{n}\ln\left(1 - x\sin\theta_j\cos\phi_j - y\sin\theta_j\sin\phi_j - \sqrt{1-x^2-y^2}\cos\theta_j\right),$$

is the Hamiltonian responsible for the interaction of the pole and the ring.

In this case

$$H_\xi = H - \xi\left(\sum_j \cos\theta_j + \kappa\sqrt{1-x^2-y^2}\right)$$

and the relative equilibrium at $x = y = 0,\ \theta_j = \theta_0,\ \phi_j = 2\pi j/n$ has angular velocity

$$\xi \;=\; \frac{(n-1)\cos\theta_0 + \kappa(1+\cos\theta_0)}{\sin^2\theta_0}$$

since $\frac{\partial H_\xi}{\partial \theta_j}(x_e) = -\frac{(n-1)\cos\theta_0+\kappa(1+\cos\theta_0)-\xi\sin^2\theta_0}{\sin\theta_0}$ must vanish.

The second derivatives at the relative equilibrium of H can be derived from those for H_r given in the previous section, together with:

$$\begin{array}{rclrcl}
\frac{\partial^2 H_p}{\partial\theta_j^2} & = & \frac{\kappa}{1-\cos\theta_0} & \frac{\partial^2 H_p}{\partial x^2} & = & \frac{n\kappa}{2} = \frac{\partial^2 H_p}{\partial y^2} \\
\frac{\partial^2 H_p}{\partial x\partial\theta_j} & = & -\frac{\kappa\cos(2\pi j/n)}{1-\cos\theta_0} & \frac{\partial^2 H_p}{\partial y\partial\theta_j} & = & -\frac{\kappa\sin(2\pi j/n)}{1-\cos\theta_0} \\
\frac{\partial^2 H_p}{\partial x\partial\phi_j} & = & -\frac{\kappa\sin\theta_0\sin(2\pi j/n)}{1-\cos\theta_0} & \frac{\partial^2 H_p}{\partial y\partial\phi_j} & = & \frac{\kappa\sin\theta_0\cos(2\pi j/n)}{1-\cos\theta_0},
\end{array}$$

while the other second derivatives all vanish. Here we have used that $\sum\cos^2(2\pi j/n) = n/2$ for $n > 2$, but for $n = 2$ the sum is 2. Thus for $n = 2$, one obtains

$$\frac{\partial^2 H_p}{\partial x^2} = \frac{2\kappa}{1-\cos\theta_0} \qquad \frac{\partial^2 H_p}{\partial y^2} = -\frac{2\kappa\cos\theta_0}{1-\cos\theta_0}.$$

The following proposition gives the symmetry adapted basis for $\mathbf{C}_{nv}(R,p)$ relative equilibria.

Proposition 5.1 *Let $n \geq 3$ and $\mu \neq 0$. With respect to the following basis for the symplectic slice:*

$$\left(e_1, e_2, e_3, e_4, \left\{\alpha_\theta^{(\ell)}, \alpha_\phi^{(\ell)}, \beta_\theta^{(\ell)}, \beta_\phi^{(\ell)} \mid 2 \leq \ell \leq [n/2]\right\}^*\right)$$

where

$$\begin{array}{rcl}
e_1 & = & \cos\theta_0\ \beta_\theta^{(1)} - \sin\theta_0\ \alpha_\phi^{(1)} - n\cos(2\theta_0)/(2\kappa)\ \delta y_n \\
e_2 & = & \sin\theta_0\ \alpha_\theta^{(1)} + \cos\theta_0\ \beta_\phi^{(1)} \\
e_3 & = & \cos\theta_0\ \alpha_\theta^{(1)} + \sin\theta_0\ \beta_\phi^{(1)} - n\cos(2\theta_0)/(2\kappa)\ \delta x_n \\
e_4 & = & \sin\theta_0\ \beta_\theta^{(1)} - \cos\theta_0\ \alpha_\phi^{(1)}
\end{array}$$

the Hessian $d^2H_\xi|_{\mathcal{N}}(x_e)$ block diagonalizes into 1×1 blocks and two 2×2 blocks, and $L_{\mathcal{N}}$ block diagonalizes into 2×2 blocks and one 4×4 block.

Proof. The proof is similar to the proof of Proposition 4.1. □

These block diagonalizations enable us to prove the following stability theorem for $n \geq 4$, illustrated by Figures 5.2 and 5.1. The cases $n = 2$ and 3 are treated afterwards.

Theorem 5.2 *A $\mathbf{C}_{nv}(R,p)$ relative equilibrium with $n \geq 4$ and $\mu \neq 0$*
(i) is spectrally unstable if and only if

$$\kappa < \kappa_0 \ \ or\ \ 8a\kappa > (n\sin^2\theta_0 + 4(n-1)\cos\theta_0)^2,$$

(ii) is Lyapunov stable if

$$\kappa > \kappa_0 \ \ and\ \ a\kappa(\kappa + n\cos\theta_0)(\kappa - \kappa_1) < 0,$$

where

$$\begin{aligned} a &= (n\cos\theta_0 - n + 2)(1+\cos\theta_0)^2 \\ \kappa_1 &= (n-1)\cos\theta_0\,(n\sin^2\theta_0 + 2(n-1)\cos\theta_0)/a \\ \kappa_0 &= (c_n - (n-1)(1+\cos^2\theta_0))/(1+\cos\theta_0)^2 \\ c_n &= \begin{cases} n^2/4 & \textit{if } n \textit{ is even,} \\ (n^2-1)/4 & \textit{if } n \textit{ is odd.} \end{cases} \end{aligned} \tag{5.1}$$

Proof. We first study the Lyapunov stability. Following the beginning of the proof of Theorem 4.2, we obtain from Proposition 5.1 that

$$d^2H_\xi|_{\mathcal{N}}(x_e) = \mathrm{diag}(A, A, D)$$

where $D = \mathrm{diag}(\{\lambda_\theta^{(\ell)}, \lambda_\phi^{(\ell)}, \lambda_\theta^{(\ell)}, \lambda_\phi^{(\ell)} \mid 2 \le \ell \le [n/2]\}^*)$,

$$A = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

and

$$\begin{aligned} \lambda_\theta^{(\ell)} &= d^2H_\xi(x_e)\cdot(\alpha_\theta^{(\ell)}, \alpha_\theta^{(\ell)}) \\ \lambda_\phi^{(\ell)} &= d^2H_\xi(x_e)\cdot(\alpha_\phi^{(\ell)}, \alpha_\phi^{(\ell)}) \\ q_{11} &= d^2H_\xi(x_e)\cdot(e_1, e_1) \\ q_{12} &= d^2H_\xi(x_e)\cdot(e_1, e_2) \\ q_{22} &= d^2H_\xi(x_e)\cdot(e_2, e_2). \end{aligned}$$

Note that D exists only for $n \ge 4$. From the previous section one has $\lambda_\phi^{(\ell)} = n\ell(n-\ell)/2$ and some additional computations give

$$\lambda_\theta^{(\ell)} = \frac{n}{2\sin^2\theta_0}\left[-(\ell-1)(n-\ell-1) + (n-1)\cos^2\theta_0 + \kappa(1+\cos\theta_0)^2\right].$$

The eigenvalues $\lambda_\phi^{(\ell)}$ are all positive, thus D is definite if $-(\ell-1)(n-\ell-1) + (n-1)\cos^2\theta_0 + \kappa(1+\cos\theta_0)^2 > 0$ for all $\ell = 2, \ldots, [n/2]$, that is if $\kappa > (([n/2]-1)(n-[n/2]-1) - (n-1)\cos^2\theta_0)/(1+\cos\theta_0)^2$ which corresponds to $\kappa > \kappa_0$.

The relative equilibrium is therefore Lyapunov stable if A is positive definite, that is if $q_{11}q_{22} - q_{12}^2 > 0$ and $q_{11} + q_{22} > 0$. Some lengthy computations give

$$\begin{aligned} q_{11}q_{22} - q_{12}^2 &= -\tfrac{n^2\cos^2 2\theta_0}{\kappa^2\sin^2\theta_0} a\kappa(\kappa + n\cos\theta_0)(\kappa - \kappa_1) \\ q_{22} &= \tfrac{n}{2(1+\cos\theta_0)^2}(\kappa - \kappa_2) \end{aligned}$$

where a, κ_1 are given in the theorem and $\kappa_2 = -2(n-1)\cos^2\theta_0/(1+\cos\theta_0)^2$. Now we show that if $q_{11}q_{22} - q_{12}^2 > 0$ and $\kappa > \kappa_0$, then $q_{11} + q_{22} > 0$: we have $q_{22}(q_{11} + q_{22}) > 0$ since $q_{11}q_{22} - q_{12}^2 > 0$, and $q_{22} > 0$ since $\kappa > \kappa_0 > \kappa_2$, hence $q_{11} + q_{22} > 0$. We proved therefore that $\mathbf{C}_{nv}(R, p)$ is Lyapunov stable if $\kappa > \kappa_0$ and $a\kappa(\kappa + n\cos\theta_0)(\kappa - \kappa_1) < 0$.

We now study the spectral stability of the relative equilibrium. It follows from Proposition 5.1 and the block diagonalization of $d^2H_\xi|_{\mathcal{N}}(x_e)$ that

$$L_{\mathcal{N}} = \mathrm{diag}\left(A_L, \left\{\begin{pmatrix} 0 & -\lambda_\phi^{(\ell)} \\ \lambda_\theta^{(\ell)} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\lambda_\phi^{(\ell)} \\ \lambda_\theta^{(\ell)} & 0 \end{pmatrix} \mid 2 \le \ell \le [n/2]\right\}^*\right)$$

where the blocks are given up to a positive scalar factor and

$$A_L = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -b & 0 & 0 \\ -c & -d & 0 & 0 \end{pmatrix}, \quad \begin{cases} a &= \beta q_{11} - \gamma q_{12} \\ b &= \beta q_{12} - \gamma q_{22} \\ c &= \alpha q_{12} - \gamma q_{11} \\ d &= \alpha q_{22} - \gamma q_{12} \end{cases}$$

and

$$\begin{aligned} \alpha &= \omega(e_1, e_3) = n\cos\theta_0 \sin^2\theta_0 - n^2\cos^2(2\theta_0)/(4\kappa) \\ \beta &= \omega(e_2, e_4) = n\cos\theta_0 \sin^2\theta_0 \\ \gamma &= \omega(e_1, e_4) = \omega(e_2, e_3) = n\sin\theta_0/2. \end{aligned}$$

The eigenvalues (up to a positive factor) of $L_{\mathcal{N}}$ are therefore

$$\pm\frac{1}{\sqrt{2}}\sqrt{\sigma \pm \sqrt{\nu}}\,, \left\{ \pm \mathrm{i}\sqrt{\lambda_\theta^{(\ell)}\lambda_\phi^{(\ell)}} \mid 2 \le \ell \le [n/2] \right\}^*$$

where $\nu = a^4 + 4a^2bc - 2a^2d^2 + 4bcd^2 + d^4 + 8adbc$ and $\sigma = -a^2 - 2bc - d^2$. The eigenvalues $\pm i\sqrt{\lambda_\theta^{(\ell)}\lambda_\phi^{(\ell)}}$ are all purely imaginary if and only if $\kappa > \kappa_0$. After some lengthy but straightforward computations we obtain that

$$\begin{aligned} \nu &= \tfrac{n^{10}}{2^{12}\kappa^4}(1+u)^2(2u^2-1)^8(\kappa+nu)^4(1-u)^2 \\ &\quad \times \left[-8(1+u)^2(nu+2-n)\kappa + (nu^2 - 4(n-1)u - n)^2\right] \\ \sigma &= -\tfrac{n^4}{2^6\kappa^2}(2u^2-1)^4(\kappa+nu)^2 \\ &\quad \times \left[-4(1+u)^2(nu+2-n)\kappa + n^2u^4 - 4n(n-1)u^3 + \right. \\ &\qquad \left. +2(3n^2-8n+4)u^2 + 4n(n-1)u + n^2\right] \end{aligned}$$

where $u = \cos\theta_0$. One can check that if $\nu \ge 0$, then $\sqrt{\nu} + \sigma \le 0$ and the eigenvalues are purely imaginary. If $\nu < 0$, then the eigenvalues have a non-zero real part. Thus the eigenvalues $\pm\sqrt{\sigma \pm \sqrt{\nu}}$ are purely imaginary if and only if $\nu \ge 0$ which is equivalent to $8a\kappa \le (n\sin^2\theta_0 + 4(n-1)\cos\theta_0)^2$. □

A spectrally stable relative equilibrium for which the Hessian $d^2H_\xi|_{\mathcal{N}}(x_e)$ is not definite is said to be *elliptic*. Note that in principle an elliptic relative equilibrium may be Lyapunov stable, but if there are more than 4 vortices then it is expected to be unstable as a result of Arnold diffusion. Moreover an elliptic relative equilibrium typically becomes linearly unstable when some dissipation is added to the system [DR02]; however adding dissipation to the point vortex system would have more profound effects, such as spreading of vorticity into vortex patches.

Corollary 5.3 *A $\mathbf{C}_{nv}(R,p)$ relative equilibrium with $n \ge 4$ and $\mu \ne 0$ is elliptic if and only if*

$$\kappa \ge \kappa_0, \;\; a\kappa(\kappa + n\cos\theta_0)(\kappa - \kappa_1) \ge 0 \;\; \textit{and} \;\; 8a\kappa \le (n\sin^2\theta_0 + 4(n-1)\cos\theta_0)^2,$$

where a, κ_0 and κ_1 are given in (5.1).

Discussion of results for $n \ge 4$ See Figure 5.1.

- If the sign of the vorticity of the polar vortex is opposite to that of the ring then there are stable configurations with $\kappa < 0$ only for $n \le 6$. Conversely configurations with $n \le 6$ and θ_0 close to π, ie with the ring close to the opposite pole, are Lyapunov stable for all $\kappa < 0$.

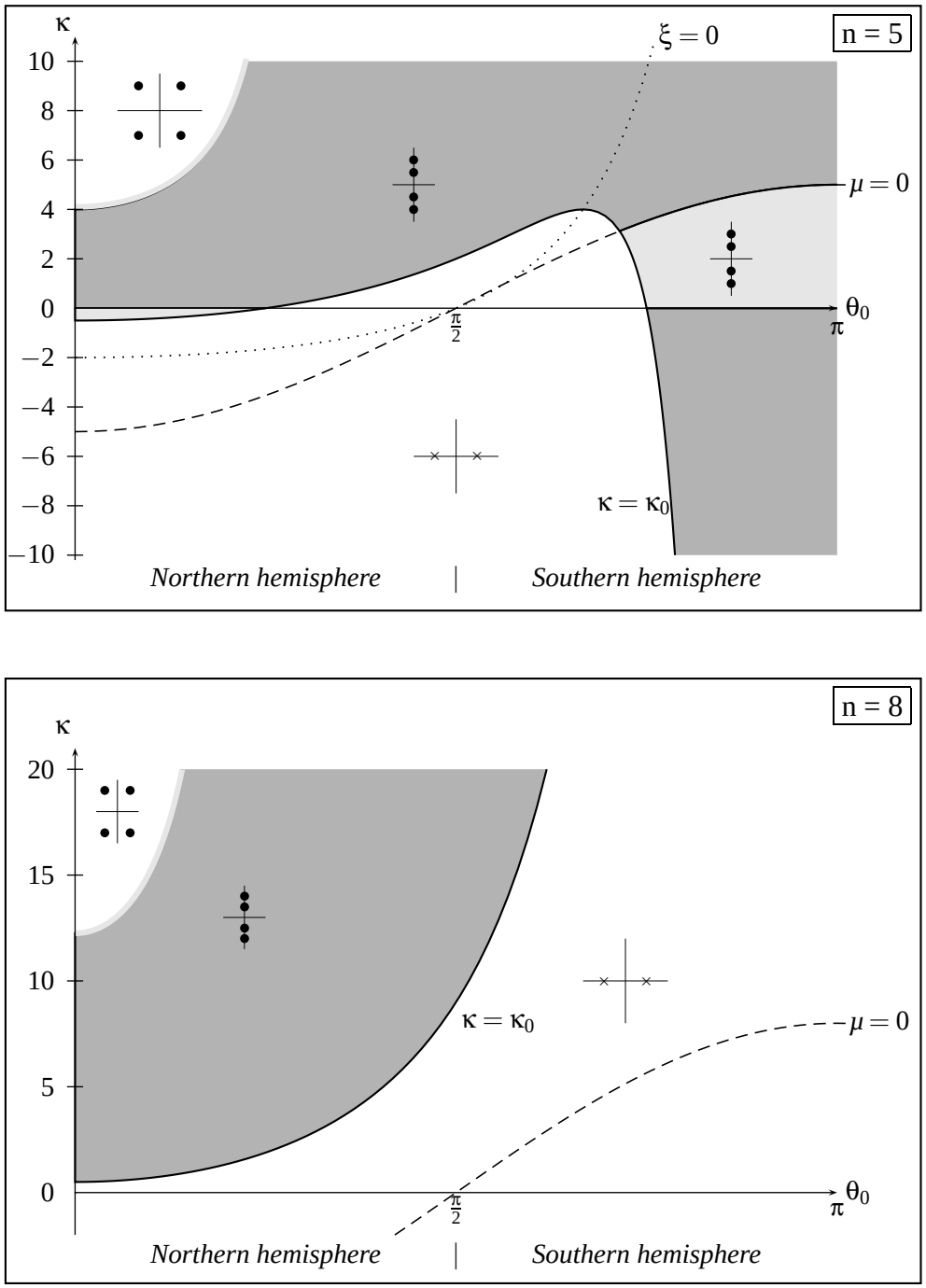

Figure 5.1: Bifurcation diagrams for $\mathbf{C}_{nv}(R,p)$ relative equilibria. The bifurcation diagrams for $n \geq 7$ are similar to that for $n = 8$, while those for $n = 4$ and 6 are similar to that for $n = 5$. The circles represent the eigenvalues of the mode $\ell = 1$, while the crosses represent those of the mode $[n/2]$. The dark regions correspond to Lyapunov stable relative equilibria, the light grey regions to elliptic ones (notice the sliver of light grey near the upper left hand corner of both diagrams: these are not drawn to scale as they are too small to appear at this scale—cf. Fig 5.2, $n = 3$, where it is drawn to scale) while the white areas correspond to unstable relative equilibria. Stability is modulo $\mathrm{SO}(2)$ rotations about the vertical axis, or modulo all rotations if $\mu = 0$—see text.

- The region of Lyapunov stability is larger when the vorticities of the pole and the ring have the same sign ($\kappa > 0$). The stability frontiers in the upper-left corners of Figure 5.1 go the infinity when θ_0 goes to $\arccos(1-2/n)$. It follows that for $n \geq 4$ and $\theta_0 > \arccos(1-2/n)$, the relative equilibria are Lyapunov stable for all sufficiently large κ. Thus, a ring of vortices is stabilized by a polar vortex with a sufficiently large vorticity of the same sign as the vortices in the ring. Note that for $4 \leq n \leq 6$ and κ positive, but sufficiently small, a ring near the opposite pole is only elliptic and may not be Lyapunov stable.

- The limiting stability results for $\theta_0 = 0$, ie when the ring is close to the polar vortex, coincide with the stability of a planar n-ring plus a central vortex, see [CS99] and [LP]. This is also true for $n = 2$ and $n = 3$.

- One of the main stability boundaries corresponds to the mode $[n/2]$ and is analogous to the stability boundary for a single ring. When n is even stability is probably lost through a pitchfork bifurcation to a relative equilibrium of type $\mathbf{C}_{\frac{n}{2}v}(R, R', p)$ consisting of two staggered $\frac{n}{2}$-rings and a pole as (κ, θ_0) passes through this boundary. When n is odd there is an analogous transcritical bifurcation to relative equilibria with only a single reflectional symmetry which fixes two vortices and permutes the others. These are denoted by $\mathbf{C}_h(\frac{n-1}{2}R, 2E)$ in [LMR01]. A nice illustration in the case $n = 3$ can be found in Figure 8 of [CMS03].

- The other stability boundary corresponds to the mode 1. Stability is lost through a Hamiltonian-Hopf bifurcation: two pairs of imaginary eigenvalues 'collide' and leave the imaginary axis. For a detailed description of the bifurcations that can be expected in this case see [vM85].

- Note that it also happens that pairs of $\ell = 1$ eigenvalues pass through zero without leaving the imaginary axis. In this case the relative equilibrium changes from being Lyapunov stable to elliptic or vice versa and these stability changes are accompanied by bifurcations.

- Finally we note that when κ crosses zero eigenvalues change sign without crossing zero due to the fact that the symplectic form becomes degenerate for $\kappa = 0$.

Discussion of the case $n = 3$ See Figure 5.2.

We assume in this discussion that the relative equilibria have non-zero momentum. For $n = 3$, by the proof of Theorem 5.2 we have $d^2H_\xi|_{\mathcal{N}}(x_e) = \mathrm{diag}(A, A)$ and $L_N = A_L$. Hence $\mathbf{C}_{3v}(R, p)$ is Lyapunov stable if $a\kappa(\kappa + 3\cos\theta_0)(\kappa - \kappa_1) < 0$, and spectrally unstable if and only if

$$8a\kappa > (3\sin^2\theta_0 + 8\cos\theta_0)^2,$$

where $a = (3\cos\theta_0 - 1)(1 + \cos\theta_0)^2$ as in Theorem 5.2. These results are illustrated in Figure 5.2. Notice that a polar vortex *destabilizes* a 3-ring if either the polar vortex is in the same hemisphere as the ring and has a sufficiently strong vorticity of the same sign as the ring, or the polar vortex has the opposite sign vorticity and the ring lies in an interval containing $\theta_0 = 2\pi/3$ that grows as the magnitude of the polar vorticity increases. Outside these regions there is a patchwork of regimes in which the relative equilibrium is either Lyapunov stable or elliptic.

The transition point where $\mu = \xi = 0$ and $\kappa = 1$ (and $\cos(\theta_0) = -1/3$) corresponds to the stable equilibrium consisting of 4 identical vortices placed at the vertices of a regular tetrahedron [PM98, LMR01, Ku04].

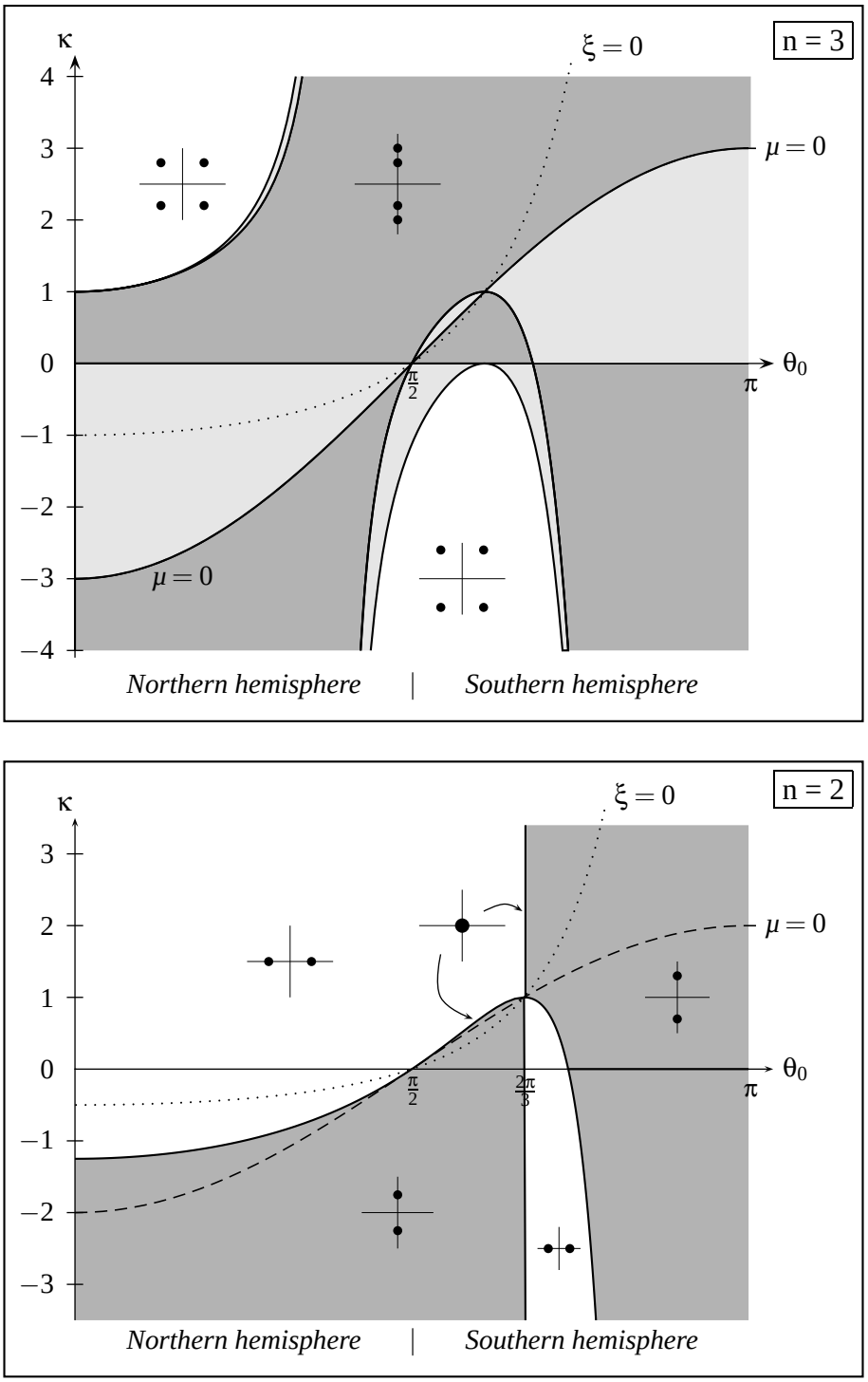

Figure 5.2: Bifurcation diagrams for $\mathbf{C}_{3v}(R,p)$ and $\mathbf{C}_{2v}(R,p)$ relative equilibria; the polar vortex of strength κ is at the North pole. The darker grey regions are where the relative equilibrium is Lyapunov stable, the pale grey regions are elliptic regions and the white regions are those where there is a real eigenvalue (spectrally unstable relative equilibria). Notice the narrow sliver of an elliptic region in the top left-hand portion of the diagram for $n = 3$. Stability is modulo $\mathrm{SO}(2)$ about the polar axis, or modulo $\mathrm{SO}(3)$ when $\mu = 0$ (see text). The circles represent the eigenvalues of the mode $\ell = 1$.

Discussion of the case $n = 2$ See Figure 5.2.

The $\mathbf{C}_{2v}(R,p)$ relative equilibria are isosceles triangles lying on a great circle, and for $\theta_0 = 2\pi/3$ the triangle becomes equilateral. We again discuss the stability of those with non-zero momenta. Indeed, any 3-vortex configuration with zero momentum is a relative equilibrium since the reduced space is just a point, and is consequently also Lyapunov stable relative to $\mathrm{SO}(3)$ [Pa92].

For $n = 2$ the symmetry adapted basis is $(\kappa\delta\theta_1 - \kappa\delta\theta_2 - 2\cos\theta_0\,\delta x, \kappa\delta\phi_1 - \kappa\delta\phi_2 - 2\sin\theta_0\,\delta y)$. Following the proof of Theorem 5.2 we obtain after some straightforward computations that $\mathbf{C}_{2v}(R,p)$ is Lyapunov stable if

$$(1+2\cos\theta_0)[(1+\cos\theta_0)^2\kappa + \cos\theta_0(2+3\cos\theta_0)] < 0,$$

and spectrally unstable if the inequality is reversed. See Figure 5.2.

- There are two stable regions. For $\theta_0 < 2\pi/3$ the relative equilibria are stable provided the polar vorticity is less than a certain θ_0 dependent critical value, while for $\theta_0 > 2\pi/3$ they are stable for all polar vorticities greater than a critical value. As $\theta_0 \to \pi$ this value goes to $-\infty$.

- For $\theta_0 = \pi/2$, where the 2-ring is equatorial and the isosceles triangle is right-angled, they are stable if and only if $\kappa < 0$. This is in agreement with [PM98, Theorem III.3], with $\Gamma_1 = \Gamma_2 = 1$, and $\Gamma_3 = \kappa$.

- *The restricted three vortex problem* The range of stability when $\kappa = 0$ does not coincide with the range of stability for a single ring. Indeed the $\mathbf{C}_{2v}(R)$ relative equilibria are Lyapunov stable for all co-latitudes (see Theorem 4.2) while $\mathbf{C}_{2v}(R,p)$ is unstable for $\kappa = 0$ and $\theta_0 \in (0,\pi/2)$. This means that if we place a *passive tracer* or *ghost vortex* at the North pole and a ring of two vortices in the Northern hemisphere, then the passive tracer will be unstable.

Remark 5.4 The stability of $\mathbf{C}_{nv}(R,p)$ relative equilibria has also been studied in [CMS03]. However our method differs significantly from theirs in that we consider the definiteness of the Hessian $d^2H_\xi|_{\mathcal{N}}(x_e)$ on the $2n$ dimensional symplectic slice, while in [CMS03] the authors determine conditions for the Hessian to be definite on the whole $2(n+1)$ dimensional tangent space. The result is that we prove the relative equilibria to be Lyapunov stable in a larger region of the parameter space. Notice in particular that for $n \leq 6$ our results say that a positive vorticity n-ring near the south pole is Lyapunov stable if the north pole has either negative or sufficiently positive vorticity. However in [CMS03] only the case of negative north polar vorticity is shown to be Lyapunov stable. In this paper we also give criteria for when the relative equilibria are *unstable* by considering the eigenvalues of the linearization $L_{\mathcal{N}}$.

6 Stability of a ring and two polar vortices: $\mathbf{C}_{nv}(R,2p)$

In this section we consider a relative equilibrium x_e of symmetry type $\mathbf{C}_{nv}(R,2p)$; that is configurations formed of a ring of n vortices of strength 1, together with two polar vortices p_N, p_S of strengths κ_N, κ_S respectively at the North and South poles. We assume without loss of generality that the ring lies in the Northern hemisphere.

We obtain analytic (in)stability criteria for the relative equilibria with respect to the $\ell \geq 2$ modes, which of course give sufficient conditions for genuine instability. A numerical investigation is needed for the $\ell = 1$ mode and hence to provide stability criteria; this is being pursued separately.

The Hamiltonian is given by

$$H = H_r + H_{p_N} + H_{p_S} + H_{NS}$$

where H_r is given in Section 4 and

$$\begin{aligned}
H_{p_N} &= -\kappa_N \textstyle\sum_{i=1}^n \kappa_i \ln(1 - \sin\theta_i \cos\phi_i\, x_N - \sin\theta_i \sin\phi_i\, y_N - \sqrt{1 - x_N^2 - y_N^2}\cos\theta_i) \\
H_{p_S} &= -\kappa_S \textstyle\sum_{i=1}^n \kappa_i \ln(1 - \sin\theta_i \cos\phi_i\, x_S - \sin\theta_i \sin\phi_i\, y_S + \sqrt{1 - x_S^2 - y_S^2}\cos\theta_i) \\
H_{NS} &= -\kappa_N \kappa_S \ln(1 - x_N x_S - y_N y_S + \sqrt{1 - x_N^2 - y_N^2}\sqrt{1 - x_S^2 - y_S^2}),
\end{aligned}$$

and the augmented Hamiltonian is:

$$H_\xi = H - \xi \left(\sum_{i=1}^n \kappa_i \cos\theta_i + \kappa_N \sqrt{1 - x_N^2 - y_N^2} - \kappa_S \sqrt{1 - x_S^2 - y_S^2} \right).$$

The angular velocity of the relative equilibrium at $x_N = y_N = x_S = y_S = 0,\ \theta_j = \theta_0,\ \phi_j = 2\pi j/n$ has angular velocity

$$\xi = \frac{(n-1)\cos\theta_0 + \kappa_N(1 + \cos\theta_0) - \kappa_S(1 - \cos\theta_0)}{\sin^2\theta_0}.$$

The second derivatives of H at the relative equilibrium can be derived from those for H_r (Section 4), those for H_p (Section 5), together with:

$$\frac{\partial^2 H_{NS}}{\partial x_N^2} = \frac{\partial^2 H_{NS}}{\partial y_N^2} = \frac{\partial^2 H_{NS}}{\partial x_S^2} = \frac{\partial^2 H_{NS}}{\partial y_S^2} = \frac{\partial^2 H_{NS}}{\partial x_N \partial x_S} = \frac{\partial^2 H_{NS}}{\partial y_N \partial y_S} = \kappa_N \kappa_S / 2$$

while the other second derivatives of H_{NS} vanish.

As in the previous sections, we can choose a symmetry adapted basis of the symplectic slice such that the matrices $d^2 H_\xi|_{\mathcal{N}}(x_e)$ and $L_{\mathcal{N}}$ block diagonalize.

Proposition 6.1 *Let $n \geq 3$ and $\mu \neq 0$. In the following basis for the symplectic slice,*

$$\left(e_1, e_2, e_3, e_4, e_5, e_6, \left\{ \alpha_\theta^{(\ell)}, \alpha_\phi^{(\ell)}, \beta_\theta^{(\ell)}, \beta_\phi^{(\ell)} \mid 2 \leq \ell \leq [n/2] \right\}^* \right)$$

where

$$\begin{aligned}
e_1 &= \cos\theta_0\, \beta_\theta^{(1)} - \sin\theta_0\, \alpha_\phi^{(1)} - n\cos(2\theta_0)/(2\kappa_N)\, \delta y_N \\
e_2 &= \cos\theta_0\, \beta_\theta^{(1)} - \sin\theta_0\, \alpha_\phi^{(1)} - n\cos(2\theta_0)/(2\kappa_S)\, \delta y_S \\
e_3 &= \sin\theta_0\, \alpha_\theta^{(1)} + \cos\theta_0\, \beta_\phi^{(1)} \\
e_4 &= \cos\theta_0\, \alpha_\theta^{(1)} + \sin\theta_0\, \beta_\phi^{(1)} - n\cos(2\theta_0)/(2\kappa_N)\, \delta x_N \\
e_5 &= \cos\theta_0\, \alpha_\theta^{(1)} + \sin\theta_0\, \beta_\phi^{(1)} - n\cos(2\theta_0)/(2\kappa_S)\, \delta x_S \\
e_6 &= \sin\theta_0\, \beta_\theta^{(1)} - \cos\theta_0\, \alpha_\phi^{(1)}
\end{aligned}$$

the Hessian $d^2 H_\xi|_{\mathcal{N}}(x_e)$ block diagonalizes into 1×1 blocks and two 3×3 blocks, and $L_{\mathcal{N}}$ block diagonalizes into 2×2 blocks and one 6×6 block.

Proof. The proof is similar to that for a single ring (see Section 3.2 and Proposition 4.1). □

The mode $\ell = 1$ gives a 3×3 block from which, unfortunately, we can not derive a useful formula for stability analogous to that for a single polar vortex. However, we can derive formulae for the stability of the other modes, and thereby obtain the following sufficient condition for instability, illustrated by Figures 6.1 and 6.2.

Theorem 6.2 *A* $\mathbf{C}_{nv}(R, 2p)$ *relative equilibrium with* $n \geq 4$ *and* $\mu \neq 0$ *is linearly unstable if*

$$\kappa_N(1+\cos\theta_0)^2 + \kappa_S(1-\cos\theta_0)^2 < c_n - (n-1)(1+\cos^2\theta_0),$$

where

$$c_n = \begin{cases} n^2/4 & \text{if } n \text{ is even,} \\ (n^2-1)/4 & \text{if } n \text{ is odd,} \end{cases}$$

and is stable with respect to the $\ell \geq 2$ *modes if this inequality is reversed.*

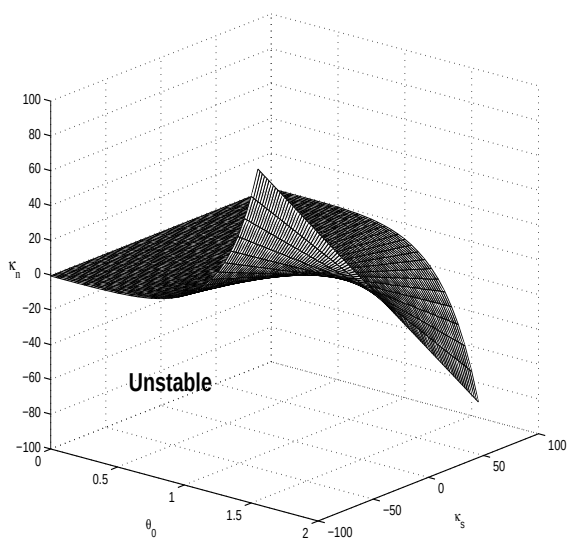

Figure 6.1: The relative equilibria $\mathbf{C}_{nv}(R, 2p)$ in the Northern hemisphere are unstable 'below' this ruled surface in $(\theta_0, \kappa_S, \kappa_N)$-space, shown in the figure for $n = 4$. Above the surface the relative equilibrium is stable with respect to all the $\ell \geq 2$ modes.

Proof. The proof is similar to that for Theorem 5.2. Following the notations of the proof of Theorem 5.2, we have $\lambda_\phi^{(\ell)} = n\ell(n-\ell)/2$ and

$$\lambda_\theta^{(\ell)} = \frac{n}{2\sin^2\theta_0}\left[-(\ell-1)(n-\ell-1) + (n-1)\cos^2\theta_0 + \kappa_N(1+\cos\theta_0)^2 + \kappa_S(1-\cos\theta_0)^2\right].$$

The relative equilibrium is linearly unstable if there exists $\ell \geq 2$, such that $\lambda_\theta^{(\ell)} < 0$. Since the highest $\lambda_\theta^{(\ell)}$ is for $\ell = [n/2]$, the relative equilibrium is linearly unstable if

$$-([n/2]-1)(n-[n/2]-1) + (n-1)\cos^2\theta_0 + \kappa_N(1+\cos\theta_0)^2 + \kappa_S(1-\cos\theta_0)^2 < 0,$$

and is stable with respect to the $\ell \geq 2$ modes if this inequality is reversed. This gives the desired criterion. □

Stability of the $\ell \geq 2$ modes From the theorem we can deduce the following results about the (in)stability of the $\mathbf{C}_{nv}(R, 2p)$ relative equilibria with respect to the $\ell \geq 2$ modes. These modes only occur for $n \geq 4$. We continue to assume the ring lies in the Northern hemisphere.

- In the limiting case as the ring converges to the North pole ($\theta_0 = 0$), for all values of κ_S the relative equilibria are linearly unstable if $\kappa_N < \frac{1}{4}(c_n - 2n + 2)$. This agrees with the instability of a ring and single pole when '$\kappa < \kappa_0$' in Proposition 5.2.

- At the opposite extreme, when the ring is at the equator ($\theta_0 = \pi/2$) they are linearly unstable if $\kappa_N + \kappa_S < c_n - n + 1$. The right hand side of this inequality is non-negative for all positive integers n, and so the 'equatorial' $\mathbf{C}_{nv}(R, 2p)$ relative equilibria are unstable if the total polar vorticity has opposite sign to that of the ring. If $\kappa_N + \kappa_S > 0$ then the critical ratio of the total polar vorticity to the total ring vorticity needed to stabilize the $\ell \geq 2$ modes grows linearly with n.

- For all $n \geq 4$ the relative equilibria are unstable for all latitudes in the Northern hemisphere if $\kappa_N < \frac{1}{4}(c_n - 2n + 2)$ *and* $\kappa_N + \kappa_S < c_n - n + 1$. In particular, for $n \geq 7$ the relative equilibria are unstable for all θ_0 if $\kappa_N < 0$ and $\kappa_S < 0$.

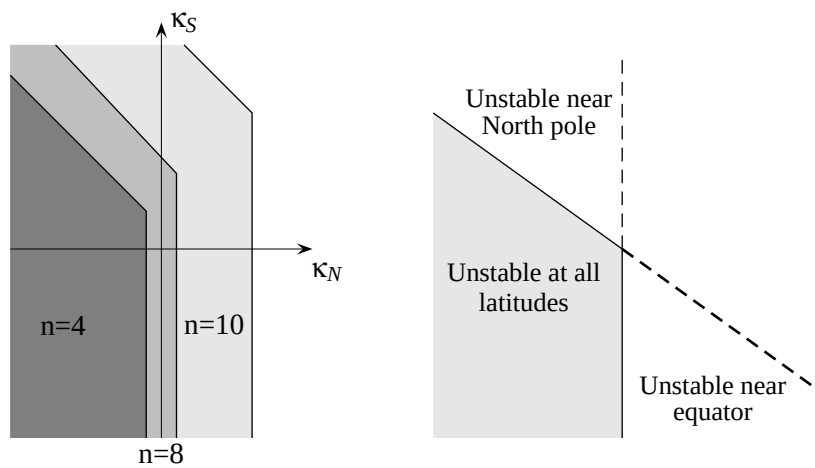

Figure 6.2: Schematic diagram showing the instabilities of the $C_{nv}(R, 2p)$ configurations due to the $\ell \geq 2$ modes: (a) The shaded regions depict the values of the polar vorticities for which *all* the relative equilibria in the Northern hemisphere are unstable: the darkest region represents $n = 4$, the next $n = 8$ and the lightest $n = 10$. (b) demonstrates that above each shaded region of (a) the corresponding relative equilibria near the North pole are unstable, while to the right it is the relative equilibria near the equator which are unstable.

To determine whether there are in fact *stable* relative equilibria it is necessary to evaluate the eigenvalues arising from the $\ell = 1$ mode. This is work in progress, and preliminary numerical investigation suggests:

- For all n and for all sufficiently large and positive polar vorticities there are ranges of θ_0 with elliptic relative equilibria;
- For all n and for κ_N sufficiently positive and $\kappa_S < 0$ there are Lyapounov stable relative equilibria in the Northern hemisphere.

7 Stability of two aligned rings: $\mathbf{C}_{nv}(2R)$

In this section we consider relative equilibria x_e of symmetry type $\mathbf{C}_{nv}(2R)$, that is configurations formed of two 'aligned' rings of n vortices each. We can assume without loss of generality that the vorticities of the vortices in the first and second ring are 1 and κ, respectively, and we denote their co-latitudes by θ_0 and θ_1. We can also assume that the ring of vorticity 1 and co-latitude θ_0 lies in the Northern hemisphere, $\theta_0 \in (0, \pi/2]$. The first question to answer is, *for which values of the parameters* $(\theta_0, \theta_1, \kappa)$ *is the configuration* $\mathbf{C}_{nv}(2R)$ *a relative equilibrium?* It was shown in [LMR01] (p. 126) that for given $\kappa > 0$ and each μ with $|\mu| < n|1+\kappa|$ there is at least one solution for (θ_0, θ_1) with $n\cos\theta_0 + n\kappa\cos\theta_1 = \mu$ and with $\theta_0 < \theta_1$ and at least one with $\theta_1 < \theta_0$. We now make this more precise.

The fixed point set $\mathrm{Fix}(G_{x_e})$ is parametrized by $x := \cos\theta_0$ and $y := \cos\theta_1$. Denote by $\tilde{F}$ the restriction of a function F to $\mathrm{Fix}(G_{x_e})$. The Hamiltonian can be split in such a way that

$$H \;=\; H_{11} + \kappa H_{12} + \kappa^2 H_{22}$$

where H_{11}, H_{12}, H_{22} do not depend on κ, $\tilde{H}_{11}$ does not depend on y and $\tilde{H}_{22}$ does not depend on x. The following proposition shows that for almost every pair (θ_0, θ_1) there exists a unique κ such that the $\mathbf{C}_{nv}(2R)$ configuration with parameters $(\theta_0, \theta_1, \kappa)$ is a relative equilibrium.

Proposition 7.1 *Let x_e be a $\mathbf{C}_{nv}(2R)$ configuration with parameters $(\theta_0, \theta_1, \kappa)$.*

1. *There exists a unique $\kappa \in \mathbb{R}^*$ such that x_e is a relative equilibrium if and only if both the following conditions hold:*

$$\left(\frac{\partial \tilde{H}_{12}}{\partial y} - \frac{\partial \tilde{H}_{11}}{\partial x}\right)(\cos\theta_0, \cos\theta_1) \neq 0, \quad \left(\frac{\partial \tilde{H}_{22}}{\partial y} - \frac{\partial \tilde{H}_{12}}{\partial x}\right)(\cos\theta_0, \cos\theta_1) \neq 0.$$

2. *The configuration x_e is a relative equilibrium for all $\kappa \in \mathbb{R}^*$ in the degenerate case when both the following conditions hold:*

$$\left(\frac{\partial \tilde{H}_{12}}{\partial y} - \frac{\partial \tilde{H}_{11}}{\partial x}\right)(\cos\theta_0, \cos\theta_1) = 0, \quad \left(\frac{\partial \tilde{H}_{22}}{\partial y} - \frac{\partial \tilde{H}_{12}}{\partial x}\right)(\cos\theta_0, \cos\theta_1) = 0.$$

3. *In both cases the angular velocity ξ of x_e satisfies*

$$\xi \;=\; \frac{1}{n}\left(\frac{\partial \tilde{H}_{11}}{\partial x}(x_e) + \kappa \frac{\partial \tilde{H}_{12}}{\partial x}(x_e)\right).$$

In particular, if in addition $\partial_x \tilde{H}_{11}(x_e)$ and $\partial_x \tilde{H}_{12}(x_e)$ are non-zero, then there exists a unique $\kappa \in \mathbb{R}^*$ such that $\xi = 0$ and so x_e is an equilibrium.

Proof. Since $H+\xi\Phi$ is a G_{x_e}-invariant function (see Section 2) the Principle of Symmetric Criticality [P79] implies that x_e is a relative equilibrium if and only if it is a critical point of $\tilde{H}+\xi\tilde{\Phi}$. It follows from $\tilde{\Phi}=n(x+\kappa y)$ that $d(\tilde{H}+\xi\tilde{\Phi})(x_e)=0$ is equivalent to the pair of equations:

$$\kappa\left(\frac{\partial\tilde{H}_{22}}{\partial y}(x_e)-\frac{\partial\tilde{H}_{12}}{\partial x}(x_e)\right)+\frac{\partial\tilde{H}_{12}}{\partial y}(x_e)-\frac{\partial\tilde{H}_{11}}{\partial x}(x_e) \;=\; 0$$

$$\xi=\frac{1}{n}\left(\frac{\partial\tilde{H}_{11}}{\partial x}(x_e)+\kappa\frac{\partial\tilde{H}_{12}}{\partial x}(x_e)\right).$$

The proposition follows easily from these. □

For example, in the case $n=4$ the degenerate case occurs when the two rings form the vertices of a cube. Hence for any values of the vorticities of the two rings the "cube configuration" is a relative equilibrium. However among this family of relative equilibria only one is an equilibrium, namely the one for which the two rings have the same vorticities, $\kappa=1$, which corresponds to the $\mathbb{O}_h(f)$ equilibrium [LMR01], a cube formed of identical vortices. See Figure 7.1.

For $\theta_1=\pi-\theta_0$, the configuration has an extra symmetry and its symmetry type is $D_{nh}(2R)$. Such a configuration is a relative equilibrium if $\kappa=-1$, the two rings have opposite vorticities. The existence and stability of such relative equilibria were studied in [LP02].

With the help of the discussion of Section 3.3, we performed a G_{x_e}-invariant isotypic decomposition and found that the symmetry adapted basis for the symplectic slice at a $\mathbf{C}_{nv}(2R)$ relative equilibrium with $n\geq 3$ and $\mu\neq 0$ is

$$\left(e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8,B_2,B_3,\ldots,B_{[n/2]}\right)$$

where

$$\begin{aligned}
e_1 &= \alpha^{(0)}_{0,\phi}-\alpha^{(0)}_{1,\phi}\\
e_2 &= \kappa\sin\theta_1\,\alpha^{(0)}_{0,\theta}-\sin\theta_0\,\alpha^{(0)}_{1,\theta}\\
e_3 &= \sin\theta_0\,\sin\theta_1\,(\kappa\cos\theta_1\,\alpha^{(1)}_{0,\theta}+\cos\theta_0\,\alpha^{(1)}_{1,\theta})\\
&\qquad +\cos\theta_0\,\cos\theta_1\,(\kappa\sin\theta_1\,\beta^{(1)}_{0,\phi}+\sin\theta_0\,\beta^{(1)}_{1,\phi})\\
e_4 &= \kappa\cos\theta_1\,\alpha^{(1)}_{0,\theta}-\cos\theta_0\,\alpha^{(1)}_{1,\theta}\\
e_5 &= \kappa\sin\theta_1\,\beta^{(1)}_{0,\phi}-\sin\theta_0\,\beta^{(1)}_{1,\phi}\\
e_6 &= \sin\theta_0\,\sin\theta_1\,(\kappa\cos\theta_1\,\beta^{(1)}_{0,\theta}+\cos\theta_0\,\beta^{(1)}_{1,\theta})\\
&\qquad -\cos\theta_0\,\cos\theta_1\,(\kappa\sin\theta_1\,\alpha^{(1)}_{0,\phi}+\sin\theta_0\,\alpha^{(1)}_{1,\phi})\\
e_7 &= \kappa\cos\theta_1\,\beta^{(1)}_{0,\theta}-\cos\theta_0\,\beta^{(1)}_{1,\theta}\\
e_8 &= \kappa\sin\theta_1\,\alpha^{(1)}_{0,\phi}-\sin\theta_0\,\alpha^{(1)}_{1,\phi}
\end{aligned}$$

and,

$$\begin{aligned}
B_\ell &= \left\{\alpha^{(\ell)}_{0,\theta},\,\alpha^{(\ell)}_{1,\theta},\,\beta^{(\ell)}_{0,\phi},\,\beta^{(\ell)}_{1,\phi},\,\alpha^{(\ell)}_{0,\phi},\,\alpha^{(\ell)}_{1,\phi},\,\beta^{(\ell)}_{0,\theta},\,\beta^{(\ell)}_{1,\theta}\right\} \quad \text{for } 2\leq\ell<[n/2]\\
B_{n/2} &= \left\{\alpha^{(n/2)}_{0,\theta},\,\alpha^{(n/2)}_{1,\theta},\,\alpha^{(n/2)}_{0,\phi},\,\alpha^{(n/2)}_{1,\phi}\right\}.
\end{aligned}$$

The adapted basis for $n=2$ is simply (e_1,e_2,e_4,e_8).

Remark. Almost all $\mathbf{C}_{nv}(2R)$ relative equilibria have a non-zero momentum. Indeed $\mu=0$ iff $x+\kappa y=0$, and from the expression of κ one can show that this last equation defines an algebraic curve in variables $(x,y)\in[0,1)\times(-1,1)\simeq\operatorname{Fix}\mathbf{C}_{nv}$.

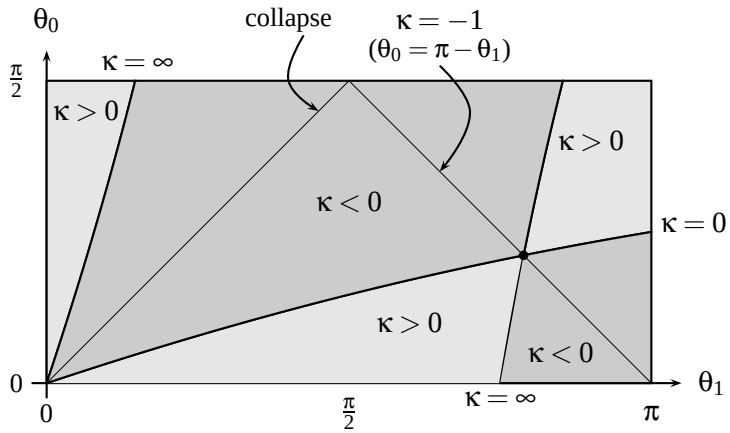

Figure 7.1: Sign of κ for $\mathbf{C}_{nv}(2R)$ relative equilibria. The degenerate case occurs where the curves $\kappa = 0$ and $\kappa = \infty$ intersect. The figure is for $n = 4$, but is similar for other values of n. The only region of stability lies at the bottom right hand corner [??], corresponding to the rings lying far apart in opposite hemispheres, and contained in the region $\kappa < 0$.

With respect to this basis $d^2H_\xi|_{\mathcal{N}}(x_e)$ block diagonalises into: two 1×1 blocks for $\ell = 0$, two 3×3 blocks for $\ell = 1$, two 4×4 blocks for each of $\ell = 2\ldots[(n-1)/2]$), together with two 2×2 blocks for $\ell = n/2$ when n is even. The linearisation $L_{\mathcal{N}}$ block diagonalises into half as many blocks of twice the size. In order to calculate the stability of the relative equilibria, we ran a MAPLE programme to compute numerically the eigenvalues of each of the blocks of $d^2H_\xi|_{\mathcal{N}}(x_e)$ and $L_{\mathcal{N}}$. The results are summarized for $n = 2\ldots 6$ in Figure 7.2. Figure 7.1 shows how the sign of κ varies for relative equilibria with different values of θ_0 and θ_1.

Discussion of results

- The numerical results suggest strongly that the relative equilibria $\mathbf{C}_{nv}(2R)$ are never stable if the two rings lie in the same hemisphere (Figure 7.2) or have the same sign vorticity (Figure 7.1).
- The stable configurations are for θ_0 close to 0 and θ_1 close to π, so the ring of vorticity 1 is close to the North pole and the other ring is close to the South pole with a vorticity close to -1. Thus the two rings 'look like' two polar vortices. It is well known that any configuration of two vortices is Lyapunov stable.
- As n increases the region of stability decreases in size. Numerical experiments with $n \geq 7$ suggest that in these cases the relative equilibria are never stable.
- For $n = 2$, 4 and 6 stability is first lost by a pair of imaginary eigenvalues of the $\ell = n/2$ block of $L_{\mathcal{N}}$ passing through 0 and becoming real. For $n = 3$ and 5 close to the $D_{nh}(2R)$ relative equilibria a pair of imaginary eigenvalues of the $\ell = (n-1)/2$ block passes through 0 but remains on the imaginary axis, so the stability changes from Lyapunov to elliptic. This imaginary pair then collides with another pair, and all move off the imaginary axis to form a complex quadruple and create instability. It seems likely that this behaviour also occurs away from the $D_{nh}(2R)$ relative equilibria, but in a region too small to be seen in the figure.

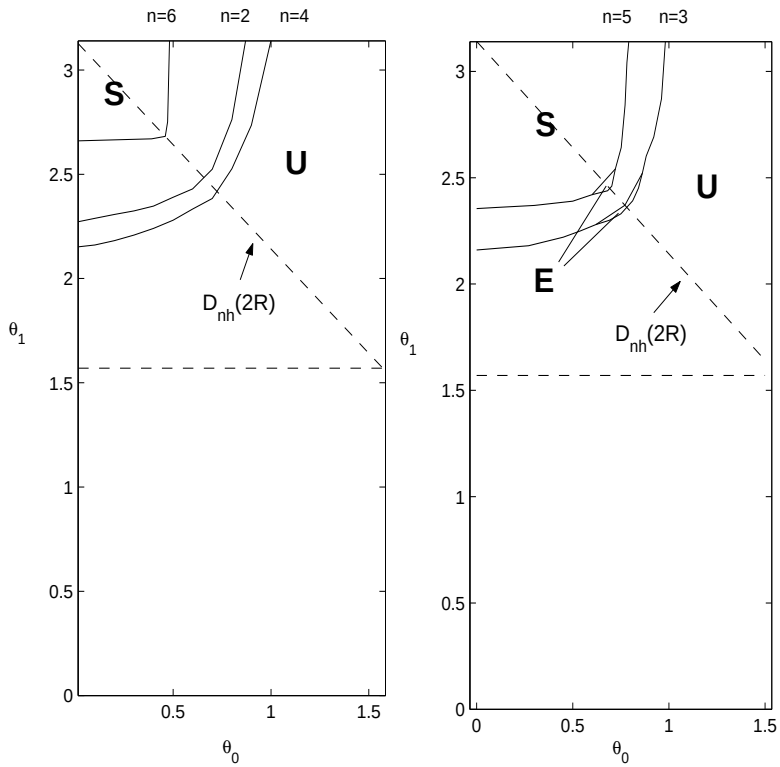

Figure 7.2: Stability results for $\mathbf{C}_{nv}(2R)$ relative equilibria. The curve plotted for each n is the stability frontier: on one side the relative equilibria are Lyapunov stable (S), while on the other side they are linearly unstable (U), with a gap of elliptic stability (E) between the two in the odd case. For $n \geq 7$, it seems likely that the relative equilibria are all unstable.

8 Stability of two staggered rings: $\mathbf{C}_{nv}(R,R')$

In this section we consider relative equilibria formed of two rings of n vortices each of strengths 1 and κ and co-latitude θ_0 and θ_1 respectively. They differ from those of the previous section in that the rings here are "staggered", that is they rotated relative to each other with an offset of π/n. Their symmetry type is $\mathbf{C}_{nv}(R,R')$. As in the previous section we can assume without loss of generality that the ring of vorticity 1 and co-latitude θ_0 lies in the Northern hemisphere.

An analogue of Proposition 7.1 also holds for $\mathbf{C}_{nv}(R,R')$ configurations: for almost every pair (θ_0,θ_1) there exists a unique κ such that the corresponding $\mathbf{C}_{nv}(R,R')$ configuration is a relative equilibrium. With the notation of the previous section, in the non-degenerate case the angular velocity and κ satisfy

$$\begin{aligned} \kappa &= -(\partial_x \tilde{H}_{11} - \partial_y \tilde{H}_{12})/(\partial_y \tilde{H}_{22} - \partial_x \tilde{H}_{12})(x_e), \\ \xi &= n^{-1}(\partial_x \tilde{H}_{11} + \kappa \partial_x \tilde{H}_{12})(x_e). \end{aligned}$$

There exist also degenerate cases. When $\theta_1 = \theta_0$ the configuration forms a single ring with $2n$ vortices with $\kappa = 1$: all the vortices have the same vorticity. These are the relative equilibria of type $\mathbf{C}_{2nv}(R)$ studied in Section 3.2. For $\theta_1 = \pi - \theta_0$, the configuration has an extra symmetry and its symmetry type is $D_{nd}(R,R')$. In this case $\kappa = -1$, the two rings have opposite vorticities. The existence and stability of such relative equilibria were studied in [LP02].

With the help of the discussion of Section 3.3, we found that a symmetry adapted basis for the symplectic slice at a $\mathbf{C}_{nv}(R,R')$ relative equilibrium with $n \geq 3$ and $\mu \neq 0$ is given by:

$$\left(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, B_2, B_3, \ldots, B_{[n/2]}\right)$$

for n odd, while for n even one is given by:

$$\begin{aligned}\big(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, \{B_\ell \mid 2 \leq \ell \leq n/2-1\}, \\ \alpha_{0,\theta}^{(n/2)} - \alpha_{1,\theta}^{(n/2)}, \alpha_{0,\phi}^{(n/2)} - \alpha_{1,\phi}^{(n/2)}, \beta_{1,\theta}^{(n/2)}, \beta_{1,\phi}^{(n/2)}\big),\end{aligned}$$

where the expressions of $e_1, \ldots, e_8$ and B_ℓ remain as in the previous section. The corresponding symmetry adapted basis for $n = 2$ is simply (e_1, e_2, e_3, e_6). As in the previous section, it can readily be seen that almost all $\mathbf{C}_{nv}(R,R')$ relative equilibria have non-zero momenta.

As for the aligned rings, we ran a MAPLE programme to determine the stability of the relative equilibria. The results are summarized in Figure 8.1 for $n = 2 \ldots 6$.

Discussion of results

- Numerical experiments suggest that stable relative equilibria only exist for $n \leq 6$.
- For $n = 5$ and 6 the relative equilibria $\mathbf{C}_{nv}(R,R')$ are stable only if the two rings lie in the same hemisphere but are sufficiently far apart.
- For $n \leq 4$ these stable regions extend to include relative equilibria with the rings in different hemispheres. However, contrary to the case $\mathbf{C}_{nv}(2R)$, the stable regions are far from the line $\theta_1 = \pi - \theta_0$ corresponding to $D_{nd}(R,R')$ relative equilibria.
- For $n = 2$ and 3 there is also a stable region with the two rings in the same hemisphere and close to each other. This includes the stable $\mathbf{C}_{4v}(R)$ and $\mathbf{C}_{6v}(R)$ relative equilibria discussed in Section 3.2.
- Note also that for $n \leq 6$, there exist stable relative equilibria (for some values of κ) in any neighbourhood of $(\theta_0, \theta_1) = (0,0)$, that is with the two rings close to the North pole.
- A study of the sign of κ shows that when $3 \leq n \leq 6$ the relative equilibria with $\kappa < 0$ are all unstable. However for $n = 2$ there exist relative equilibria with κ positive in the stable region corresponding to the two rings both being relatively close to the equator, but in opposite hemispheres.
- The tetrahedral equilibrium with all 4 vortices identical is Lyapounov stable, and in Fig. 8.1 (with $n = 2$) lies at the point where the two stable regions meet the two unstable regions on the $D_{2d}(R,R')$-locus. The analogous point with $n = 3$ corresponds to the stable equilibrium consisting of 6 identical vortices lying at the vertices of an octahedron $\mathbb{O}_h(v)$.

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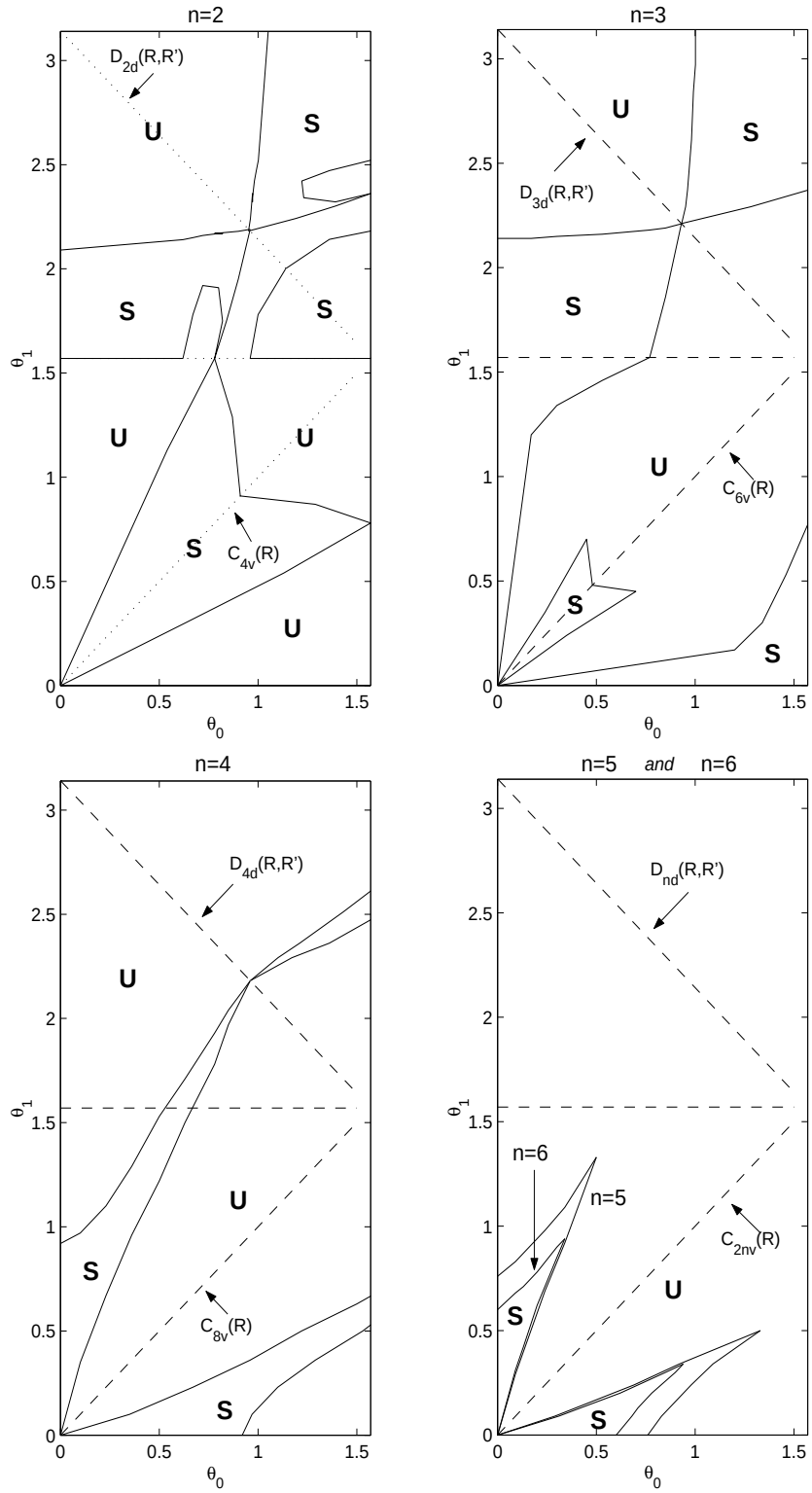

Figure 8.1: Stability results for $\mathbf{C}_{nv}(R,R')$ relative equilibria. The curves plotted are stability frontiers: on one side the relative equilibria are Lyapunov stable (S), while on the other side they are linearly unstable (U). For $n \geq 7$, it seems likely that the relative equilibria are all unstable.

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